Shocking Remarks on Stellar Pulsation

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Abstract. Smoothly varying sound waves steepen as they propagate, at a rate that is an increasing function of the amplitude of the wave. If they are not first absorbed or otherwise dissipated by diffusion, the waves eventually shock. One might expect, therefore, that the steepening process in the outwardly propagating component of a normal acoustic pulsation mode in a star might permit the wave to escape more easily into the atmosphere, thereby leaking energy in an amplitude-dependent manner. Is this the process that limits low-amplitude intrinsically overstable pulsations such as occur in roAp stars? Even if the waves do not shock below the location of the acoustic cutoff associated with the basic frequency of the wave, which is likely in low-amplitude pulsators, the transmitted component might shock subsequently, heating the atmosphere and perhaps even producing a chromosphere.

1. Introduction

When Hiromoto Shibahashi asked me to discuss shocks in stars, he might have had in mind further development of the work that we had carried out on shock waves in $\gamma$ Equ with Don Kurtz and Eiji Kambe (Shibahashi et al. 2008). Or, perhaps, he wanted me to review what has been written more generally about shock waves in stellar atmospheres, particularly, I presume, in the atmospheres of cool stars. I prefer to concentrate on stars such as the roAp stars, which is actually what I am going to do here, partly because roAp stars are of greater pertinence to much of what has been discussed at this conference, and partly because it affords me the opportunity to address what is arguably the most outstanding unresolved question concerning our understanding of the dynamics of low-amplitude pulsators, namely: what limits the oscillation amplitudes?

Very broadly speaking, stellar pulsations can be divided into two categories: there are the intrinsically stable pulsations which are excited stochastically to very low amplitude by the turbulent convection in the envelope (and, strictly speaking, by the turbulence in a convective core, if there is one), and there are the intrinsically overstable pulsations. I choose to divide the latter into two subclasses: the high-amplitude pulsations of the so-called classical variables such as Cepheids, RR Lyrae stars and Miras, and the low-amplitude pulsations of stars such as the roAp stars. Amplitude limitation of the former is thought to be caused by a saturation of the driving process at high pulsation amplitude; in some cases that is understood at least qualitatively, and has been simulated quantitatively by numerical computation. That process cannot account for the low-amplitude pulsators, because nonlinearity in the driving region is too weak. One must instead look higher in the envelope, or even in the atmosphere, for an explanation, for only there is adequately nonlinear behaviour likely to arise. It is with
this in mind that I have sought an explanation, although I had never been moved sufficiently to investigate the matter in any detail until Hiromoto delivered his request. That request came only a few days before I left home for Japan, so I have had time to provide no more than the preliminary sketch that follows. What I present here is therefore necessarily only a report on work in progress.

2. A Simplified Summary of Shock Formation

The development of shocks in a simple adiabatic wave propagating into a uniform, inviscid, perfect gas was explained long ago in a beautiful analysis by Riemann (1858). I shall not reproduce that analysis here, but instead offer a non-rigorous description of the physics, which I hope will enlighten those who have not come across it before. It is illustrated in Fig. 1, in which the thick solid curve represents a wavelengthworth of a sinusoidal disturbance having wavenumber $k$ and moving from left to right. It is plotted with respect to a spatial coordinate $\xi = z - c_0 t$ moving with the wave. Here $z$ is distance, $t$ is time, and $c_0 = \sqrt{\gamma p_0/\rho_0}$ is the adiabatic sound speed in the unperturbed state, which is also the mean speed of the wave. The diagram is intended to represent snapshots of the spatial variation of the waveform (taken at different times and depicted with different line styles); it can also represent temporal variation at a fixed point in space (observed at different locations). The ordinate is any wave quantity $\psi$, be it density perturbation $\rho' := \rho - \rho_0$, fluid velocity $w$, or perhaps the perturbation $c' := c - c_0$ to the sound-speed. Consider it first to be $c'$.

The disturbance propagates everywhere with the local sound speed $c$. Therefore fluid near the crest of the wave moves faster than the mean wave, whereas fluid near the trough moves more slowly. After some time $\Delta t$ the sinusoidal wave form will have distorted into the short-dashed curve, the distortion being described by a displacement in $\xi$ by an amount $c' \Delta t$, as indicated by the arrows. This development continues until at some point the slope becomes infinite (resulting in the dot-dashed curve), after which time an increasingly larger region of the waveform achieves the vertical (representing a discontinuity in all wave quantities, and constituting a shock). The magnitude of the discontinuity grows until it is met by the wave crest (as the thin solid curve illustrates), after which time it declines as energy is dissipated in the shock. A waveform a short time later still is denoted by the long-dashed curve; note that it is almost (but not exactly) straight, indicating that the deceleration behind the shock is almost constant.

Before continuing I should point out an important obvious consequence of this description. It has often been said that a shock wave occurs when the fluid velocity in the wave reaches the sound speed. It is quite evident that, in the case considered here, this is not necessarily true. If $\psi$ is considered now to represent the fluid velocity $w(\xi, t)$, it is clear that the velocity in any part of the waveform can never exceed the maximum velocity of the initial wave, and indeed, as I have already mentioned, after the wave crest meets the shock the maximum velocity declines. In fact, in the development of the initially sinusoidal wave considered here, the shock first forms where the fluid velocity $w$ is zero! Moreover, were it not for thermal conduction and viscous dissipation (ignored here), any acoustic wave would eventually shock, however small the initial amplitude.

I should remind the reader that what I have said is a mere description of the import of Riemann’s analysis. It is not a proof, nor even a demonstration. The actual analysis rests on the establishment of invariants of the flow, now called Riemann invariants,
Figure 1. Nonlinear evolution of an initially sinusoidal simple wave (i.e. a wave travelling in only one direction) having fluid velocity \( w = w_0 \sin(k\xi) \), where \( \xi = z - c_0 t \), and associated excess sound speed \( \psi = c - c_0 = \psi_0 \sin(k\xi) \) at a sequence of instances separated by a time interval \( \Delta t \). The initial wave is represented by the thick solid curve. After time \( \Delta t \) the waveform is represented by the short-dashed curve, which can be constructed from the initial wave by a displacement \( \Delta \xi = (c - c_0)\Delta t \) in the moving frame; the displacement is indicated by the arrows, whose length is everywhere proportional to \( \psi \). The dot-dashed line is the waveform at time \( \Delta t \) later still, at the moment where the greatest slope (in magnitude) is \(-\infty\). To continue the geometrical construction further would evidently lead to a multivalued waveform, which is physically impossible. Instead, a discontinuity develops, which grows until the wave crest meets the discontinuity, and subsequently declines as energy is dissipated. This construction is (essentially) valid for any (non-sinusoidal) initial waveform.

which permit the geometrical construction that I have just described to be applied to wave slowness, which is a vector whose components are equal to the inverse of the phase speed in the corresponding directions.

About a century after Riemann, Lighthill (1978) showed that there are corresponding adiabatic invariants for simple waves propagating into a slowly varying medium, and subsequently Shibahashi et al. (2008) generalized that analysis to vertical propagation of (high-wavenumber) waves through an atmosphere stratified under gravity \( g \), which they applied to the roAp star \( \gamma \) Equ. It should be noticed in this case that the amplitudes of upwardly propagating waves increase with height, at least before they shock, because density decreases upwards. (The velocity amplitude \( U \) of a high-frequency low-amplitude unshocked acoustic wave satisfies \( \rho U^2 c \approx \text{constant} \).) Therefore it is no longer necessarily the case that the amplitude of an initially subsonic flow remains subsonic when the shock develops. In fact, Shibahashi et al. found that the shocking amplitude \( U \) of a wave having frequency \( \omega \) in a typical roAp star is about \( (1 + \gamma^{-1})^{-1} g / \omega \), where \( g = |g| \), irrespective of its amplitude in the photosphere. For a wave with period 12 min, which is typical, \( U \approx 8 \text{ km s}^{-1} \), which is coincidently not...
very different from the sound speed. It is evident from the formula, however, that the physical shocking criterion is not simply $U \approx c$.

### 3. Mode-amplitude Limitation

For me to acquire fully nonlinear understanding is difficult, if not impossible, as is the case, I believe, for many others too. I shall therefore continue to argue non-rigorously, in only a quasi-nonlinear sense, by thinking in terms of small, essentially linear, perturbations to a nonlinear wave, as indeed did Lighthill in deriving his Riemann adiabatic invariants. So I first point out that a linear acoustic mode of oscillation of a star can be thought of as a (non-interacting) superposition of outwardly and inwardly propagating waves (with the same horizontal structure). Moreover, since I am interested particularly in roAp stars, whose oscillations appear to be of only low degree, and because I am interested in the dynamics in only the outer layers of the stars where the wave amplitude is relatively high, the degree of the mode is irrelevant because near the stellar surface all low-degree p-modes of a given frequency $\omega$ look alike. Therefore I presume the modes to propagate vertically. An upwardly propagating wave experiences a diminution of the pressure and density of the gas on ever-decreasing scales $H$ until eventually $H$ becomes less than the local wavelength of the wave (actually, half the inverse wavenumber), at which point the wave is reflected downwards, interfering with itself to form a standing wave. In the light of the description in the previous section, nonlinearity of that upward wave might be expected to cause it to steepen, if not all the way to a shock, reducing the characteristic wavelength (in at least part of the waveform) and thereby permitting it to propagate to levels higher in the envelope. Put another way, the steepening wave, akin to that illustrated in Fig. 1, develops higher spatial (and temporal) harmonics which can propagate beyond the reflecting level of the fundamental. If those harmonics then encounter the photosphere, above which is an essentially isothermal atmosphere, having constant scale height $H$, they can propagate out of the star and leak energy away. The steepening process is nonlinear, proportional to the square (and higher powers) of the amplitude $U$, so the energy loss rate, which is essentially quadratic in wave quantities, is proportional to $U^4$. The rate of gain of energy from the background state of an over-stable mode is essentially a linear process, for it occurs sufficiently deep in the star (mainly in the hydrogen ionization zone for roAp stars) for the amplitude to be relatively low; it is therefore proportional to the energy itself, which is proportional to only $U^2$. Balancing losses and gains therefore results in an equation from which $U$ cannot be factored out (as it can for linear waves), and thus determines the limiting value of $U$.

It must be appreciated that this argument is not strictly correct, because it is based essentially on linear reasoning, aside from the wave-steepening presumption. For nonlinear waves, simple additive superposition does not apply, so the mode cannot actually be regarded as the sum of oppositely directed, non-interacting, propagating waves. Any outwardly propagating wave cannot be considered as a simple wave, isolated from any downward component (indeed, it is not even meaningful to imagine an unambiguous separation into outward and inward components), so the wave-steepening discussion of the previous section cannot validly be applied. Nevertheless, if the nonlinearity is weak, one might reasonably expect the coherent intrinsic nonlinear simple-wave-like steepening tendency to dominate over the less coherent, rapidly oscillating, nonlinear interaction between what one might loosely consider to be the upwardly and down-
wardly propagating wave components, so that the basic idea should be more-or-less correct. It is with this in mind that I consider the simple model described below.

4. Equations of Motion

To keep matters simple I consider the dynamics of vertical adiabatic motion \( w \) of an inviscid gas under constant gravity \( g \). The equations of motion are

\[
\rho \left( \frac{\partial w}{\partial t} + w \frac{\partial w}{\partial z} \right) = - \frac{\partial p}{\partial z} + \rho g, \tag{1}
\]

\[
\frac{\partial p}{\partial t} + \frac{\partial (\rho w)}{\partial z} = 0, \tag{2}
\]

\[
\frac{\partial p}{\partial t} + w \frac{\partial p}{\partial z} = c_0^2 \left( \frac{\partial \rho}{\partial t} + w \frac{\partial \rho}{\partial z} \right), \tag{3}
\]

where \( z \) is a vertical co-ordinate (measured downwards), \( t \) is time, \( p \) and \( \rho \) are pressure and density, and \( c_0^2 = \gamma p/\rho \) is the square of the sound speed, in which the (first) adiabatic exponent is \( \gamma = (\partial \ln p/\partial \ln \rho)_s \), the derivative being taken at constant specific entropy \( s \). For simplicity I take \( \gamma \) to be constant. In accord with my only mildly non-linear thinking, I separate the wave disturbance from the static background hydrostatic state \( (p_0, \rho_0) \) according to

\[
p = p_0(z) + p'(z, t), \quad \rho = \rho_0(z) + \rho'(z, t), \quad w = w(z, t). \tag{4}
\]

I then eliminate \( p \) and \( \rho \) from Eq. (1) using Eqs. (2) and (3) to obtain the following wave equation:

\[
\frac{\partial^2 w}{\partial z^2} + \frac{1}{H_{\rho 0} \rho_0} \frac{\partial w}{\partial z} - \frac{1}{c_0^2} \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial z} \left( \gamma p' \frac{\partial w}{\partial z} + w \frac{\partial p'}{\partial z} - gw' \right) - \frac{\partial}{\partial t} \left( \rho' \frac{\partial w}{\partial t} + \rho w \frac{\partial w}{\partial z} \right) \tag{5}
\]

in which \( H_{\rho 0} = p_0^{-1} dp_0/dz = \gamma g/c_0^2 \) is the pressure scale height of the equilibrium state and \( c_0 \) is the unperturbed sound speed. This equation is exact. I have assembled all the linear disturbance terms on the left, and the nonlinear terms on the right.

Ignoring the nonlinear terms leads to the familiar linearized wave equation in normal form:

\[
\frac{\partial^2 u}{\partial z^2} = \frac{1}{c_0^2} \left( \frac{\partial^2 u}{\partial t^2} + \omega_c^2 u \right), \tag{6}
\]

where \( u = p_0^{-1/2} w \) and

\[
\omega_c^2 = \frac{c_0^2}{4H_{\rho 0}^2} \left( 1 - \frac{1}{2} \frac{dH_{\rho 0}}{dz} \right) \tag{7}
\]

is the square of an acoustic cutoff frequency.*

*It differs from the more familiar expression in which \( H_{\rho 0} \) is replaced by the density scale height, and the sign of the second term in the parentheses is reversed (Deubner & Gough 1984) because here it is the (scaled) velocity, rather than Lagrangian pressure perturbation, that is the dependent variable (e.g. Gough 2007).
Because the background is independent of \( t \), separated solutions of the form 
\[ u(z) \cos(\omega t) \] can be found, where 
\[ d^2 u/dz^2 + k^2 u = 0 \] with 
\[ k^2 = (\omega^2 - \omega_c^2)/\epsilon_0^2. \] In regions 
where \( \omega_c^2 < \omega^2 \), \( u \) is sinusoidal, and the wave can propagate; where \( \omega_c^2 > \omega^2 \) the wave 
is evanescent, and it cannot propagate if \( \cos(\omega t) \), with \( \omega \) real, truly represents the time 
dependence.

5. A Toy Stellar Model

I represent the star by a model consisting of a plane-parallel polytrope of index \( m \) supporting 
an isothermal atmosphere. The center of the star is represented by a rigid base 
at depth \( Z \) beneath the seismic surface, and the isothermal atmosphere is taken to match 
continuously (with respect to \( p \) and \( \rho \)) at depth \( z_s \), which I regard as the photosphere.

For \( z > z_s \), \( p \propto z^{m+1} \) and \( \rho \propto z^m \); \( c_0^2 = \gamma g z/(m+1) \) and \( H_{p0} = z/(m + 1) \).

For \( z < z_s \), \( p \) and \( \rho \) both decay exponentially with height as \( \exp[(m + 1)(z/z_s - 1)] \);
\( c_0^2 = \gamma g z_s/(m+1) =: c_{0a}^2 \) and \( H_{p0} = z_s/(m + 1) =: H \) are both constant, as is the acoustic 
cutoff frequency \( \omega_{ca} \) given by
\[ \omega_{ca}^2 = \frac{(m + 1)\gamma g}{4z_s}. \] (8)

The sound speed is, of course, continuous, but the acoustic cutoff frequency is not: \( \omega_{ca}/\omega_{ph} = \sqrt{(m + 1)/(m - 1)} \), where \( \omega_{ph} \) is the value of \( \omega_c \) at the top of the envelope.

The seismic surface is at \( z = 0 \), and the effective acoustic depth of the model (corresponding 
to the acoustic radius of the star) is \( \tau_R = 2[(m + 1)Z/\gamma g]^{1/2} \). The upper 
turning point of a mode with fiducial frequency \( \omega_0 \) is at \( z_0 = (m - 1)\gamma g/4\omega_0^2 \), provided 
that it lies in the polytropic region. Thus I require that \( z_s < z_0 \), to ensure that the region 
of propagation is confined beneath the photosphere. In fact, for carrying out the analysis I shall assume that \( z_s \) is substantially smaller than \( z_0 \) so that the wave amplitude has 
diminished sufficiently in the evanescent region to render insignificant the influence of the atmosphere 
on the solution in the polytropic envelope. As will become evident later, 
that simplifies the analysis considerably.

To compute the low-amplitude pulsations it is adequate to assume adiabatic motion and adopt Eqs. (4) and (5). Nonadiabaticity, and the influence of the Reynolds 
stresses associated with convection, will be assumed to have been taken into account in 
computing the linearized growth rate \( \eta \) of the mode. I carry out weakly nonlinear theory by 
imagining wave quantities to have been expanded in powers of the amplitude; I write
\[ p' = p_1 + p_2 + \ldots, \quad \rho' = \rho_1 + \rho_2 + \ldots, \quad w = w_1 + w_2 + \ldots. \] (9)

These expressions are substituted into Eq. (5), and terms of like order are balanced.

6. First-order Pulsation Theory

In the polytropic envelope, the first-order terms in Eq. (5) permit the velocity \( w_1 \) to be 
expressed in the form \( \Phi_1(z)\cos(\omega t) \), where
\[ \frac{d^2 \Phi_1}{dz^2} + \frac{m + 1}{z} \frac{d\Phi_1}{dz} + \frac{(m + 1)\omega^2}{\gamma g z} \Phi_1 = 0. \] (10)
After making the substitution $z = γgξ^2/4(m + 1)ω_0^2 =: Kξ^2$ and $Φ_1 = ξ^{-m}ψ_1$, in terms of a fiducial frequency $ω_0$ which is characteristic of the modes under consideration ($ξ$ here is not to be confused with the variable $z − c_0t$ with the same name adopted in Sect. 2), Eq. (10) reduces to Bessel’s equation:

$$L_1ψ_1 := \frac{d^2ψ_1}{dξ^2} + \frac{1}{ξ} \frac{dψ_1}{dξ} + \left(σ^2 - \frac{m^2}{ξ^2}\right)ψ_1 = 0,$$

where $σ = ω/ω_0$ is a dimensionless measure of order unity of the frequency of the mode of oscillation. The value $ξ_1$ of $ξ$ at the upper turning point of the mode is $\sqrt{(m^2 − 1)/σ}$. The photosphere is at $ξ = ξ_s$.

Strictly speaking, the solution $ψ_1$ is a linear combination of Bessel functions $J_m(σξ)$, $Y_m(σξ)$ of the first and second kinds, whose ratio should be determined by matching onto a causal solution in the atmosphere. But in view of my assumption that the photosphere is acoustically far from the upper turning point, the coefficient of $Y_m$ is small, so I neglect it. That simplifies the formulae. In particular, the first-order velocity eigenfunction becomes

$$w_1 = W(σξ)^{-m}J_m(σξ) \cos(ωt).$$

Note that near the photosphere where $ξ$ is small, $w_1$ is almost independent of height. Application of the condition $w_1 = 0$ on the lower rigid boundary at $ξ = ξ = ω_0τ_R$ yields the eigenfrequencies, given by $σ = ξ^{-1}j_{m,n}$, where $j_{m,n}$ is the $n$th zero of $J_m$. For the high-order modes of relevance to roAp stars, $j_{m,n} ∼ β − (4m^2 − 1)/β + ...$ where $β = (n + m/2 − 1/4)π$. The constant $W$ approximates the velocity amplitude of the mode in the photosphere.

7. Second-order Theory

The first-order solution $w_1$, together with its associated variables $p_1$ and $ρ_1$ which are obtained immediately from it by using Eqs. (2) and (3), can be substituted into the right-hand side of Eq. (5), retaining terms of only second order. That generates an inhomogeneous term in the equation for $w_2$, which is a second temporal harmonic of the fundamental. In fact the entire nonlinear term is proportional to $\sin(2ωt)$, and I denote it by $N_2(w_1; z, ωt) =: F_2(ψ_1; σξ)\sin(2ωt)$. It admits a particular integral $\tilde{w}_2 = ξ^{-m}\hat{ψ}_2(σξ) \sin(2ωt)$ with the same time dependence, where $\hat{ψ}_2$ satisfies

$$L_2\hat{ψ}_2 = F_2(ψ_1(σξ); σξ) =: \mathcal{F}_2(σξ)$$

in which $L_2$ is obtained from $L_1$ by replacing $σ$ with $2σ$.

An appropriate integral yields

$$\tilde{w}_2 = 4W^2σ^2e^{-m}\int_{ξ_a}^{ξ} ξ' \left[Y_m(2σξ')J_m(2σξ) − J_m(2σξ')Y_m(2σξ)\right] \mathcal{F}_2(σξ')dξ' \sin(2ωt).$$

To this must be added the complementary function

$$\tilde{w}_2 = Wξ^{-m}\left[AJ_m(2σξ) + BY_m(2σξ)\right] \sin(2ωt) + \left[CJ_m(2σξ) + DY_m(2σξ)\right] \cos(2ωt),$$

(15)
where $A$, $B$, $C$ and $D$ are constants yet to be determined.

The complete second-order solution in the polytropic envelope is $w_2 = \hat{w}_2 + \tilde{w}_2$. This has a substantial amplitude above $\xi_1$ because it has twice the frequency of the fundamental component.\(^\dagger\) It must therefore be matched at $\xi_s$ to an outwardly propagating wave in the atmosphere, which has the form

\[ w_2 = W^2 \left[ E \cos(k \xi + 2\omega t) + F \sin(k \xi + 2\omega t) \right] e^{(\zeta_s-c \xi)/2H} \quad (z \leq z_s), \tag{16} \]

where

\[ \kappa^2 = \left( 4\omega^2 - \omega_{ca}^2 \right) / c_{0a}^2 \tag{17} \]

in which $\omega_{ca}$ is given by Eq. (8). Matching $w_2$ and the Lagrangian pressure perturbation (which is equivalent to matching $\partial w_2 / \partial z$) at $z = z_s$ ($\xi = \xi_s$) and demanding that $w_2$ vanishes at $\xi = \Xi$ provides six simultaneous linear equations to determine the six unknown constants of integration: $A$, $B$, $C$, $D$, $E$ and $F$, thereby completing the solution to this order. The equations have sufficient symmetry to permit them to be solved readily by hand in terms of the coefficients in the boundary and matching conditions. Those coefficients contain Bessel functions and weighted integrals of products of Bessel functions displayed implicitly in Eq. (14) – the nonlinear function $F_2$ is a bilinear function of Bessel functions and their derivatives – and has been evaluated numerically. In so doing I replaced the lower limit of integration by zero, to render the integrals $\sigma$-independent.

The unwary reader might have wondered why I appeared not to have contemplated taking into account a possible nonlinear perturbation to $\sigma$. Actually, I didn’t mention it in order to keep the discussion superficially simple. But now I should explain. In common with many other finite-amplitude expansions (e.g., Veronis 1959; Schütler et al. 1965), to this order the perturbation to $\sigma$ vanishes. That is evident because the nonlinear inhomogeneity $N_2(w_1; z, \omega t)$ is orthogonal to $w_1$. If one were to take the expansion to the next order the situation would be different, because the corresponding nonlinearity $N_3(w_1, w_2; z, \omega t)$ contains resonating terms proportional to $\cos(\omega t)$ which threaten to cause an artificial secular instability unless $\sigma$ is adjusted to produce a term $2\sigma_0\sigma w_1$ to quench it.

Finally I point out that the weakly nonlinear solution to Eq. (5) has been developed in a consistent amplitude expansion whose validity does not rely on the motivating approximate quasi-linear physical arguments that I presented in Sect. 3.

### 8. Determination of the Limiting Amplitude

The analysis summarized in the previous four sections estimates the relative amplitude of a propagating second-harmonic component of a nonlinear stellar oscillation in terms of its fundamental velocity amplitude $W$. Provided that $\omega$ is not too low, that component escapes through the atmosphere and so drains energy from the mode. The energy flux is approximately

\[ F_w \approx \frac{\rho w^2 v_{ga}}{2} \approx 1 \frac{\rho_w W^4}{E^2 + F^2} c_{0a} \sqrt{1 - \omega_{ca}^2/4\omega^2}, \tag{18} \]

\(^\dagger\)I remind the reader that the expansion of the single nonlinear mode leads to a harmonic series in time. The harmonics are not to be confused with the independent overtone modes of a multi-periodic pulsating star, which are often miscalled harmonics.
where $\rho_0$ is the equilibrium density at the envelope-atmosphere interface and where the overbar denotes time average. The flux is approximately the product of what is essentially the group velocity $v_{ga} = \sqrt{1 - \omega_{ca}^2/4\omega^2c_0a}$ of the second harmonic in the isothermal atmosphere and the energy density, which is composed of the kinetic energy density $\frac{1}{2} \rho_0 w^2$ and the potential energy density, which, in a simple linear acoustic wave, has the same magnitude as the kinetic energy density.

In oscillatory balance the energy flux through the atmosphere is exactly compensated by the extraction of energy from the equilibrium stratification of the envelope beneath, which is believed to occur principally in the hydrogen ionization zone. In principle it can be estimated as twice the growth rate $\eta$ of a linear mode – because well beneath the photosphere the oscillation amplitude is quite small – multiplied by the total energy $E_w$ of the mode which can again be estimated by linear theory:

$$E_w \approx \rho_0 W^2 \sigma^{-2m-2} K_\xi^{-2m} \int_0^{\ln z_s} x \left[ J_m(x) \right]^2 \, dx =: \rho_0 I W^2 \sigma^{-2(m+1)}$$

(19)

for a mode of order $n$. The lower limit of the inertia integral $I$ should have been $\sigma^{-1} \xi_s$, but because it is small, and because for $\xi \lesssim \xi_s$ — recall that $\omega_0$ has been chosen to be close to the eigenfrequency $\omega$ of the mode, so that $\sigma \approx 1$; the reason $\omega$ itself was not used is because a frequency-dependent transformation to $\xi$ renders the solution somewhat more cumbersome to interpret — the integrand is proportional to $x^{2m+1}$, extrapolation to the origin makes little difference, yet neatly renders $I$ independent of $\sigma$.

The energy balance thus becomes

$$\frac{F_w}{E_w} \approx \frac{1}{2I} \left( E^2 + F^2 \right) c_{0a} \sqrt{1 - \omega_{ca}^2/4\omega^2\sigma^2} W^2 \approx 2\eta,$$

(20)

implying a photospheric velocity amplitude

$$W = \frac{2}{\sigma^{m+1}} \left[ \frac{I \eta}{(E^2 + F^2) c_{0a}} \right]^{1/2} \left( 1 - \frac{\omega_{ca}^2}{4\omega^2} \right)^{-1/4}.$$

(21)

9. Application to a Rapidly Oscillating Ap Star

A faithful transference of this result to a real roAp star is not possible, because the outermost subphotospheric layers of the envelope, near where the oscillations are normally observed, are not accurately represented by a polytrope. Nevertheless, my hope is that my analysis has taken some reasonable account of the principal mechanisms at work in the determination of the oscillation amplitudes. For a star of prescribed mass and radius, $g$ is known in the surface layers; it remains to choose representative values of $m$, $\gamma$ and the photospheric level $z_s$. For illustration I take $\gamma = 5/3$, which is not far from reality at least near the photosphere. However, the effective $m$ (defined locally in terms of $\Gamma := \frac{d \ln p}{d \ln p}$) varies dramatically with height, so I have adopted what I hope are a few representative values. Finally, $z_s$ defines the effective temperature $T_{\text{eff}}$ of the toy star in terms of $g$, $m$, $\gamma$ and the mean molecular mass $\mu$ in the isothermal atmosphere, since $g z_s/(m+1) = c_{0a}^2 \approx \gamma R_{\text{eff}} T_{\text{eff}} / \mu$, where $R$ is the gas constant.

There is yet no complete, or even approximate, calculation of $\eta$. Balmforth et al. (2001) have represented a star with two antipodal circular spots in which convection is
considered to be suppressed by the magnetic field, and permitting convection to operate elsewhere, both in determining the equilibrium stratification and through the interaction of the modulated convective heat flux and Reynolds stress with the global oscillations, as had Dolez & Gough (1982) before them. Both spots and ‘quiet’ regions were modelled with segments of spherically symmetrical stellar models. Direct interaction with the magnetic field was ignored, so the growth rates were intended to apply only to modes far from the highly damping resonances discussed by Cunha & Gough (2000) and Saio & Gautschy (2004). More recently, Saio et al. (2012) have taken the direct interaction of the oscillations with the magnetic field into account, but they ignored the convection entirely, thereby modelling the entire star as though it were a spot. Since the convection tends to damp the oscillations, they, like Balmforth et al., are likely to have overestimated \( \eta \), but for a different reason. In any case, both studies found growth rates of order \( 10^{-5} \text{s}^{-1} \), so I adopt that value here, recognizing that it might be too high.

Table 1. Formal photospheric velocity amplitudes \( W \) (in \( \text{km s}^{-1} \)) of a representative roAp mode. The amplitudes were computed from Eq. (21), with \( \gamma = 5/3 \) and \( \eta = 10^{-5} \text{s}^{-1} \), and polytropic index \( m \) beneath the photosphere. The parameter \( \alpha \) is the mode frequency in units of the photospheric acoustic cutoff in the envelope: \( \alpha = \sqrt{(m+1)/(m-1)}\omega/\omega_{ca} \); given that \( \omega \) is determined essentially by the stellar parameters \( M \) and \( R \), and the order \( n \) (for given low degree) of the mode, \( \alpha \) can be regarded as a measure of the location \( z_s \) of the photosphere in the toy model, according to Eq. (8). The condition \( z_s < z_t \), assumed to justify the neglect of \( Y_m \) in the formula (12) for \( w_1 \), requires that \( \alpha < 1 \). However, values greater than unity have been included in the hope of giving some idea of what a consistent analysis would yield. The condition that the second harmonic can propagate in the atmosphere is \( \alpha > \sqrt{(m+1)/(m-1)/2} \), which is not satisfied for \( \alpha = 0.8 \) when \( m = 2 \). For the fundamental to be evanescent in the atmosphere, \( \alpha < \sqrt{(m+1)/(m-1)} \), which is satisfied for all the entries in the table.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( m = 2 )</th>
<th>( m = 3 )</th>
<th>( m = 4 )</th>
<th>( m = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>–</td>
<td>177</td>
<td>2.6</td>
<td>171</td>
</tr>
<tr>
<td>0.9</td>
<td>20</td>
<td>390</td>
<td>600</td>
<td>22</td>
</tr>
<tr>
<td>1.0</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>1.1</td>
<td>9</td>
<td>0.4</td>
<td>8</td>
<td>0.1</td>
</tr>
<tr>
<td>1.2</td>
<td>35</td>
<td>7</td>
<td>33</td>
<td>14</td>
</tr>
</tbody>
</table>

In Table 1 I list velocity amplitudes \( W \), according to Eq. (21), for several values of \( m \) and \( z_s \) for a star of mass \( 2 M_{\odot} \) and radius \( 2 R_{\odot} \). In all cases I take the oscillation frequency to be \( 1.5 \text{mHz} \), which corresponds to a period of about 11 minutes. The constants \( E, F \) and \( I \) were obtained from the analysis described in Sects. 7 and 8. On the whole the amplitudes are high for low values of \( \alpha \). That is hardly surprising because the upper turning point of the mode is relatively deep in the envelope, and only little of the oscillation energy can reach the very outer layers of the star where the second harmonic is expected to be most easily generated. So why are such low-frequency modes not observed to dominate the acoustic spectrum? I presume that the reason is that they are intrinsically stable (cf. Balmforth et al. 2001; Saio et al. 2012). When \( \alpha \) increases above unity, \( W \) is undulatory; there are regions in \( \alpha-m \) parameter space in which \( W \) has quite high values. It is not unlikely that that behaviour is an artefact of my
having ignored the Bessel function of the second kind in my approximate representation (12) of $w_1$, which, for such values of $\alpha$, is not strictly valid.

![Light curve](image)

**Figure 2.** Two periods of the light curve (in units of the amplitude of the fundamental component) computed for an envelope of polytropic index $m = 3$, with $\alpha = 1.1$.

In Fig. 2, I present a very rough representation of a light curve from the leaky nonlinear mode, for the case $m = 3$, $\alpha = 1.1$, obtained simply as four times the relative temperature fluctuation $T'/T_{\text{eff}}$ in the photosphere (estimated by $4(\gamma - 1)\rho'/\rho_s$ obtained from my adiabatic analysis). It is evidently very approximate. Nevertheless, it does exhibit the steepening characteristic of the simple wave illustrated in Fig. 1, and it is superficially similar to at least some observations of roAp light curves. That result provides some support to the physical picture that I described in Sect. 3.

### 10. Brief Discussion

I suggest that even a low-amplitude pulsator, such as a rapidly oscillating Ap star, develops incipient shocking characteristics in its outer layers, steepening the eigenfunction which can then be thought of as a temporally harmonic series. Although the first harmonic may be regarded as being well confined beneath its acoustic cutoff (thinking quasi-linearly), high harmonics can be imagined to propagate farther out of the envelope into the atmosphere, and thereby leak energy from the mode. The amplitude of the $n$th harmonic is of the order of the $(n-1)$th power of the amplitude $W$ of the fundamental, at least when $W$ is small, implying that the energy loss rate is $O(W^2)$ of the energy itself. In contrast, the excitation of the mode is essentially linear, so the mode extracts energy from the background state at a rate proportional to just its energy. Consequently
a balance is achieved, which is nonlinear in \( W \), and which thus determines \( W \), unlike any purely linear energy balance.

The fate of the wave propagating into the atmosphere is to suffer further steepening, and possibly to shock high in the atmosphere, as described by Shibahashi et al. (2008). However, in oscillatory equilibrium (i.e., when energy input and output balance, resulting in an oscillation at its limiting amplitude), there is no inward propagation of information from the atmosphere, so it was not necessary for me to consider the shocking behaviour when matching the envelope and atmospheric representations of the mode. That is not to say that conditions in the photospheric regions are completely oblivious of the upper-atmospheric shocks: the shocks enhance energy locally and heat the atmosphere, possibly creating high altitude chromospheric and coronal conditions. Evidence for classical chromospheric activity has been sought in roAp atmospheres, and has not been found. But that would not necessarily mean that chromospheres do not exist in these stars, if what was looked for was the kind of activity seen in cool stars; maybe it is necessary to seek evidence in a different region of the optical spectrum.

The preliminary analysis that I have presented here is not wholly consistent. For example, I ignored the Bessel function of the second kind in my representation of the first-harmonic component of the mode, merely to ease the algebra. I suspect that that is not a bad approximation, at least when \( \alpha \) is small, but I cannot yet be sure. Of course, representing the star as a plane-parallel polytrope supporting an isothermal atmosphere is bound to have weakened my prediction. However, I suspect that that too is not the dominant source of uncertainty. It is noteworthy that, unlike the highly nonlinear classical variables, roAp stars, which oscillate at relatively low amplitudes, are amenable to a straightforward amplitude expansion of the kind described in this report.

On the whole, the velocity amplitudes obtained from the model exceed typical radial velocity measurements. This could be either because the estimate of the growth rate \( \eta \) that I have adopted is too high, or because I have underestimated the nonlinear steepening. It is probably both. It should also be recognized that merely estimating the optical flux variation as an Eulerian variation in the photosphere of \( T^4 \) computed from an adiabatic calculation, as I did for producing Fig. 2, is extremely crude. And finally, estimating \( \eta \) from a calculation in which wave-steepening near the upper turning point (again, arguing quasi-linearly) has been ignored is dangerous (notwithstanding the fact that a complete linear calculation hasn’t even been carried out) because, as realized by Balmforth et al. (2001), the driving region is so close to the photosphere that conditions in the photosphere modify the eigenfunction in the hydrogen ionization zone enough to have a significant impact on the calculated growth rate. There is still much more to be investigated in this fascinating subject.

Acknowledgments. It is not normally my wont to thank organizers of conferences for imposing work upon me. But exceptionally I do thank Hiromoto Shibahashi this time, not only for organizing an excellent meeting in a wonderful location, but also for inducing me to think about a problem that I have had on my ‘to-do’ list for many years, yet had never been moved sufficiently to work on it: relative progress from a situation in which there is no result is high – indeed initially it is formally infinite – which cannot but give the executor some satisfaction. I thank Paula Younger for typing the manuscript. Additionally I thank the Japan Society for the Promotion of Science for an Invitation Fellowship, during part of the tenure of which this work was carried out.
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