Polarized Line Formation with Angle-Dependent Partial Frequency Redistribution

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Abstract. The linear polarization of spectral lines seen in the solar limb observations is created by the scattering of the anisotropic radiation field by atoms and molecules. The partial frequency redistribution (PRD) effects in line scattering are necessary ingredients for interpreting the linear polarization observed in strong resonance lines. This polarization is sensitive to the form of the PRD function used in the polarized line transfer equation. The use of angle-averaged PRD function is quite common in the literature on polarized transfer, as it greatly reduces the computing efforts. In this paper we present our recent work on the importance of the angle-dependent PRD in polarized line transfer. First we present a brief historical background on polarized transfer with angle-dependent PRD. To simplify the numerical work needed to handle angle-dependent PRD function a Stokes vector decomposition technique was developed by Frisch (2009, 2010). After briefly recalling this technique, we present two numerical methods developed to solve the polarized line transfer with angle-dependent PRD. These are (1) polarized accelerated lambda iteration method and (2) the scattering expansion method. Through illustrative examples, we show that while angle-dependent effects are somewhat less important for scattering polarization in the absence of magnetic fields, they play an important role in the presence of a weak magnetic field.

1. Introduction

Several different factors influence in an intricate way the polarization of resonance lines. This resonance polarization is sensitive to the anisotropy of the radiation field, geometry of the medium, temperature structure of the atmosphere, collisional rates, weak magnetic fields, non-LTE mechanisms such as frequency redistribution, quantum interferences between different atomic levels, etc. In particular partial frequency redistribution (PRD) plays a fundamental role in shaping the inner wings of linearly polarized profiles of resonance lines. Here we summarize the importance of the angle-dependent PRD to resonance polarization both in the presence and absence of magnetic fields. We limit our treatment to two-level atom with unpolarized lower level and consider an one-dimensional isothermal atmosphere. For effects of angle-dependent PRD in multidimensional media we refer the reader to Anusha & Nagendra (2011, 2012). Further the angle-dependent PRD effects on line transfer in a two-term atom or a two-level atom with hyperfine structure are discussed in Supriya et al. (2013b).

A scattering process on a two-level atom is in general described by a PRD matrix. One of the difficulties in using this PRD matrix in polarized line transfer computations stems from the fact that the PRD matrix couples in an intricate way the angles and frequencies of the incident and scattered rays. In particular angular dependencies occur not only in polarization phase matrices, but also in PRD functions. The angular dependence
of the PRD function arises from the thermal motion of the scattering atom. Therefore, to reduce the numerical work, one often replaces angle-dependent PRD functions by their angle-averaged versions as suggested by Rees & Saliba (1982). This so-called hybrid approximation has been used in the one-dimensional polarized line transfer computations in the absence of magnetic fields by Rees & Saliba (1982), Nagendra (1986), Faurobert (1987, 1988), Nagendra (1988, 1994), Paletou & Faurobert-Scholl (1997), Sampoorna & Trujillo Bueno (2010), Sampoorna et al. (2010), Anusha et al. (2010), Supriya et al. (2012), Sowmya et al. (2012), and Nagendra & Sampoorna (2012). In the presence of a magnetic field (Hanle effect), a similar angle-averaged approximation has been used by Faurobert-Scholl (1991), Nagendra et al. (1999), Faurobert-Scholl et al. (1999), Fluri et al. (2003), Nagendra et al. (2002), Sampoorna et al. (2008), Anusha et al. (2011), and Sowmya et al. (2012). The angle-averaged approximation is quite reasonable for intensity profiles, but is questionable for linear polarization profiles. This is because unlike intensity, linear polarization is directly controlled by the anisotropy of the incident radiation field.

First polarized line transfer computations with angle-dependent PRD and in the absence of magnetic field was by Dumont et al. (1977), who used the type-I* PRD function of Hummer (1962). Subsequently McKenna (1985) and Faurobert (1987, 1988) considered the effects of angle-dependent type-II† and type-III‡ PRD functions of Hummer (1962) on linear polarization profiles of resonance lines. The case of Hanle effect with angle-dependent PRD was considered by Nagendra et al. (2002). The numerical method used by these authors to solve the concerned transfer problem was however computationally expensive. Therefore, Frisch (2009, 2010) developed a Stokes vector decomposition technique to reduce the computational cost needed to handle angle-dependent PRD. Using this decomposition technique efficient iterative methods were developed in Sampoorna et al. (2011), Sampoorna (2011b), and Nagendra & Sampoorna (2011) to solve polarized line transfer equation with angle-dependent PRD both in the absence and presence of magnetic fields. Here we present a summary of the results presented in the above-mentioned papers.

2. Rayleigh and Hanle PRD Matrices

A scattering event is characterized by a redistribution matrix that describes the correlation between frequencies, directions, and polarization of the incident and scattered rays. The problem of redistribution of resonance radiation including the effects of collisions was investigated by Omont et al. (1972). Based on this work, Domke & Hubeny (1988) derived the explicit form of the PRD matrix for resonance scattering on a two-level atom with unpolarized lower level. A more elegant and sophisticated expression of this

*Type-I redistribution represents the case of infinitely sharp lower and upper levels (or pure Doppler redistribution in the laboratory frame).

†Type-II redistribution represents the case of infinitely sharp lower level and radiatively broadened upper level (coherent scattering in the atomic frame).

‡Type-III redistribution represents the case of infinitely sharp lower level and radiatively as well as collisionally broadened upper level (complete redistribution in the atomic frame).
Rayleigh PRD matrix was derived by Bommier (1997a) and it is given by

\[
\hat{R}_{ij}(x, \Omega, x', \Omega') = \sum_{kQ} T^K_Q(i, \Omega) R^K(x, x', \Theta)(-1)^Q T^K_Q(-j, \Omega'), \quad i, j = 0, 1, 
\]

where

\[
R^K(x, x', \Theta) = W_K(J_l, J_u) \left\{ \alpha R_{II}(x, x', \Theta) + \left[ \beta^{(K)} - \alpha \right] R_{III}(x, x', \Theta) \right\}, \quad (2)
\]

Here, \( T^K_Q(i, \Omega) \) are the irreducible tensors (Landi Degl’Innocenti 1984) with \( K = 0, 2 \) and \( Q \in [-K, +K] \). The type-II and type-III PRD functions of Hummer (1962) are denoted respectively by \( R_{II} \) and \( R_{III} \). They depend on the scattering angle \( \Theta \) between the incident (\( \Omega' \)) and scattered (\( \Omega \)) rays. \( x \) and \( x' \) are the non-dimensional frequencies of scattered and incident rays. The atomic polarizability factor \( W_K \) depends on the total angular momentum quantum numbers of the lower (\( J_l \)) and upper (\( J_u \)) levels. The branching ratios are given by

\[
\alpha = \frac{\Gamma_R}{\Gamma_R + \Gamma_I + \Gamma_E}, \quad \beta^{(K)} = \frac{\Gamma_R}{\Gamma_R + \Gamma_I + D^{(K)}}, \quad (3)
\]

where \( \Gamma_R \) is the upper level spontaneous deexcitation radiative rate, \( \Gamma_I \) and \( \Gamma_E \) are the inelastic and elastic collision rates, and \( D^{(K)} \) is the collisional depolarization rate such that \( D^{(0)} = 0 \).

The effects of a weak magnetic field \( B \) on resonance polarization were considered by Omont et al. (1973). Although they did not derive an explicit form of the Hanle PRD matrix, they showed that Hanle effect is affective only in the line core and vanishes in the line wings. Based on this suggestion, a heuristic form for the Hanle PRD matrix was proposed. It is given by

\[
\hat{R}_{ij}(x, \Omega, x', \Omega', B) = R(x, x', \Theta) \times \begin{cases} \hat{P}_{ij}(\Omega', \Omega', B), & \text{for } x \leq x_c \\ \hat{P}_{ij}(\Omega, \Omega'), & \text{for } x > x_c, \end{cases} \quad (4)
\]

where now \( i, j = 0, 1, 2 \), \( x_c \) is the transition frequency between the line core and wings, \( \hat{P}_{ij}(\Omega, \Omega', B) \) and \( \hat{P}_{ij}(\Omega', \Omega', B) \) are the elements of the Hanle (Stenflo 1994) and Rayleigh (Chandrasekhar 1950) phase matrices respectively. \( R(x, x', \Theta) \) is either type-II or type-III PRD function or a linear combination of them. We refer to Eq. (4) as 1D cut-off approximation.

The explicit form of the PRD matrix that takes into account the effects of elastic collisions in the presence of arbitrary magnetic fields was derived by Bommier (1997b). The Hanle PRD matrix is obtained by going to the weak field limit (Zeeman splitting much smaller than the Doppler width of the line), of this general PRD matrix. In the weak field limit circular polarization decouples from both intensity and linear polarization. This is referred to as Approximation-I by Bommier (1997b), wherein the coupling of frequency redistribution with the angular phase matrix is kept intact. Thereby, no artificial frequency domains are required to represent the transition from Hanle in the line core to Rayleigh scattering in the line wings. The angle-dependent redistribution matrix under Approximation-I may be written as (see Bommier 1997b)

\[
\hat{R}_{ij}(x, \Omega, x', \Omega', B) = \sum_{kQ'Q} T^K_Q(i, \Omega) N^K_Q(x, x', \Theta, B)(-1)^Q T^K_Q(-j, \Omega'). \quad (5)
\]
Here the magnetic kernel is of the form

\[
N_{QQ}^{K}(x, x', \Theta, B) = N_{QQ}^{K,II}(x, x', \Theta, B) + N_{QQ}^{K,III}(x, x', \Theta, B),
\]  

(6)

where the type-II and type-III magnetic kernels are given by

\[
N_{QQ}^{K,II/III}(x, x', \Theta, B) = \sum_{Q'} d_{QQ'}^{K}(\theta_B)d_{Q'Q}^{K}(-\theta_B)Z_{KQ}^{II/III}(x, x', \Theta, B).
\]  

(7)

Here \( B \) and \( (\theta_B, \chi_B) \) represent respectively the strength and orientation of the vector magnetic field \( B \) about the atmospheric normal. Explicit expressions for the reduced rotation matrices \( d_{QQ'}^{K} \) can be found in Landi Degl’Innocenti & Landolfi (2004, Table 2.1, p. 57). For a two-level atom with unpolarized lower-level and a \( J_l \rightarrow J_u \rightarrow J_l \) scattering transition, \( Z_{KQ}^{II/III} \) take the forms

\[
Z_{KQ}^{II}(x, x', \Theta, B) = \frac{\Gamma_R}{\Gamma_R + \Gamma_I + \Gamma_E + i\omega_L g_{J_u} Q} w_{J_u J_l}^{(K)} R_{Q,II}^{K,K}(x, x', \Theta, B),
\]  

(8)

\[
Z_{KQ}^{III}(x, x', \Theta, B) = \frac{\Gamma_R}{\Gamma_R + \Gamma_I + \Gamma_E + i\omega_L g_{J_u} Q} \Gamma_R \Gamma_I + \Gamma_E + i\omega_L g_{J_u} Q \times R_{Q,III}^{K,K,K}(x, x', \Theta, B).
\]  

(9)

Here \( \omega_L \) is the Larmor frequency, and \( g_{J_u} \) is the Landé factor of the upper level. The quantities \( w_{J_u J_l}^{(K)} \) are defined in Eq. (10.11) of Landi Degl’Innocenti & Landolfi (2004). The composite redistribution functions of type-II \( (R_{Q,II}^{K,K}) \) and type-III \( (R_{Q,III}^{K,K,K}) \) can be found respectively in Nagendra & Sampoorna (2011) and Sampoorna (2011a, see also Sampoorna et al. 2007a,b).

Hanle PRD matrix given by Approximation-I is numerically expensive to compute due to the inherent coupling between the frequency redistribution and the polarization. Therefore to further simplify the problem, Bommier (1997b) introduced two more levels of approximations in which the frequency space \((x, x')\) is divided into several domains. In each domain the frequency redistribution is decoupled from the polarization. Angle-dependent and angle-averaged PRD functions are respectively used in Approximation-II and Approximation-III. Thus the domains are both angle \( (\Theta) \) and frequency dependent in Approximation-II, while they are only frequency dependent in Approximation-III. Under Approximation-II, the Hanle PRD matrix is given by Eq. (5), but with the replacement

\[
N_{QQ}^{K}(x, x', \Theta, B) = N_{QQ}^{K}(m, B)R_{II}(x, x', \Theta) + N_{QQ}^{K}(m, B)R_{III}(x, x', \Theta),
\]  

(10)

in Eq. (6). The index \( m \) denotes different frequency domains, \( m = 1, 2, 3 \) represent the domains for the type-III redistribution and \( m = 4, 5 \) represent the domains for the type-II redistribution. Under Approximation-III the Hanle PRD matrix is given by Eqs. (5) and (10), but with the angle-dependent PRD functions replaced by the angle-averaged ones. The analytic form of \( N_{QQ}^{K}(m, B) \) in different angle-dependent (Approximation-II) and angle-averaged (Approximation-III) frequency domains can be found in Bommier (1997b, see also Nagendra et al. 2002; Anusha et al. 2011).
In the case of 1D cut-off approximation, \( N^{K}_{QQ} \), takes the form

\[
N^{K}_{QQ}(x, x', \Theta, B) = R(x, x', \Theta) \times \begin{cases} 
N^{K}_{QQ}(B), & \text{for } x \leq x_c, \\
(1 - \epsilon)W_K(J_l, J_u)\delta_{QQ'}, & \text{for } x > x_c,
\end{cases}
\]

where \( \epsilon = \Gamma_1/(\Gamma_R + \Gamma_1) \) is the thermalization parameter.

### 3. A Decomposition Technique for the Angle-Dependent PRD Problems

Here we present the essential steps of the decomposition technique of Frisch (2009) developed for angle-dependent PRD problems. We also briefly discuss the modifications necessary to handle Hanle PRD matrix under Approximation-II. We remark that the decomposition technique developed in Frisch (2009, see also other connected papers) cannot deal with polarized transfer problems with an incident polarized radiation. Because in such cases a rigorous treatment of the boundary condition becomes necessary. A consistent way of dealing with such type of problems has been recently presented by Faurobert et al. (2013).

The polarized transfer equation for the Stokes vector component has the form

\[
\frac{\partial I_i}{\partial \tau} = [\varphi(x) + r] [I_i(\tau, x, \Omega) - S_i(\tau, x, \Omega)], \quad i = 0, 1, 2,
\]

where \( \Omega (\theta, \chi) \) is the ray direction with respect to the atmospheric normal and \( \mu = \cos \theta \). The line optical depth is denoted by \( \tau \) and \( \varphi(x) \) is the normalized Voigt function. \( r \) is the ratio of continuum to line absorption coefficient. The total source vector is given by

\[
S_i(\tau, x, \Omega) = \frac{\varphi(x)S_{Li}(\tau, x, \Omega) + rS_{c,i}}{\varphi(x) + r},
\]

where the unpolarized continuum source vector components are : \( S_{c,0} = B_{v0} \), with \( B_{v0} \) the Planck function at the line center, and \( S_{c,1} = S_{c,2} = 0 \). The line source vector is

\[
S_{Li}(\tau, x, \Omega) = G_i(\tau) + \int_{-\infty}^{+\infty} \frac{1}{4\pi} \sum_{j=0}^{2} \frac{R_{ij}(x, \Omega, x', \Omega', B)}{\varphi(x)} I_j(\tau, x', \Omega') \sin \theta' d\theta' d\chi' dx',
\]

where \( \Omega' (\theta', \chi') \) is the incident ray direction with respect to the atmospheric normal. The unpolarized primary source vector components are : \( G_0(\tau) = \epsilon B_{v0} \) and \( G_1(\tau) = G_2(\tau) = 0 \). Following Frisch (2009) we decompose the Stokes source vector into six irreducible components \( S^K_Q \) as

\[
S_i(\tau, x, \Omega) = \sum_{KQ} T^K_Q(i, \Omega)S^K_Q(\tau, x, \Omega),
\]

with a similar decomposition for the Stokes vector \( I_i \) in terms of \( I^K_Q \). The irreducible line source vector components are then given by

\[
S^K_{Li}(\tau, x, \Omega) = G^K_Q(\tau) + \int_{-\infty}^{+\infty} \frac{1}{4\pi} \sum_{Q'} \frac{N^K_{QQ'}(x, x', \Theta, B)}{\varphi(x)} \sum_{K'Q''} \Gamma_{KQ', K'Q''}(\Omega') \times I^K_{Q'}(\tau, x', \Omega') \sin \theta' d\theta' d\chi' dx',
\]
functions, and are given by

\[ k \text{ with } \]

Because of the presence of an angle-dependent PRD function the irreducible components of the vector magnetic kernel in the Fourier basis \( \hat{\rho} \) can also be expanded over the azimuth \( \chi \) as

\[ R(x, x', \Theta) = \frac{1}{2} \sum_{k=\pm \infty} \hat{r}^{(k)}(x, \theta, x', \theta') \exp[i(k - k')], \quad (18) \]

where the Fourier coefficient

\[ \hat{r}^{(k)}(x, \theta, x', \theta') = (1 + \delta_{0k})(2 - \delta_{0k}) \frac{2}{2\pi} \int_{0}^{2\pi} R(x, x', \Theta) \cos[k(\chi - \chi')] \, d(\chi - \chi'). \quad (19) \]

It is easy to verify that \( \hat{r}^{(k)} = \hat{r}^{(-k)} \). Using Eq. (18) in Eq. (16), it can be shown that \( I_{Q}^{K} \) can also be expanded over the azimuth \( \chi \) as

\[ I_{Q}^{K}(\tau, x, \Omega) = \frac{1}{2} \sum_{k=\pm \infty} \hat{r}^{(k)}(\tau, x, \theta) \exp[ik\chi], \quad (20) \]

with similar expansions for \( G_{Q}^{K} \) and \( S_{Q}^{K} \) (see Frisch 2009, for details). The Fourier coefficients \( \hat{r}^{(k)K} \) satisfy a transfer equation similar to Eq. (12), with the source term given by Eq. (13), but with \( \tilde{S}_{l,Q}^{(k)K} \) and \( \tilde{S}_{c,\Omega}^{(k)K} \) instead of \( S_{l,i} \) and \( S_{c,j} \). Here \( \tilde{S}_{c,\Omega}^{(k)K} = 2\delta_{0k}\delta_{0K}\delta_{0Q}B_{0} \). The Fourier coefficients of the line source vector are given by

\[ \tilde{S}_{l,Q}^{(k)K}(\tau, x, \theta) = G_{Q}^{(k)K}(\tau) \int_{-\infty}^{\infty} \frac{1}{2} \sum_{Q} \sum_{B} \frac{\hat{N}^{(k)K}_{Q} Q'(x)}{Q} \sum_{k'=\pm \infty} \hat{r}^{(k-k')} Q''(\theta') \times \tilde{r}^{(k')K'}(\tau, x', \theta') \sin \theta' \, d\theta' \, dx', \quad (21) \]

where \( \hat{G}_{Q}^{(k)K}(\tau) = 2\delta_{0k}\delta_{0K}\delta_{0Q}G_{0}(\tau) \), and \( k' = k + Q' - Q'' \). Clearly, for a given value of \( k \), the values of \( k' \) restricted to \( k - 4 \leq k' \leq k + 4 \). The non-zero azimuthal Fourier coefficients are of the form

\[ \hat{r}^{(Q'-Q'')}_{K,\Omega,\Omega'}(\theta') = \sum_{i=0}^{2} (-1)^{Q'} \hat{T}_{Q,\Omega}^{K}(i, \theta') \hat{T}_{\Omega'}^{K'}(i, \theta'), \quad (22) \]

with \( K \) and \( K' \) both even or both odd. \( \hat{T}_{Q}^{K}(i, \theta) \) are a combination of trigonometric functions, and are given by \( T_{Q}^{K}(i, \Omega) = \hat{T}_{Q}^{K}(i, \theta) e^{iQ\tau} \). The analytic form of the magnetic kernel in the Fourier basis \( \hat{N}_{Q}^{(k)K} = \hat{N}_{Q}^{(k)K} \) depends on the level of approximation used to
represent the Hanle PRD matrix (see Section 2 for details). In the case of 1D cut-off approximation

\[
\tilde{N}_Q^{(k)K}(x, \theta, x', \theta') = \tilde{r}^{(k)}(x, \theta, x', \theta') \times \begin{cases} 
N_{QQ}^K(B), & \text{for } x \leq x_c, \\
(1 - \epsilon)W_K(J, J_a)\delta_{QQ'}, & \text{for } x > x_c.
\end{cases}
\]

(23)

In the case of Approximation-II the frequency domains depend on \((\chi - \chi')\) through the scattering angle \(\Theta\), while the Fourier basis introduced above is inherently azimuth-angle independent. Thus to implement Approximation-II in the decomposition presented above, Sampoorna (2011b) used the angle-averaged frequency domains of Approximation-III for the angle-dependent problem. With this simplification \(\tilde{N}_Q^{(k)K}\) for Approximation-II is given by

\[
\tilde{N}_Q^{(k)K}(x, \theta, x', \theta', B) = N_{QQ}^K(m, B) \tilde{r}_{II}^{(k)}(x, \theta, x', \theta') + N_{QQ}^K(m, B) \tilde{r}_{III}^{(k)}(x, \theta, x', \theta').
\]

(24)

We remark that the use of angle-averaged frequency domains for handling angle-dependent Hanle transfer problem leads to some loss of information as pointed out by Anusha & Nagendra (2012). To overcome this inconsistency they proposed an azimuthal Fourier decomposition technique, wherein one expands the angle-dependent Hanle PRD matrix over only the outgoing azimuth angle \(\chi\), instead of expanding the PRD function over the azimuth difference \((\chi - \chi')\) as done here. This alternative decomposition technique allows one to use the angle-dependent frequency domains for angle-dependent Hanle transfer problem and gives self-consistent solution as demonstrated in Supriya et al. (2013a).

In the case of Hanle PRD matrix under Approximation-I, the decomposition technique presented here does not pose any such problems as it did in the case of Approximation-II. Thus we continue to use this decomposition technique for Approximation-I (non-domain based theory). In this case the \(\tilde{N}_Q^{(k)K}\) is given by Eqs. (6)–(9), but with \(\tilde{R}_Q^{K,K}\) and \(\tilde{R}_Q^{K,K,K}\) replaced respectively by \(\tilde{R}_Q^{(k)K,K,K}\) and \(\tilde{R}_Q^{(k)K,K,K}\).

In the non-magnetic case, the Stokes vector and source vector are axisymmetric. Thus the above decomposition simplifies considerably in this case (see Frisch 2010). In particular, \(I_i\) can be decomposed into four irreducible components as follows:

\[
I_i(\tau, x, \mu) = \sum_{k, q=0} \tilde{F}_Q^K(i, \mu) \tilde{I}_Q^K(\tau, x, \mu), \quad i = 0, 1,
\]

(25)

with a similar expansion for \(S_i\). The irreducible line source vector components \(S_{l,Q}^{K}(\tau, x, \mu)\) may then be written as

\[
S_{l,Q}^{K}(\tau, x, \mu) = \delta_{K0} \delta_{Q0} G_0(\tau) + \int_{-\infty}^{+\infty} \int_{-1}^{+1} \tilde{R}_Q^{K}(x, \mu', \mu') \varphi(x) \sum_{K', q' = 0} \tilde{I}_Q^{K'}(\mu') \times I_Q^{K'}(\tau, \mu', \mu') d\mu' d\mu',
\]

(26)

where

\[
\tilde{R}_Q^{K}(x, \mu', \mu') = \sum_{j=0,1} \tilde{F}_Q^K(j, \theta') \tilde{F}_Q^K(j, \theta'),
\]

(27)

\[
\tilde{R}_Q^{(k)K}(x, \mu', \mu') = W_K(J, J_a) \left[ \alpha \tilde{r}_{II}^{(Q)}(x, \theta, \theta') + \beta^{(K)} - \alpha \right] \tilde{r}_{III}^{(Q)}(x, \theta, \theta').
\]

(28)
4. Numerical Methods of Solution

Here we present two iterative methods to solve the angle-dependent PRD problems.

4.1. Scattering Expansion Method for Angle-Dependent PRD Problems

In this method Stokes $I$ is computed by neglecting its coupling with linear polarization, which is a good approximation when linear polarization is small. Further, we assume that Stokes $I$ is axisymmetric and is given by the component $\tilde{I}^{(0)}_{10}$ itself to an excellent approximation. This component is the solution of a non-LTE unpolarized radiative transfer equation with the line source function given by

$$S^0_{l0}(\tau, x, \theta) = \epsilon B_{v_0} + (1 - \epsilon) \int_{-\infty}^{+\infty} \frac{\pi r^{(0)}(x, \theta, x', \theta')}{\varphi(x)} I^0_0(\tau, x', \theta') \sin \theta' d\theta' dx', \quad (29)$$

where $r^{(0)} = \tilde{r}^{(0)}/2$, $I^0_0 = I^{(0)0}/2$, and $S^0_{l0} = \tilde{S}^{(0)0}/2$. Equation (29) can be solved using an accelerated lambda iteration (ALI) method (Sampoorna et al. 2011).

For the computation of the polarization, an integral equation is established for the $(k, Q)$ component of the source term. Its solution can then be written as a series expansion in the mean number of scattering events (Frisch et al. 2009). The first term in this expansion gives the single scattered solution, while the following terms take into account multiple scattering events. The single scattering approximation to each component $\tilde{S}^{(k)2}_{lQ}$ is obtained by keeping only the contribution of $\tilde{I}^{(0)0}_{0}$ on the RHS of Eq. (21) to the $K = 2$ Fourier coefficients. It is given by

$$\tilde{S}^{(k)2}_{lQ}(\tau, x, \theta) \approx \int_{-\infty}^{+\infty} \frac{\pi \tilde{K}^{(k)2}(x, \theta, x', \theta', B)}{\varphi(x)} \tilde{I}^{(k)}_{2-4,00}0(\theta') I^0_0(\tau, x', \theta') \sin \theta' d\theta' dx', \quad (30)$$

where the superscript (1) stands for the single scattering. It can be shown that, the values of $k$ now get limited to $k = 0, \pm 1$, and $\pm 2$ (Sampoorna 2011b). The single scattered radiation field $\tilde{I}^{(k)2}_{Q}$ for $k = 0, \pm 1, \pm 2$ is calculated using a formal solver. It is the starting solution for calculating the higher-orders of scattering.

To get the higher-order terms we substitute for $\tilde{I}^{(k)K'}_{Q''}$ appearing in the RHS of Eq. (21), from $\tilde{I}^{(k)K'0}_{Q''}$. Thus we now include the coupling of $(K = 2, Q)$ components with other polarization components $(K' = 2, Q')$, but continue to set $k' = 0$. As a result $k = Q'' - Q'$, which takes values 0, $\pm 1$, $\pm 2$, $\pm 3$, and $\pm 4$. Clearly the number of the $K = 2$ Fourier coefficients that contribute to the source vector has increased from 25 for single scattering to 45 for higher orders of scattering. The $\tilde{S}^{(k)2}_{lQ}$ at order $n$ are therefore given by (Nagendra & Sampoorna 2011)

$$\tilde{S}^{(k)2}_{lQ}(\tau, x, \theta)^{(n)} = \tilde{S}^{(k)2}_{lQ}(\tau, x, \theta)^{(1)} + \sum_{Q'} \int_{-\infty}^{+\infty} \frac{\pi \tilde{K}^{(k)2}(x, \theta, x', \theta', B)}{\varphi(x)} \tilde{I}^{(k)2}_{2-4,00}0(\theta') \tilde{I}^{(k)0}_{Q''}(\tau, x', \theta') \sin \theta' d\theta' dx', \quad (31)$$

with $k = Q'' - Q'$. We recall that $\tilde{S}^{(k)2}_{lQ}(\tau, x, \theta)^{(1)}$ are zero for $k = \pm 3$ and $\pm 4$. The higher-order terms given by Eq. (31) are calculated iteratively, until convergence is reached.
In the non-magnetic case Eqs. (30) and (31) simplify considerably (see Eqs. (25)–(28)). In particular, the single scattering approximation for each component \( S^n_{Q,l} \) (with \( Q = 0, 1, 2 \)) is given by

\[
\left[ S^n_{l,Q}(\tau, x, \mu) \right]^{(1)} = \int_{-\infty}^{+\infty} \frac{1}{2} \int_{-1}^{+1} \frac{\tilde{R}^2_Q(x, \mu, x', \mu')}{\varphi(x)} \Gamma_{Q0}^{00}(\mu') I_0^0(\tau, x', \mu') \, d\mu' \, dx'.
\] (32)

The higher-order scattering terms are given by

\[
\left[ S^n_{l,Q}(\tau, x, \mu) \right]^{(n)} = \left[ S^n_{l,Q}(\tau, x, \mu) \right]^{(1)} + \int_{-\infty}^{+\infty} \frac{1}{2} \int_{-1}^{+1} \frac{\tilde{R}^2_Q(x, \mu, x', \mu')}{\varphi(x)} \sum_{Q' \geq 0} \Gamma_{QQ'}^{22}(\mu')
\times \left[ \tilde{I}_{Q'}^2(\tau, x', \mu') \right]^{(n-1)} \, d\mu' \, dx'.
\] (33)

4.2. Polarized Accelerated Lambda Iteration Method

Polarized ALI (PALI) method to solve polarized radiative transfer equation with angle-dependent PRD in the absence of magnetic fields has been developed by Sampoorna et al. (2011). Here we briefly present the essential steps of this iterative method.

In the non-magnetic case, we have only four irreducible source vector and Stokes vector components. Therefore we can introduce four-component vectors \( S(\tau, x, \mu) = [S^0, S^1, S^2]^\top \) and \( I(\tau, x, \mu) = [I^0, I^1, I^2]^\top \). Equation (26) can now be written in vector form as

\[
S_l(\tau, x, \mu) = G(\tau) + \int_{-\infty}^{+\infty} \frac{1}{2} \int_{-1}^{+1} \frac{\tilde{R}(x, \mu, x', \mu')}{\varphi(x)} \Gamma(\mu') I(\tau, x', \mu') \, d\mu' \, dx',
\] (34)

where \( G(\tau) = [G^0(\tau), 0, 0, 0]^\top \). The \( 4 \times 4 \) matrix \( \tilde{R} \) is diagonal, namely \( \tilde{R} = \text{diag}[\tilde{R}_0^0, \tilde{R}_1^0, \tilde{R}_1^1, \tilde{R}_2^1] \). Its elements are defined in Eq. (28). The \( 4 \times 4 \) matrix \( \Gamma \) is a full matrix with elements \( \Gamma_{KK'}^{QQ'} \) defined in Eq. (27) (see also Frisch 2010).

The formal solution of the transfer equation is \( I_{xy} = \Lambda_{xy} [S_{xy}] \), where \( \Lambda_{xy} \) is the frequency- and angle–dependent (4 \times 4) integral operator. Introducing the operator splitting \( \Lambda_{xy} = \Lambda_{xy}^* + (\Lambda_{xy} - \Lambda_{xy}^*) \), where \( \Lambda_{xy}^* \) is the diagonal approximate operator (see Olson et al. 1986), it is possible to set up an iterative scheme to compute the source vectors as

\[
S^n_{x,y} = S^n_{x,y} + \delta S^n_{x,y}, \quad S^n_{l,x,y} = S^n_{l,x,y} + \delta S^n_{l,x,y},
\] (35)

where \( n \) is the iteration index. The line source vector corrections are given by

\[
\delta S^n_{l,x,y} = \int_{-\infty}^{+\infty} \frac{1}{2} \int_{-1}^{+1} \frac{\tilde{R}_{y,x',\mu'} \Gamma_{\mu'} p_x \Lambda_{x',\mu'}^* \left[ \delta S^n_{y,x',\mu'} \right] \, d\mu' \, dx' = G(\tau) + \bar{J}^n_{xy} - S^n_{l,x,y},
\] (36)

where \( p_x = \varphi_x/(\varphi_x + r) \) and

\[
\bar{J}^n_{xy} = \int_{-\infty}^{+\infty} \frac{1}{2} \int_{-1}^{+1} \frac{\tilde{R}_{y,x',\mu'} \Gamma_{\mu'} \Lambda_{x',\mu'} \left[ S^n_{y,x',\mu'} \right] \, d\mu' \, dx'.
\] (37)

The line source vector corrections can be computed using either the frequency-angle by frequency-angle (FABFA) method or a core-wing separation method (Sampoorna.
et al. 2011). In the FABFA method $\delta S^n_{l,\mu}$ are computed using $\delta S^n_{l} = A^{-1} r^n$, where the residual vector $r^n$ is given by the right-hand side of Eq. (36). At each depth point, $r^n$ and $\delta S^n_{l}$ are vectors of length $4N_x 2N_\mu$, where $N_x$ is the number of frequency points in the range $[0, x_{\text{max}}]$ and $N_\mu$ is the number of angle points in the range $[0 < \mu \leq 1]$. The matrix $A$ thus has dimensions $(4N_x 2N_\mu \times 4N_x 2N_\mu)$. In the case of isothermal atmospheres, matrix $A^{-1}$ is computed only once before the iteration cycle.

The core-wing separation method, on the other hand avoids the numerically expensive computation and inversion of $A$ matrix (Paletou & Auer 1995). This method, involves making the following approximation

$$\tilde{R}_{x',\mu} \approx R_{x',\mu}^{AA}$$

(38)

where $R_{x',\mu}$ are $(4 \times 4)$ diagonal matrices defined by $R_{x,\mu} = \text{diag}[1, 1, 0, 0]$ and $R_{x',\mu}^{AA} = \text{diag} \left[ \left( R_{x,\mu}^{AA} \right)^0, \left( R_{x',\mu}^{AA} \right)^2, 0, 0 \right]$. The elements $\left( R_{x,\mu}^{AA} \right)^K$ have a form similar to $\tilde{R}_{x,\mu}^K$ given in Eq. (28), but with $\tilde{R}_{x,\mu}^{(0)}$ replaced by corresponding angle-averaged type-II and type-III PRD functions. Using Eq. (38) in Eq. (36), one can establish two separate equations for computing $\delta S^n_{l,\mu}$, one for the core ($x \leq x_c$) and one for the wings ($x > x_c$), wherein only $4 \times 4$ matrices alone need to be handled (Sampoorna et al. 2011).

5. Results and Discussions

We consider isothermal, self-emitting plane-parallel atmospheres with no incident radiation at the boundaries. These slab models are characterized by $(T, a, \epsilon, r)$, where $T$ is the optical thickness of the slab. The Planck function at the line center $B_\nu$ is taken as unity. Further we neglect the effects of elastic collisions ($\Gamma_E = 0$). The polarizability factor $W_2$ is taken as unity. For angle, frequency, and depth grid discretization, we have used the quadratures of the same order as those used by Nagendra & Sampoorna (2011).

5.1. Stokes Profiles Calculated with Angle-Dependent and Angle-Averaged PRD Matrices

In order to evaluate the importance of angle-dependent PRD, here we compare the emergent Stokes profiles computed with angle-dependent and angle-averaged PRD matrices. The Stokes parameters calculated with angle-dependent PRD matrix are shown as solid lines and those calculated with angle-averaged PRD matrix as dashed lines.

5.1.1. Non-Magnetic Resonance Scattering Case

Here we consider the case of resonance scattering in the absence of magnetic fields. In Fig. 1 we compare the emergent $I$ and $Q/I$ profiles computed with angle-averaged (dashed lines) and the angle-dependent (solid lines) PRD matrices for different values of optical thickness $T$. We cover a range of optical thickness from thin slabs ($T = 20$) to very thick slabs ($T = 2 \times 10^8$). We also show a plot of the absolute difference $\Delta(Q/I) = ||(Q/I)_{AD} - (Q/I)_{AA}||$ in percentage, where the acronyms AD for angle-dependent and AA for angle-averaged.

As the optical thickness $T$ increases, the amplitude of the near wing peak in $Q/I$ initially increases until $T = 2 \times 10^4$. For $T > 2 \times 10^4$ the near wing peak amplitude
Figure 1. The emergent $I$ and $Q/I$ profiles at $\mu = 0.11$ and $\gamma = 0^\circ$ computed for the angle-averaged (dashed lines) and the angle-dependent (solid lines) PRD. Panel (a): thin lines $T = 20$, medium thick lines $T = 200$, and thick lines $T = 2 \times 10^3$. Panel (b): thin lines $T = 2 \times 10^4$, medium thick lines $T = 2 \times 10^6$, and thick lines $T = 2 \times 10^8$. Other model parameters are $(a, \epsilon, r) = (10^{-3}, 10^{-4}, 0)$. The absolute differences $\Delta(Q/I)$ in percentage are also shown.

decreases. This trend is also followed by the absolute difference $\Delta(Q/I)$. The biggest differences are however noted for $T = 2 \times 10^3$. Clearly the $Q/I$ profiles computed with angle-dependent and angle-averaged PRD show significant differences both around the line core and near wing maximum for $T \leq 2 \times 10^3$ (the relative differences being between 10\% and 30\%). For $T \geq 2 \times 10^4$ the differences are small and appear mainly near the line core (the relative difference less than 10\%). Therefore angle-averaged PRD can safely be used to compute $I$ and $Q/I$ profiles in the absence of magnetic fields.

5.1.2. Magnetic Hanle Scattering Case

Here we consider the case of Hanle scattering. The vector magnetic field is characterized through the field strength parameter $\Gamma = g_L \omega_L / \Gamma_R$, and the field inclination $\theta_B$ and $\chi_B$. For our study on the comparison between Stokes profiles computed with angle-averaged and angle-dependent PRD, we consider the Hanle PRD matrix given by Approximation-I of Bommier (1997b). An angle-averaged version of this Approximation-I is presented in Sampoorna et al. (2008). In Fig. 2 we present the emergent $Q/I$ and $U/I$ profiles computed with angle-averaged (dashed lines) and the angle-dependent (solid lines) non-domain-based PRD matrix for different values of optical thickness $T$. The model parameters are exactly the same as those in Fig. 1, but for the presence of a weak magnetic field.

The Stokes $I$ is nearly insensitive to weak fields, and hence is not displayed. The behavior of $Q/I$ for different optical thickness and its comparison between angle-averaged and angle-dependent cases are similar to the corresponding non-magnetic case. Only difference being the presence of a depolarization in the line core. The $U/I$ profiles computed with angle-averaged and angle-dependent PRD show considerable differences in their shapes themselves for all values of $T$. Only for $T = 20$ the angle-
dependent and angle-averaged solutions match near the line center. For other values of $T$ differences are observed for all frequencies in the range $0 \leq x \leq 5$. The absolute difference $\Delta(U/I)$ is most pronounced for $T = 20$. Consequently, angle-dependent PRD effects are more important for $U/I$ profiles than for $Q/I$ profiles.

### 5.2. Stokes Profiles Calculated with Different Forms of Hanle PRD Matrix

In Fig. 3, we compare $Q/I$ and $U/I$ profiles computed using various forms of angle-dependent Hanle PRD matrix discussed in Section 2. For this comparison, we consider a self-emitting slab atmosphere of parameters $(T, a, e, r) = (2 \times 10^3, 10^{-3}, 10^{-3}, 0)$ and magnetic field parameters $(\Gamma, \theta_B, \chi_B) = (1, 30^\circ, 0^\circ)$.

Solutions computed using Approximation-I are shown as solid lines, those computed with Approximation-II are shown as dotted, dashed, and dot-dashed lines, and finally solutions computed with 1D cut-off approximation are shown as dash-triple-dotted lines. Dotted lines are obtained using the perturbation code of Nagendra et al. (2002). This code solves the Hanle transfer equation with Approximation-II in the Stokes vector basis. Here polarization is treated as a perturbation to intensity, and an iterative method is set up. Since Nagendra et al. (2002) solve the problem in Stokes vector basis, the angle-frequency couplings are highly intricate, so that the iterative method is numerically expensive. Dashed line is obtained using the code of Supriya et al. (2013a) which is based on scattering expansion method. This code solves the Hanle transfer equation with Approximation-II in the reduced Fourier basis, wherein the decomposition technique of Anusha & Nagendra (2012) is used. As a result actual angle-dependent domains are used for solving angle-dependent problem. The dot-dashed line is computed using the decomposition technique and scattering expansion method presented in Sections 3 and 4.1 respectively. As already discussed in these sections, angle-averaged domains are used to solve the angle-dependent problem.

From Fig. 3, we see that $Q/I$ is nearly insensitive to the type of frequency domains used, while $U/I$ is quite sensitive, particularly in the transition region $(3 \leq x < 5)$. The slight differences seen in the line core between the perturbation code results and
Figure 3. A comparison of $Q/I$ and $U/I$ profiles computed using various forms of angle-dependent Hanle PRD matrix. Model parameters are $(T, a, \epsilon, r) = (2 \times 10^3, 10^{-3}, 10^{-3}, 0)$ and magnetic field parameters $(\Gamma, \theta_B, \chi_B) = (1, 30^\circ, 0^\circ)$. Solid line corresponds to the solution computed using Approximation-I, dotted, dashed and dot-dashed lines to Approximation-II, while dash-triple-dotted line to 1D cut-off approximation. See Section 5.2 for details.

6. Conclusions

We have developed new efficient iterative methods to solve the polarized line transfer equation including angle-dependent PRD, both in the presence and absence of magnetic fields. Our iterative methods are based on the decomposition technique of Frisch (2009, 2010). In this technique a non-axisymmetric transfer problem is converted to an axisymmetric one through (i) the decomposition of the Stokes vector and redistribution matrix in terms of the irreducible spherical tensors, and (ii) the Fourier azimuthal expansion of the redistribution functions. Resulting axisymmetric polarized transfer equation is then solved using (a) a scattering expansion method, and (b) a polarized acceler- ated lambda iteration method. We have considered both the exact (Approximation-I of Bommier 1997b) and approximate (Approximation-II of Bommier 1997b, 1D cut-off approximation) forms of the Hanle PRD matrix.

We show that the angle-averaged PRD is a good approximation when computing $I$ and $Q/I$ profiles in the absence of magnetic fields. In the presence of a weak magnetic field, $I$ and $Q/I$ profiles continue to be somewhat insensitive to angle-dependence of the PRD function, while $U/I$ profiles are highly sensitive. $U/I$ profiles are also quite sensitive to the frequency domains used in their computation. Thus angle-dependent
PRD should be taken into account for magnetic field determination based on Hanle effect.

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