Gravitational and Magnetohydrodynamic Waves

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Abstract. Gravity is known to have important effects on electromagnetic fields in the environs of compact objects. After a brief introduction to electromagnetism in general relativity, we focus on the interaction between gravitational waves (GWs) and magnetohydrodynamic (MHD) waves. We use the ideal MHD approximation, and the familiar 1+3 split of the reference frame, which allows a direct interpretation of the results by the physical observer. We show how fast magneto-sonic waves and Alfvén waves can grow in a magneto-plasma by action of gravitational waves on the ambient magnetic field in the far-field approximation for various configurations of ambient field and wave directions.

1. Introduction

The effect of gravity on electromagnetism near compact objects has been pointed out in two seminal papers, on black holes (Blandford & Znajek 1977) and radio pulsars (Muslimov & Tsygan 1992). This talk addresses the transfer of energy from GWs into MHD waves. The basic idea is that in case of a binary merger a large amount of the gravitational binding energy of the system is radiated as gravitational wave energy (see Figure 1). Since GWs are difficult to observe while MHD waves lead to (an imprint on) electromagnetic radiation from the source it is of interest to find out whether GWs can transfer some of their energy to a magnetized plasma in the form of MHD waves.

The first part serves to convey the flavor of the physics and the methods involved. In the second part, we zoom in on the results of our own work, mainly a PhD project by Joachim Moortgat (Moortgat 2006).

2. ‘An Illuminating Application of General Relativity to Electrodynamics’

You are all familiar with the appearance of virtual mechanical forces in a rotating system which are absent in an inertial frame. Now, similarly but less known, rotation also affects electromagnetism. In 1939 J. Robert Oppenheimer asked his student Leonard I. Schiff to solve a problem, which he published as ‘an illuminating application of general relativity to electrodynamics’ (Schiff 1939).

Consider two concentric, hollow and static, metallic spheres, and put equal but opposite charges on the two. Now both the electric and the magnetic field external to the outer sphere are found to be zero by a static observer O (see left side of Figure 2). Of course, an electric field does exist in the shell between both spheres but not outside of the larger one. Next, consider the situation in which the two charged spheres are
Figure 1. Schematic drawing of two strongly magnetized neutron stars with magnetic moments $m_1, m_2$, orbiting each other and with spin axes $\Omega_1, \Omega_2$ aligned with each other and with the orbital rotation. Here, we focus on the interaction between GWs emitted along the orbital rotation axis with the far-field magnetized wind.

Figure 2. On the left, two concentric, hollow, metallic spheres with equal but opposite electric charges are standing still and the observer $O$ is either static or rotating. On the right, the spheres are rotating at equal rate while the observer is standing still.
rotating around a common axis and with the same period. In this case an external static observer does measure a magnetic field since the electric current formed by the large rotating charged sphere is not cancelled by that from the smaller one (see right side of Figure 2). Finally, let us return to the situation on the left in which the spheres are at rest but now let the observer orbit the spheres with the same period as the spheres on the right but in the opposite sense. The question posed to Schiff was: does the orbiting observer measure a magnetic field or not?

Schiff went to solve the problem within general relativity (GR). And, of course, we know that if the electromagnetic field vanishes in a region of space-time it cannot become finite by any coordinate transformation. Let us follow Schiff’s approach as it offers a neat perspective on the kind of results we may expect to find later on when solving the coupling problem between MHD waves and GWs.

2.1. Covariant Formulation of Maxwell’s Laws

A rotating frame is an accelerating frame and we therefore need the framework of GR to write down Maxwell’s laws in a coordinate invariant or, as it is called, covariant way. We will do this in two steps: First we take the step to special relativity (SR), and then to GR. We will use Gaussian units but put $c$, the speed of light, equal to one.

In the first step, from Galilean physics to SR, we rewrite Maxwell’s equations - which, of course, are already valid under SR - in the form of 4-vectors and 4-tensors. We first express the force equation and the energy equation for a particle of rest-mass $m$ and charge $e$ which moves with speed $\vec{v}$ in an electric ($\vec{E}$) and a magnetic field ($\vec{B}$), in 4-notation by introducing the Faraday two-form for the electromagnetic fields ($u^a = (\gamma, \gamma \vec{v})$ is the 4-velocity, $p^a = mu^a$ the 4-momentum):

$$\frac{dp_a}{d\tau} = eF_{ab}u^b$$

$$||F_{ab}|| = 
\begin{bmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & B_z & -B_y \\
E_y & -B_z & 0 & B_x \\
E_z & B_y & -B_x & 0
\end{bmatrix}$$

Then Maxwell’s equations can be written in a compact form in terms of the Faraday tensor $F$ and its dual $F^*$, where $*F_{ab} = \frac{1}{2} \epsilon_{abcd} F^{cd}$ and $\epsilon_{abcd}$ is the Levi-Civita tensor:

$$\frac{\partial *F^{ab}}{\partial x^b} = 0$$

$$\frac{\partial F^{ab}}{\partial x^b} = 4\pi j_m^a$$

Here the indices $a, b$ take on values 0 (time component), and 1,2,3 (space components).

In the second step, we generalize these to any system of coordinates including accelerating frames. We do this by replacing partial derivatives by covariant ones, and now $*F_{ab} = \frac{1}{2} |g|^{1/2} \epsilon_{abcd} F^{cd}$:

$$\nabla_b *F^{ab} = 0$$

$$\nabla_b F^{ab} = 4\pi j_m^a$$
Finally, in a coordinate basis, we can write out Maxwell’s laws in components again in a form valid for any metric $g$:

$$\frac{\partial F_{ab}}{\partial x^c} + \frac{\partial F_{bc}}{\partial x^a} + \frac{\partial F_{ca}}{\partial x^b} = 0$$  \hspace{1cm} (7)

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^b} \left( \sqrt{-g} F^{ab} \right) = 4\pi j_m^a$$  \hspace{1cm} (8)

where $a, b, c$ take on values 0, 1, 2, 3. Note that they do not differ much from the non-relativistic form. But of course, curved space as given by the metric tensor does enter through the appearance of upper and lower indices and the determinant $g \equiv \|g_{ab}\|$.

As a last step, we add the Einstein field equations (EFE), since the electromagnetic fields are coupled to the gas motion through the energy tensor in the EFE.

$$G_{ab} = 8\pi G T_{ab}$$  \hspace{1cm} (9)

### 2.2. Electromagnetism in a Rotating System

The next step Schiff took was to go to a rotating frame, write down the line element,

$$ds^2 = g_{ab}dx^a dx^b = -\left[1 - \Omega^2(x^2 + y^2)\right]dt^2 + dx^2 + dy^2 + dz^2 + 2\Omega y dx - 2\Omega x dy,$$  \hspace{1cm} (10)

read off the elements of the metric tensor, and use these to write out Maxwell’s laws Eqs. (7, 8) explicitly:

$$\vec{\nabla} \cdot \vec{B} = 0,$$  

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_m + \rho_E(\vec{\Omega}, \vec{B})$$  \hspace{1cm} (11)

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0,$$  

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi \vec{j}_m^\tau + \vec{j}_E(\vec{\Omega}, \vec{E}, \vec{B})$$  \hspace{1cm} (12)

As you can see, Schiff recovered Maxwell’s laws with two modifications: a virtual charge density in Poisson’s law ($\rho_E$) and a virtual current density in Ampère’s law ($\vec{j}_B$). The appearance of these virtual terms is not surprising in view of our earlier discussion of the appearance of virtual terms in the equation of motion. The ‘virtual’ terms depend exclusively on the rotation speed $\vec{\Omega}$ of the rotating frame and on the electric and magnetic fields $\vec{E}$, $\vec{B}$.

#### 2.2.1. Solution of Paradox of Rotating Shells

Applying these results to the rotating, charged spheres, Schiff demonstrated that the virtual current cancels the electric current exactly:

$$4\pi \vec{j}_m^\tau = -4\pi \rho_m \vec{\Omega} \times \vec{r} = -\vec{j}_B.$$  \hspace{1cm} (13)

This completes the proof that, for an observer orbiting the charged fixed spheres, there is no magnetic field, as is the case for the fixed observer.

#### 2.2.2. Pulsars ‘Avant la Lettre’

Schiff’s solution has an interesting second application, to pulsars, (of course this was not recognized at the time since pulsars had not yet been discovered). The explicit expression for the virtual charge density in the rotating frame reads:

$$\rho_E(\vec{\Omega}, \vec{B}) = 2 \vec{\Omega} \cdot \vec{B} - (\vec{\Omega} \times \vec{r}) \cdot (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \cdot [(\vec{\Omega} \times \vec{r}) \times \vec{B}]$$  \hspace{1cm} (14)
One immediately recognizes the (negative of the) familiar Goldreich-Julian density. The modified Poisson equation (11) for a rotating frame leads to a well-known result in pulsar physics: Consider a rotating star and assume that the interior is well-conducting. Then, the co-moving electric field inside the star vanishes. This is just the result of the ideal MHD assumption. In the fixed lab frame, we have $\mathbf{E} = -\mathbf{\Omega} \times \mathbf{r}$ and ordinary Poisson’s law in the fixed frame then dictates a charge density $\rho = \nabla \cdot \mathbf{E}/4\pi$ in the fixed frame. In the co-rotating frame, the charge density is equal to the Goldreich-Julian density.

3. General Relativistic MHD

We will now sketch a familiar route to MHD in GR. First, one assumes that the matter can be described in a fluid approximation: at each point in space-time the 4-velocity $u^a$ characterizes the fluid motion.

3.1. 1+3 Split in Ortho Normal Frame

Next, in order to keep the quantities in equations as close as possible to measurements, one chooses an orthonormal frame (ONF) at each point. The observer is chosen to move with the local fluid velocity, which is taken as the time direction. This choice allows a 1 + 3 split of the equations. The ONF is characterized locally by a flat metric. Note that in a coordinate basis the metric would not be flat but the basis vectors commute. Now, the change in basis vectors with coordinates is expressed by the so-called Ricci rotation coefficients which take the place of Christoffel symbols.

The electric and magnetic fields are now defined as

$$E_a \equiv F_{ab} u^b, \quad B_a \equiv \ast F_{ab} u^b = \frac{1}{2} \epsilon_{abc} B^c \tag{15}$$

where $\epsilon_{abc} \equiv \frac{|g|^{1/2}}{2} \epsilon_{abcd} u^d$. It is easy to check that these fields are orthogonal to the time-direction and purely space-like. Indeed, in a frame co-moving with the fluid, the 4-velocity reduces to $u^a = (1, 0, 0, 0)$ and the definitions (15) take the form of the usual electric and magnetic field: $E_a = (0, \mathbf{E}), B_a = (0, \mathbf{B})$. Conversely, the Faraday tensor can now be expressed with these 4-vectors as:

$$F_{ab} = u_a E_b - u_b E_a + \epsilon_{abc} B^c \tag{16}$$

3.1.1. Maxwell’s Equations in ONF

We are now ready to write down Maxwell’s equations in a form which is suitable to the astronomical observer, those in the ONF. In 3-form, the equations are Marklund et al. (2000):

$$\nabla \cdot \mathbf{B} = \rho_B, \quad \nabla \cdot \mathbf{E} = 4\pi \rho_m + \rho_E \tag{17}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{j}_B, \quad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{j}_m + \mathbf{j}_E \tag{18}$$

There are now four extra, or virtual terms: a charge density in Poisson’s equation, a magnetic charge density which acts as a source for the magnetic field, a current density...
in Faraday’s equation, and another current density in Ampère’s equation. All of these virtual terms are well-determined functions of the electric and magnetic fields and of the properties of the ONF as determined by the Ricci rotation coefficients:

\[
\rho_E \equiv -\Gamma_{\beta\alpha}^\gamma E^\beta - \epsilon^{\alpha\beta\gamma} \Gamma^0_{\alpha\beta} B_\gamma,
\]

(19)

\[
\rho_B \equiv -\Gamma_{\beta\alpha}^\gamma B^\beta + \epsilon^{\alpha\beta\gamma} \Gamma^0_{\alpha\beta} E_\gamma
\]

(20)

\[
\vec{j}_E \equiv [-\left(\Gamma_{\beta\gamma}^\alpha - \Gamma_{\beta}\delta\right) E^\beta + \Gamma^\beta_{\alpha\gamma} E^\alpha - \epsilon^{\alpha\beta\gamma} \left(\Gamma^0_{\beta\gamma} B_\delta + \Gamma^\delta_{\beta\gamma} B_\delta\right) e_\alpha,}
\]

(21)

\[
\vec{j}_B \equiv [-\left(\Gamma_{\beta\gamma}^\alpha - \Gamma_{\beta\gamma}\rho_0\right) B^\beta + \Gamma^\beta_{\alpha\gamma} B^\alpha + \epsilon^{\alpha\beta\gamma} \left(\Gamma^0_{\beta\gamma} E_\delta + \Gamma^\delta_{\beta\gamma} E_\delta\right) e_\alpha
\]

(22)

3.1.2. The Ideal MHD Assumption

The ideal MHD assumption of perfect conductivity - that is the vanishing of the electric field in a frame co-moving locally with the fluid - which results in \( \vec{E} + \vec{v} \times \vec{B} = 0 \) for non-relativistic motion, is now expressed in GR by putting the electric 4-vector equal to zero:

\[
E_a = F_{ab} u^b = 0
\]

(23)

4. Interaction between GW and Ambient Magnetic Field

For a short history of the excitation of waves, both in a magneto-plasma and in a vacuum with an ambient magnetic field, we refer to Moortgat & Kuijpers (2003). Having in mind the late phase of a binary merger when a large amount of GWs escapes and hits a previously ejected magnetized wind (see Figure 1), we have studied the conversion of GWs into MHD waves in a simplified setting where a monochromatic GW of arbitrary linear polarization is launched into a half-space filled with a homogeneous, hot plasma and a uniform, ambient magnetic field.

4.1. Extra Currents in the Presence of GWs and an Ambient Magnetic Field

We have applied the results of the previous section for the varying metric of two monochromatic GWs of different polarization in the presence of an ambient, uniform, magnetic field and a homogeneous, hot plasma. Strictly speaking, the metric is a solution of the EFE in the absence of magnetic field or plasma. We construct an ONF for an observer who is hit by a GW. The basis determines the Ricci coefficients. In our case, the only non-vanishing virtual terms are an electric and a magnetic current. Putting these into Maxwell’s equations and the equations of motion we find growing MHD waves. This growth occurs through the virtual currents in Maxwell’s equations; the equation of motion still only has the Lorentz force from the material current, and GR does not give rise to an extra virtual Lorentz force (see Figure 3a).

4.2. Growth of MHD Waves

We choose the ambient magnetic field in the \( x - z \) plane and the GW direction in the \( z \)-direction (Figure 4). The \( + \)-polarized GW excites a fast magneto-sonic wave, which grows linearly in space, here along the \( z \)-coordinate. The excitation works best if the ambient magnetic field is transverse to the incident GW direction. As long as the mismatch between GW and MHD wave remains small \( (z \Delta k \ll 1) \) the growth (see Figure
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Figure 3. a. The scheme on the left shows that coupling between plasma and GWs is not through the momentum equation for the gas but via Maxwell’s equations; b. The curve on the right shows an example of the growth of the transverse magnetic field component $B_x(z)$ of a fast magneto-sonic wave with distance $z$.

Figure 4. MHD mode polarisation in an ambient magnetic field $B_0$ which lies in the $x-z$ plane ($u_A$ Alfvén speed; $k_g$ GW vector). The wave perturbations of velocity, electric and magnetic fields, and current are given by respectively $v^1, E^1, B^1, j^1$. The fast MHD mode $k_f$ is on the left, and the Alfvén mode $k_A$ on the right.

3b) is (Moortgat & Kuijpers 2004):

$$\frac{B^1_y(z,t)}{B^0} = \frac{h_x}{2} \sin \theta \ \omega z \ \cos[\omega(z - t)]$$ (24)

We also find that an Alfvén wave can be excited, which again grows linearly, now excited by the $x$-polarized GW. This coupling exists only for small angles $\theta$ between ambient magnetic field and incident GW, with a growth:

$$\frac{B^1_x(z,t)}{B^0} = \frac{h_x}{4} \left[ 1 + u_{A||} \ e^{i\omega(z/u_{A||} - t)} - \frac{1 + u^2_{A||}}{1 - u_{A||}^2} 2 e^{i\omega(z-t)} \right] \approx \frac{\theta h_x}{2} \omega z \ \cos[\omega(z - t)]$$ (25)

Our results can be understood intuitively as follows. First let us consider the fast MSW. As a GW propagates out of the paper in Figure 5a, particles arranged in a circle undergo periodic distortions which are divergence-free, i.e. there is no compression of the fluid (see Figure 5b for a 3-D version). Figure 6 demonstrates how a transverse magnetic field is periodically compressed and diluted by the divergence-less GW propagating out of the paper. These magnetic perturbations then couple to the fluid and excite a MSW.
Figure 5. Left: Sketch of the relative separations of a number of test particles on a circle as a GW propagates out of the paper; Right: A 3-D sketch of the space and magnetic field distortions as a GW propagates in the vertical direction.

Figure 6. Sketch of the relative gas motion and the magnetic field change as a GW propagates out of the paper transverse to an ambient magnetic field (Papadopoulos et al. 2001).

One can also look at the particle excursions under the action of the GW as in a GW detector. If one considers the particles as foot-points of magnetic field lines one obtains
Figure 7. 3D-sketch of magnetic field changes for a +polarized GW and a fast MHD wave (a. left), and for a ×-polarized GW and an Alfvén wave (b. right).

\[
\delta x = \frac{1}{2} (h_x x_0 + h_x y_0), \quad \delta y = \frac{1}{2} (h_x x_0 - h_x y_0)
\]  

for a + and respectively ×-polarized wave. As our ambient magnetic field has no y-component, obviously a MSW is excited by a +polarized GW and an Alfvén wave by a ×-polarized GW, respectively:

\[
\delta B_x \propto \frac{1}{2} h_x B^0_x \quad \quad \delta B_y \propto \frac{1}{2} h_x B^0_x
\]  

4.3. Self-Consistent Solution

As a next step, we have considered the self-consistent solution when the growth of the MHD waves leads to feedback and damping of the GW. We now find two mixed wave modes, each of which has characteristics of both an MHD wave and a GW. The GW is slightly modified by the magneto-plasma and the MSW is slightly modified by the GW. Their phase speeds are given by, respectively:

\[
v_{\text{phase, GW}}^2 = 1 + \frac{2(B^0_y)^2}{\omega^2} \frac{1 + u_A^2}{1 - u_A^2}, \quad v_{\text{phase, MSW}}^2 = 1 - \frac{(2B^0_x)^2}{\omega^2} \frac{u_A^2}{1 - u_A^2}
\]

For the AW coupling, a similar result is obtained.

5. Conclusion

We have studied the conversion of gravitational wave energy into MHD waves when a monochromatic GW impinges on a uniform, hot and streaming magnetized plasma in the MHD approximation. This idealized computation serves as a pointer in the more difficult problem of a merging neutron star binary where GWs are launched into a surrounding magnetized wind. In the application of our results to the latter problem we
have approximated the wind with a relativistically outflowing, magnetized electron-positron pair plasma with a transverse magnetic field which is uniform over a certain scale height.

Assuming that both the GW amplitude and the ambient magnetic field strength in the wind decrease inversely with distance, we find that the volume integrated energy of fast magneto-sonic waves grows linearly with distance. For an interaction region of 0.03 pc, a gravitational wave frequency of $\omega_g = 2\Omega_{\text{binary}} = 4\pi \times 10^3$ rad/s, a GW amplitude of $10^{-3}$ at a distance of 10 light cylinder radii (500 km), a magnetic field strength of $10^7$ T at the light cylinder (or $10^9$ T at the stellar surface), and a Lorentz factor of 100 for the relativistic wind, the magnetic component of the excited fast magneto-sonic wave has a total energy of

$$T_B^{(1)} = V \frac{B_0 B^{(1)}}{4\pi} \sim 10^{37} \text{J},$$

(29)

which is only a fraction $10^{-6}$ of the wind magnetic energy in the same volume, and about $10^{-10}$ of the binding energy of the binary.

Of course, there is a large freedom in the chosen values for the various parameters (Moortgat & Kuijpers 2003). One effect is that the excited magnetic wave amplitude decreases inversely with the wind Lorentz factor squared, due to special relativistic weakening of the field in an outflowing wind. Had we applied our results to a mildly relativistic wind and to a magnetar-like magnetic field strength of $10^{11}$ T at the stellar surface, the estimate Eq. (29) would have gone up with a factor $10^4$.

Finally, once the MHD waves are excited they may lead to low-frequency radio emission within the bandwidth of a radio telescope such as LOFAR, at the same time as GWs arrive at a GW detector (Moortgat & Kuijpers 2006).

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References

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