Theoretical Approaches to the Physics of Spectral Line Polarization

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**Abstract.** Due to the continuous developments in polarimetric instrumentation, which will become even more dramatic in the near future with the availability of new generation solar telescopes, we are now severely confronted with a variety of new detailed observations of high diagnostic potential, whose interpretation requires a firmly established theoretical framework. In this contribution, I review the fundamental physical processes that underlie the generation and transfer of polarized radiation in stellar atmospheres, and I discuss the present status of the theoretical schemes now available, pointing out their main successes and limitations. I also present some ideas about the theoretical improvements that I consider necessary to achieve a correct interpretation of the complex phenomenology shown by polarimetric observations, focusing particularly on the second solar spectrum, which can be considered as one of the most important test benches of the theory.

1. Introduction

Polarized radiation is expected whenever the interaction process between matter and radiation is characterized by any kind of symmetry-breaking (see, for example, the review by Casini & Landi Degl’Innocenti 2007). The symmetry-breaking can be intrinsic to the interaction itself, or due to external causes (e.g., presence of a magnetic field). In order to get more insights on this fundamental statement, let us consider a quantum transition between an upper level $J_u$, and a lower level $J_\ell$, $J$ being the quantum number associated with the total angular momentum operator. According to quantum mechanics, the atomic system in the upper (or lower) level can occupy any of the $(2J + 1)$ possible $M$ magnetic substates, which are degenerate if no magnetic field is present. The radiation emitted by the various transitions between the $M_u$ substates of the upper level and the $M_\ell$ substates of the lower level (generally referred to as Zeeman components), is characterized by well defined polarization properties which depend on the angle between the emission direction and the quantization axis, and on the value of $\Delta M \equiv M_u - M_\ell$. In the electric dipole approximation the only allowed transitions are those with $\Delta M = 0$ ($\pi$ components), and those with $\Delta M = \pm 1$ ($\sigma_b$ and $\sigma_r$ components, respectively). The polarization properties of the $\pi$ and $\sigma$ components emitted along various directions are schematically shown in Fig. 1.

In a de-excitation process, each Zeeman component contributes to the emitted radiation with a particular weight which takes into account the relative population of the upper $M$-sublevel of the component, the relative strength of the component, and the emission direction. If the atomic system is naturally (isotropically) excited, then the
upper magnetic sublevels are equally populated, and it can be shown that the weights of the various Zeeman components assume values such that, if no magnetic field is present, the total radiation emitted in any direction is completely unpolarized.

There are two basic scenarios (not mutually exclusive) where the different polarization properties of the Zeeman components manifest themselves in emission of polarized radiation. The first scenario corresponds to the possibility that the upper substates may be separated in energy, so that the contributions of the various components fall at slightly different wavelengths. In this case, even if the atomic system is naturally excited, the emitted radiation has polarization properties varying with wavelength. The most common circumstance for this condition to occur is when the atomic system interacts with an external magnetic field (Zeeman effect). The second scenario corresponds to the possibility that the upper substates are unevenly populated, so that the various components no longer contribute to the emitted radiation with those particular weights that make the total polarization vanish. This condition generally occurs whenever the atomic system is excited through a physical process which, for any reason, is not spatially isotropic. An atom whose magnetic sublevels are not evenly populated and/or are characterized by well-defined phase relations is said to be polarized. The radiation emitted by a polarized atomic system is in general polarized, as it can be shown through the following clarifying example. Let us consider a two level atom, the lower level having $J_l=0$ and the upper level having $J_u=1$, and let us suppose to excite it with a collimated unpolarized radiation beam propagating along the quantization axis. In the presence of such an incident radiation field, it can be shown that only the $M_u=\pm 1$ magnetic sublevels will be populated, so that only the $\sigma$ components will contribute to the emitted radiation. From Fig. 1 it is then clear that if the radiation emitted perpendicularly to the incident beam is considered, this will be totally linearly polarized perpendicularly to the scattering plane. It should be observed that this is basically the physical mechanism responsible for the linearly polarized spectrum of the solar radiation coming from quiet regions close to the limb (the so-called Second Solar Spectrum). Of course, in the solar atmosphere the radiation field impinging on the atoms is not perfectly collimated along the local vertical, nevertheless it has a certain degree of anisotropy that provides for the radiation scattered at $90^\circ$ a linear polarization degree of the order of 1%. Polarization produced through resonance scattering processes like the one here described is usually referred to as scattering polarization.

The quantum theory of the Zeeman effect was developed in the first half of the last century, and we can safely say that today it is absolutely well established. Indeed, the
difficulties related to the interpretation of spectropolarimetric signals produced through this mechanism are essentially due to the complexity of the line formation problem in the solar atmosphere, the physical origin of the polarization being, on the other hand, absolutely clear. On the contrary, as soon as the first scattering polarization signals, detected at the solar limb since the early 1940s (see Redman 1941), could be observed with a sufficient accuracy, it was immediately clear that the theoretical interpretation of the peculiar profiles shown by several of them would have required a substantial improvement of our understanding of the physics of resonance scattering. The incomplete comprehension of this fundamental physical process has become particularly evident during the last decades when, thanks to the development of polarimeters like ZIMPOL (see Povel 1995), able to reach sensitivities of the order of 1 part over $10^5$, many new apparently “enigmatic” scattering polarization signals have been detected (see Stenflo & Keller 1997). Given the high number of physical “ingredients” involved in the generation of scattering polarization signals, and given their consequent enormous diagnostic content, since the 1970s their interpretation has been representing one of the most important and exciting theoretical challenges in the field of solar polarimetry.

Though our understanding of the physics of resonance scattering has substantially improved during the last 30 years, and a robust theoretical scheme, based on the principles of quantum electrodynamics has been developed (see Landi Degl’Innocenti & Landolfi 2004, hereafter LL04), and successfully applied for the interpretation of a wide range of signals (see Trujillo Bueno 2009, for a recent review), our comprehension of the rich phenomenology shown by scattering polarization signals remains rather fragmentary.

In the present contribution I will review some of the most important theoretical approaches to the physics of scattering polarization developed so far. The aim of the paper is to point out strengths and limitations of the various approaches by reviewing the physical processes and phenomena they are able to describe, and by discussing the most important hypotheses and approximations they are based on (for a detailed derivation of each approach the reader will be referred to the original papers or to more exhaustive monographs). Examples of the main successes of each theory, as well as a discussion of unsolved problems, will also be presented. Of course, the review is by no means exhaustive, and I apologize in advance to all whose work is not mentioned here.

2. The second solar spectrum

The linearly polarized solar limb spectrum, today known as the second solar spectrum, is probably the clearest manifestation of scattering polarization in astrophysics, and it is no coincidence that it played, and is still playing a fundamental role in the development of the theory of polarization. Its importance can be clearly highlighted through a few remarkable examples of signals whose interpretation required fundamental advances in our understanding of the physics of resonance scattering.

The interpretation of the peculiar polarization pattern observed across the H and K lines of Ca II (and, similarly, across the D1 and D2 lines of Na I), which shows a sign reversal between the two lines, required the comprehension of the fundamental role played by quantum interferences (or coherences) between different quantum levels, as first showed by Stenflo (1980). In 1997, Stenflo explained the triplet-peak structure of the second solar spectrum of the Ba II D2 line in terms of hyperfine structure (HFS),
pointing out how this physical aspect, generally negligible in the usual intensity spectrum, can be extremely important when polarization phenomena are taken into account (see Stenflo 1997). In the same year, Trujillo Bueno & Landi Degl’Innocenti (1997) pointed out the importance of lower level polarization, previously systematically neglected due to the qualitative idea that atomic polarization in long-lived levels should be always completely destroyed by depolarizing collisions and magnetic fields. By taking into account this new ingredient it has been possible to interpret the polarization signals observed in the Ca ii infrared triplet line at 8662 Å (see Manso Sainz & Trujillo Bueno 2003), as well as in the Mg ii b-lines (see Trujillo Bueno 1999, 2001).

There are now clear evidences that the interpretation of a large class of signals (like the one observed in the Ca ii line at 4227 Å) requires the effects of partial redistribution in frequency to be taken into account. The development of theoretical frameworks able to account for these effects is nowadays one of the hottest topics in this research field. Laboratory experiments are now being performed in order to clarify various aspects of this spectrum, like the presence of line-integrated linear polarization in the D1 line of sodium and potassium (see Thalmann et al. 2006, 2009), and in order to verify the main results derived from the theoretical approaches developed so far.

The importance of the second solar spectrum is not limited to single signals, but also concerns some of its general properties, as recently pointed out by Belluzzi & Landi Degl’Innocenti (2009, 2010). Using the atlas “The Second Solar Spectrum” (Gandorfer 2000, 2002, 2005), Belluzzi & Landi Degl’Innocenti (2009) identified the strongest signals of this spectrum, and divided them into five different classes according to the shape of their profile. Through such a systematic analysis, the authors discovered a series of spectroscopic properties common to the spectral lines responsible for such signals, properties that have been summarized into the following three empirical laws:

1. **First law:** the transitions producing the strongest polarization signals of the second solar spectrum are either resonance transitions, or subordinate transitions whose lower level is the upper level of a resonance transition producing a strong polarization signal.

2. **Second law:** all the strong polarization signals that are produced by spectral lines having a small equivalent width (i.e. lines having $W_{\lambda}/\lambda < 20 F$) are of type “S”.

3. **Third law:** the spectral lines producing the strongest polarization signals ($Q/I > 0.17\%$) are due to transitions having either $\Delta J \equiv J_u - J_f = +1$ or $\Delta J = 0$.

The theoretical interpretation of these laws is already under investigation, and it will probably represent a further important test bench for the theoretical approaches that will be developed in the future.

3. **A unifying theoretical approach in the formalism of quantum electrodynamics**

The unifying, self-consistent theoretical approach to the physics of polarization, formulated by Landi Degl’Innocenti (1983) within the framework of the density-matrix

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1. All the signals whose profile shows a single sharp peak have been classified as “S” signals.
formalism, starting from the principles of quantum electrodynamics, represents one of the most important theoretical accomplishments obtained so far in this research field. A clear and detailed derivation of this theoretical scheme can be found in LL04, and will not be repeated here. In this section, in order to point out strengths and weaknesses of this approach, I will present only the fundamental equations the theory is based on, discussing their physical meaning, and focusing the attention on the most important hypotheses and approximations that are introduced (see also Casini & Landi Degl’Innocenti 2007; Trujillo Bueno 1990). I will then review some of the most important successes of this theory, and I will discuss its main limitations and their consequences for the interpretation of particular classes of signals.

Other theoretical approaches, developed with different assumptions and suitable for investigation of those signals which cannot be interpreted under the approximations of the scheme described here, will be presented in the next section.

3.1. The density operator

The first problem to be addressed in developing a quantum theory for scattering polarization is to find a suitable way to describe atomic polarization (i.e. the presence of population unbalances and/or of quantum interferences between pairs of magnetic sublevels). A very good solution, as proposed by Bommier & Sahal-Bréchot (1978) and by Bommier (1980) is through the density operator, an extremely powerful tool for describing any physical system that is in a statistical mixture of states. The density operator is defined by

$$\hat{\rho} = \sum_{\alpha} p_{\alpha} |\psi_{\alpha}\rangle\langle\psi_{\alpha}|,$$

(1)

where $p_{\alpha}$ is the probability for the system to be in the (pure) dynamical state described by the vector $|\psi_{\alpha}\rangle$, and where the sum is extended to all the pure states in which the system can be found. The action of the density operator is completely specified once its matrix elements, evaluated on a given basis of the Hilbert space associated to the quantum system, are known. Such density-matrix elements contain all the accessible information about the system; through them it is possible to describe the dynamical state of the system and its temporal evolution in a very compact way. For an atomic system, the most natural basis for defining the matrix elements of the density operator is the basis of the eigenvectors of the total angular momentum. On this basis the elements of the density matrix are given by

$$\langle \alpha JM | \hat{\rho} | \alpha' J' M' \rangle = \rho (\alpha JM, \alpha' J' M'),$$

(2)

where $J$ and $M$ are the angular momentum quantum numbers previously introduced, while $\alpha$ represents a set of inner quantum numbers. As it can be shown (see, e.g., LL04), the diagonal elements represent the populations of the magnetic sublevels, while the off-diagonal elements describe the quantum interferences between different magnetic sublevels. The expectation value of a given dynamical variable $\hat{O}$ is given by (see, e.g., LL04)

$$\langle \hat{O} \rangle = \text{Tr} (\hat{\rho} \hat{O}),$$

(3)

with $\hat{O}$ the quantum operator associated to the dynamical variable $O$. 
3.2. The statistical equilibrium and the radiative transfer equations

The time evolution of the density operator associated to a physical system described by the Hamiltonian $H$ is given by the Liouville equation

$$\frac{d}{dt} \rho(t) = -\frac{2\pi i}{\hbar} [H, \rho(t)].$$

If the physical system is composed by one atom and by the radiation field, then the Hamiltonian $H$ is given by the sum $H = H_A + H_R + V$, with $H_A$ the atomic Hamiltonian (which eventually includes the interaction with an external magnetic field), $H_R$ the Hamiltonian of the radiation field (generally described within the formalism of the second quantization), and $V$ the interaction Hamiltonian (for its explicit expression see, for example, Cohen-Tannoudji et al. 1977).

When the total Hamiltonian can be expressed as the sum of an unperturbed Hamiltonian and an interaction Hamiltonian, it is generally useful to work in the so-called interaction picture. In this representation, in fact, only the interaction Hamiltonian appears in the commutator in the right-hand side of the Liouville equation, which assumes the form

$$\frac{d}{dt} \rho_I(t) = -\frac{2\pi i}{\hbar} [V_I, \rho_I(t)],$$

where the label $I$ means that the corresponding quantity is defined in the interaction picture. Equation (5) has the formal solution

$$\rho_I(t) = \rho_I(0) - \frac{2\pi i}{\hbar} \int_0^t [V_I(t'), \rho_I(t')] dt'.$$

This equation can be used again to express the quantity $\rho_I(t')$ that appears in the integral. By repeating this procedure, a perturbative development of the formal solution of Eq. (5) is obtained. If the development is stopped at the $k$-th term, by substituting the quantity $\rho_I(t')$ in the integral with $\rho_I(0)$, the matter-radiation interaction is said to be described at the $k$-th perturbative order.

The equation describing the time evolution of the expectation value of a given dynamical variable $O$ can be obtained calculating the time derivative of Eq. (3), and using Eq. (6). After some algebra one obtains

$$\frac{d}{dt} O(t) = \text{Tr} \left\{ \left( \frac{d}{dt} \hat{O}_I(t) \right) \rho_I(t) \right\} - \frac{2\pi i}{\hbar} \text{Tr} \left\{ \left[ \hat{O}_I(t), V_I(t) \right] \rho_I(0) \right\}$$

$$- \frac{4\pi^2}{\hbar^2} \text{Tr} \left\{ \int_0^t \left[ \left[ \hat{O}_I(t), V_I(t) \right], V_I(t') \right] \rho_I(t') dt' \right\}. $$

This is an exact equation, and represents the starting point to derive the equations describing the evolution of the atomic system and of the radiation field resulting from their mutual interaction. In particular, if in place of $O(t)$ and $\hat{O}_I(t)$ one puts the density matrix elements associated to the atomic system, and the corresponding quantum operator, then the statistical equilibrium equations for the density matrix are obtained. If in place of $O(t)$ and $\hat{O}_I(t)$ one puts the Stokes parameters describing the radiation field, and the corresponding quantum operators, then the radiative transfer equations are obtained.
The approximation that is now introduced is to substitute the quantity $\rho_I(t')$ that appears in the integral in the right-hand side of Eq. (7) with $\rho_I(0)$, so that it can be extracted from the integral. Since this is equivalent to performing the same substitution in the integral on the right-hand side of Eq. (6), this approximation is equivalent to consider a 1-st order perturbative development of the matter-radiation interaction, which means that only 1-st order processes (processes involving only one photon) can be described.

Within this approximation, scattering, which is intrinsically a 2-nd order process, has to be treated as a succession of independent 1-st order absorption and emission processes (i.e. in the limit of complete redistribution in frequency, CRD). It should be observed that this is correct either in the presence of collisions that perturb the atom during the interaction with the radiation field to the point of completely relaxing the coherence of the scattering process, or when the radiation field illuminating the atom is spectrally flat (i.e. independent of frequency). Since polarization phenomena are generally negligible when the first condition occurs (because of the effect of depolarizing collisions), the latter (usually referred to as the flat-spectrum approximation) is generally assumed when the discussed theoretical approach is applied. More precisely, this approximation requires the radiation field to be flat across spectral intervals larger than the Bohr frequency connecting pairs of magnetic sublevels between which quantum coherences are considered, and larger than the inverse lifetime of the various magnetic sublevels.

The statistical equilibrium equations for the density matrix were first derived (under the previously discussed approximation) by Bommier & Sahal-Bréchot (1978) and Bommier (1980). The equations describe the transfer and relaxation of populations and coherences due to absorption and emission processes, and due to the presence of a magnetic field (Hanle effect). The same equations were then rederived, under the same approximation, in the seminal paper by Landi Degl’Innocenti (1983). In this work Landi Degl’Innocenti derived within the same formalism both the statistical equilibrium equations for the density matrix, and the radiative transfer equations for polarized radiation (described in terms of the four Stokes parameters), thus “closing” the Non-LTE problem in the presence of polarization phenomena (Non-LTE problem of the 2nd kind). The equations include the effects of a magnetic field (Zeeman and Hanle effects), and of collisions (elastic and inelastic). A detailed derivation of these equations, as well as a complete discussion of the important aspect (here neglected) of velocity/density-matrix correlations, can be found in LL04.

3.3. **Strengths and successes of the theory**

The main strength of the theoretical scheme described above is clearly its self-consistency, since all the equations are derived within the same elegant formalism from the principles of quantum electrodynamics. Going to more specific aspects, particularly important strength points of the theory are the following:

- possibility to take into account lower level polarization;
- possibility to take into account coherences within the same $J$-level (multi-level atoms) and between different $J$-levels (multi-term atoms);
- possibility to treat atoms with hyperfine structure;
possibility to describe the interaction with a magnetic field in every regime, from the Zeeman effect regime to the complete Paschen-Back effect regime;

possibility to take into account both the Zeeman and the Hanle effects.

The theory has been extensively and very successfully used for the determination of magnetic fields in filaments and prominences, where the flat-spectrum approximation is generally well justified (see, e.g., Bommier et al. 1981; Athay et al. 1983; Bommier et al. 1994; Trujillo Bueno et al. 2002). Thanks to the possibility of taking into account quantum coherences between different $J$-levels, through this theoretical scheme it is possible to reproduce the peculiar polarization patterns (see Sect. 2) observed in the second solar spectrum across the H and K lines of Ca II, and across the D$_1$ and D$_2$ lines of Na I (see LL04). It was within the framework of this theoretical approach that Trujillo Bueno (1999, 2001) and Manso Sainz & Trujillo Bueno (2003), as mentioned in Sect. 2, first explained, in terms of lower level polarization, the “enigmatic” signals produced in the same spectrum by the Mg I b-lines and by the Ca II infrared triplet, respectively. The possibility of treating multi-level atoms has made this approach particularly attractive for investigating complex atoms. Very interesting results have been obtained, for example, for the interpretation of the second solar spectrum of Ti I (Manso Sainz & Landi Degl’Innocenti 2002) and Ce II (Manso Sainz et al. 2006).

An interesting result, which could be obtained thanks to the possibility for the theory to model atoms with HFS, and to take into account both the Hanle and Zeeman effects in every magnetic regime, is the discovery of a differential magnetic sensitivity of the triple-peak signal shown in the second solar spectrum by the Ba II D$_2$ line (see Belluzzi et al. 2006, 2007). Such differential magnetic sensitivity, which has great diagnostic potentialities, has already been confirmed by various observational campaigns (see, e.g., Ramelli et al. 2009; López Ariste et al. 2009). It should be observed that the possibility of considering coherences between different $J$-levels, or between different HFS $F$-levels, in the presence of a magnetic field in the incomplete Paschen-Back effect regime allows the theory to account for quantum mechanisms like the level-crossing and the anti-level-crossing effects (see, e.g., Belluzzi et al. 2007).

Within the framework of this theory, Belluzzi et al. (2009) investigated the physical origin of the weak polarization signal observed by Stenflo et al. (2000) in the second solar spectrum of the faint lithium doublet at 6708 Å, and concluded, in disagreement with this observation showing a triplet-peak structure, that only a two-peak profile can be theoretically expected, and that only a one-peak profile should be measurable unless the temperature of the atmospheric layer the radiation comes from is sufficiently low. On the other hand, improved observations (like those presented by Stenflo (2011), and others which are still unpublished) seem to agree with the one-peak profile predicted by Belluzzi et al. (2009). The theoretical result of Belluzzi et al. (2009) may thus be considered as an example of a theoretical prediction that anticipated an observational finding, an important indication of the adequacy of the theory. The soundness of the theory can be also appreciated in that it has allowed to predict new physical mechanisms like the so-called “alignment-to-orientation conversion mechanism” (see LL04), or the “atomic-polarization-induced Faraday pulsation effect” (see Landi Degl’Innocenti et al. 2011).

\footnote{This expression has been proposed by Bommier (1980).}
It should be finally pointed out that the results obtained by applying this theory to different multiplets are in good qualitative agreement with the third law formulated by Belluzzi & Landi Degl’Innocenti (2009). Other examples of successful applications of this theoretical scheme can be found in Trujillo Bueno (2009).

3.4. Limitations and open problems

The main limitation of this theoretical approach is the impossibility to take into account effects of partial redistribution (PRD) in frequency. While in the intensity spectrum such effects are important (mainly in the wings) only for a limited number of very strong resonance lines (e.g., Ca II H&K, Mg II h&k, Hα, Lyα), in the second solar spectrum, as pointed out by several works, starting with the pioneering one by Omont et al. (1972), they seem to be the key ingredient to interpret most of the linear polarization profiles that Belluzzi & Landi Degl’Innocenti (2009) have classified as “M” signals\(^3\) (typical examples are the signals produced by the Na I D\(_2\) line or by the Ca I line at 4227 Å).

The open problems, however, are not limited to the interpretation of the “M” signals in terms of PRD effects. An interesting example of an unsolved problem concerning the second solar spectrum is the physical origin of the triplet-peak profile shown by the Sc II line at 4247 Å. This signal is very similar to the one produced by the Ba II D\(_2\) line, which has been clearly interpreted in terms of the HFS shown by two of the seven stable isotopes of barium (see Stenflo 1997; Belluzzi et al. 2007). Given that the only stable isotope of scandium shows HFS, and noticing the similarities between the two lines in the intensity spectrum, it was believed that this signal could also be interpreted in terms of HFS\(^4\). However, as shown by Belluzzi (2009), this is not the case: indeed the theory unexpectedly fails to model this signal, even though the effects of HFS are correctly taken into account, and the same modeling assumptions used for barium are considered. Such a failure has important consequences: if the Sc II signal is produced by a physical mechanism neglected by Belluzzi (2009), or that the theory does not account for (for example PRD effects), then not only it will be important to identify this mechanism, but it will be also interesting to understand why it seems not to be necessary in the case of barium. Perhaps the problem on the physical origin of the triplet-peak structure of the Ba II signal is still waiting for a complete answer!

4. Theoretical approaches accounting for PRD effects

The most natural way to describe PRD effects is by means of a redistribution function, a powerful tool which naturally appears in the “traditional” scattering approach to the physics of resonance polarization (see, e.g., Stenflo 1994). This approach, which has been successfully applied for treating PRD phenomena, has, on the other hand, the

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\(^3\)According to this classification the “M” signals are all those signals which show positive polarizing peaks (or lobes) in the wings of the line, a depolarization approaching to the line center, and (eventually) a sharp polarizing peak in the line core.

\(^4\)It should be observed that only 18% of the barium isotopes and 100% of scandium show HFS. This is in perfect agreement with the fact that the central peak of the barium signal is much higher than the other peaks, while the three peaks of the scandium signal have the same amplitude.
limitation of being able to describe only two-level atoms with unpolarized, infinitely sharp lower level. The redistribution function $R$ is defined so that

$$R(v_2, \vec{\Omega}_2, \vec{e}_2; v_1, \vec{\Omega}_1, \vec{e}_1) \, dv_1 \, dv_2 \, \frac{d\Omega_1}{4\pi} \, \frac{d\Omega_2}{4\pi}$$

(8)

represents the probability that in a scattering process an incoming photon of frequency $v_1$, propagation direction $\vec{\Omega}_1$, and polarization state $\vec{e}_1$ is scattered in a photon with frequency $v_2$, propagation direction $\vec{\Omega}_2$, and polarization state $\vec{e}_2$. If the scattering of a radiation beam, whose polarization properties are described through the 4 Stokes parameters, is considered, then the redistribution function is replaced by a $4 \times 4$ redistribution matrix $R_{ij}(v_2, \vec{\Omega}_2; v_1, \vec{\Omega}_1)$.

When the frequency dependence can be ignored, the redistribution matrix reduces to the so-called scattering phase matrix (or Rayleigh phase matrix) $P_{ij}(\vec{\Omega}_2; \vec{\Omega}_1)$ (see, e.g., Hamilton 1947; Stenflo 1994, or LL04). In the atomic rest frame the angular and frequency dependencies can be separated, so that the redistribution matrix can be factorized into the product of the scattering phase matrix times a scalar function describing the frequency redistribution. This is no more true in the laboratory frame where the Doppler effect, due to the motion of the atoms, has to be taken into account. In this case, the scalar function describing the frequency redistribution also depends on the directions of the incoming and scattered beams, and an intricate coupling between frequency and angle dependencies appears (see Hummer 1962, for the unpolarized case).

In order to simplify the problem, the so-called hybrid approximation has often been used (e.g., Rees & Saliba 1982). This approximation, which is justified only by heuristic arguments, consists in calculating the average over the angles of the scalar function describing the frequency redistribution, so that the angle and frequency dependencies can be factorized as in the atomic rest frame.

From the 1970s, the effects of PRD in frequency have been investigated using different approximate forms of the redistribution matrix, generally based on the angle averaged type I and type II redistribution functions of Hummer (1962) (see Sect. 1 of Sampoorna et al. 2010, for a historical review). Particularly useful results were obtained by Faurobert (1987, 1988) who investigated the role of PRD effects considering both angle-averaged and angle-dependent redistribution functions, also including the effect of a magnetic field. These investigations pointed out the limitations of approximate forms of the redistribution matrix previously proposed, and showed that the approximation of complete redistribution (CRD) is generally adequate to describe the line core polarization. The fact that PRD effects (i.e. coherent scattering) are important in the wings of strong resonance lines, while they are usually negligible in the line core has been then confirmed by several other investigations (see, e.g., Frisch 1996).

From the theoretical point of view, the most tangled physical aspect in a scattering process of polarized radiation is the role of collisions (elastic and inelastic). A first, fundamental investigation on this aspect was carried out by Omont et al. (1972) who introduced the role of collisions in the quantum theory of Raman scattering developed by Fiutak & Van Kranendonk (1962). Starting from this work, Domke & Hubeny (1988) derived a very general analytic expression of the redistribution matrix for scattering of arbitrarily polarized radiation by an atom undergoing collisions. Using this expression Faurobert-Scholl (1992) pointed out the sensitivity to the elastic collisional rate of the line-wing polarization, while Nagendra (1994) showed how depolarizing collis-
sions affect the line-core polarization. A detailed investigation on the role of various atmospheric parameters on the emergent linear polarization, calculated including PRD effects, has been recently carried out by Samporn et al. (2010). To this end, Samporn & Trujillo Bueno (2010) developed new, very efficient numerical approaches for PRD radiative transfer based on the Gauss-Seidel and Successive-Overrelaxation iterative methods.

Since the frequency redistribution matrix does not explicitly appear in the statistical equilibrium equations and in the radiative transfer equations, in order to take into account PRD effects within the theoretical approach described in Sect. 3, it is necessary to consider the next terms in the perturbative expansion of the atom-radiation interaction (i.e. second and higher order processes). Unfortunately, such a generalization of the theory has revealed to involve many difficulties, since the equations that come out when higher order terms are included become immediately extremely complex and practically unmanageable, unless further approximations are introduced.

Nevertheless, first important results have already been obtained. Under the simplifying assumption of a two-level atom with unpolarized lower level, and neglecting stimulation effects, Bommer (1997a) was able to perform all the calculations including higher order terms in Eq. (7), thus deriving a generalized expression for the radiative transfer coefficients. With respect to the expression obtained when only first-order processes are taken into account (valid under the complete redistribution approximation), the new emission coefficient contains an extra term (the so-called Rayleigh scattering term) which describes the contribution of coherent scattering. Remarkably, the redistribution matrix of Domke & Hubeny (1988) is recovered if the generalized emission coefficient derived by Bommer (1997a) is expressed in terms of a redistribution matrix:

\[ \epsilon_i(v_2, \tilde{\Omega}_2) = \gamma_i(0) v_i^4 \int dv_1 \int d\Omega_1 4\pi \sum_{j=0}^{3} R_{ij}(v_2, \tilde{\Omega}_2; v_1, \tilde{\Omega}_1) S_j(v_1, \tilde{\Omega}_1) \]

\[ \quad + \gamma_i(0) v_i^4 \epsilon B_F(v_0) \phi(v_0 - \nu) \delta_{i,0} , \tag{9} \]

with \( i = 0, 1, 2, 3 \) (corresponding respectively to the Stokes parameters \( I, Q, U, \) and \( V)\), \( \gamma_i(0) \) the line integrated absorption coefficient, \( \nu_0 \) the line frequency, and \( S_j \) \( (j = 0, \cdots, 3) \) the four Stokes parameters. The last term describes the contribution to the emitted radiation coming from collisionally excited atoms, the branching ratio being \( \epsilon = \frac{\Gamma_f}{\Gamma_{in}} \), with \( \Gamma_f \) and \( \Gamma_{in} \) the inelastic collisions de-excitation rate and the radiative de-excitation rate, respectively. The \( \delta_{i,0} \) shows that this contribution is unpolarized (collisions are assumed to be isotropic). The quantity \( B_F \) represents the Planck function in the Wien limit, while \( \phi(v_0 - \nu) \) is a normalized Lorentzian profile. The redistribution matrix is given by (see Bommer 1997b)

\[ R_{ij} = \sum_{K=0}^{2} W_K(J_i, J_o) \left[ \phi^{(K)}(\tilde{\Omega}_2; \tilde{\Omega}_1) \right]_{ij} \left[ \alpha r_{II}(v_2, v_1) + \left[ \beta^{(K)} - \alpha \right] r_{III}(v_2, v_1) \right] . \tag{10} \]

This expression is valid in the atomic rest frame, where, as can be observed, the frequency and angular dependencies can be factorized. The quantity \( W_K \) is a numerical factor introduced by Landi Degl’Innocenti (1984), the matrix \( \phi^{(K)} \) is the \( K \)-th multipole component of the Rayleigh phase scattering matrix, also introduced by Landi
Degl’ Innocenti (1984). The functions $r_{II}(v_2, v_1)$ and $r_{III}(v_2, v_1)$ are given by

$$r_{II}(v_2, v_1) = \phi(v_0 - v_1) \delta(v_2 - v_1), \tag{11}$$

$$r_{III}(v_2, v_1) = \phi(v_0 - v_1) \phi(v_2 - v_1), \tag{12}$$

where $\phi(v_0 - v_1)$ is a Lorentzian absorption profile, while $\delta(v_2 - v_1)$ is the Dirac delta. The first function describes the coherent contribution to the scattering, the second one the contribution in complete redistribution. The branching ratios are given by $\alpha$ and $[\beta^K - \alpha]$, respectively, with

$$\alpha = \frac{\Gamma_R}{\Gamma_R + \Gamma_I + \Gamma_E}, \quad \text{and} \quad \beta^K = \frac{\Gamma_R}{\Gamma_R + \Gamma_I + D^{K}}, \tag{13}$$

where $\Gamma_E$ is the elastic collisions rate (responsible for line-broadening and for destruction of frequency correlations between incoming and scattered photons), and $D^{K}$ is the $K$-multipole depolarizing rate due to elastic collisions (responsible for depolarization). The quantity $\alpha$ is the probability that a radiative decay takes place before any kind of collision: it thus represents the branching ratio for the scattering processes that are coherant in the atom rest frame. The quantity $\beta^K$, on the other hand, is the probability that a radiative decay occurs before an elastic depolarizing collision. The difference $[\beta^K - \alpha]$ thus gives the probability that the radiative decay takes place after the atom undergoes an elastic collision that redistributes the photon frequency, but that does not destroy atomic polarization: it thus represents the branching ratio for the scattering processes that are completely redistributed in frequency in the atom rest frame. It should be observed that a branching ratio term $(1 - \epsilon)$, taking into account the fraction of atoms that are radiatively excited, is already included in the branching ratios $\alpha$ and $[\beta^K - \alpha]$ (for a discussion on the physical meaning of the various quantities appearing in Eq. (10), see also Sampoorna et al. 2010). In this context, the CRD limit is obtained when the elastic collisions are so efficient that any frequency correlation between the incoming and outgoing photons is completely destroyed. Indeed, when $\Gamma_E >> \Gamma_R$ the coherent term disappears ($\alpha \rightarrow 0$), and the resulting redistribution matrix corresponds to the one which can be obtained within the theoretical framework described in Sect. 3 (valid under the flat spectrum approximation). As already observed in Sect. 3, when the elastic collisional rate is very high, depolarizing collisions are also very efficient, and polarization phenomena become negligible. For this reason, when the CRD limit due to high collisional rates is considered, it is customary to assume the rates $\Gamma_E$ and $D^{K}$ as independent, though they have the same physical origin. On the other hand, in the limit of no collisions ($\Gamma_E = D^{K} = 0$) the branching ratio $[\beta^K - \alpha]$ vanishes, and the scattering is completely coherent, as implied by energy conservation.

Within the same approach, Bommier (1997b) also investigated the effect of a magnetic field of arbitrary strength (which is found to mix angle and frequency dependencies also in the atom rest frame), as well as the transition to the laboratory frame.

Another interesting theoretical approach, based on the density matrix formalism, which can be applied to multi-level atoms, in the presence of arbitrary magnetic fields, and that does not require the flat spectrum approximation to be assumed, has been proposed by Landi Degl’ Innocenti et al. (1997). In this approach the density-matrix formalism is generalized to the framework of the so-called metalvels theory, in which the atomic levels are assumed to be composed by a continuous distribution of infinitely sharp sublevels. The formulation presented in Landi Degl’ Innocenti et al. (1997) does
not take collisions into account, and thus describes the limit of pure coherent scattering. This scheme has been applied by Landi Degl’Innocenti (1998) for the interpretation of the peculiar signal observed in the second solar spectrum of the sodium doublet.

More recently, “intermediate” theoretical schemes, exploiting the strength points both of the scattering approach, and of the density matrix formalism have been proposed. An interesting example is the approach proposed by Holzreuter et al. (2005) for the investigation of the physical origin of the triplet-peak structure shown by several signals of the second solar spectrum. Although these approaches are able to account for many fundamental physical aspects, in general they are not derived in a self-consistent way from basic physical principles. This makes extremely important, and in some cases also very difficult, to clearly establish their limits of applicability, which might often result to be fairly restrictive.

5. Conclusions

While until the end of the 1990s, the various branches of solar polarimetry, from the instrumentation to the observational techniques, from the theory of polarization to its solar application, were developed hand in hand, in the last ten years, as it can also be noticed from the references mentioned in this paper, progresses in the field of pure theory are proceeding significantly slower than in the other research branches.

In 1995, when the first “Solar Polarization Workshop” was organized, the need of developing theoretical tools suitable for describing the effects of frequency redistribution was already clear, and the problem of deriving a general analytical form of the redistribution function was strongly debated. Today, fifteen years later, our understanding of this physical aspect has substantially improved, different promising approaches have been proposed, and for the case of a two-level atom (with unpolarized lower level) the PRD problem seems to be solved. Nevertheless, the development of a self-consistent, general theory for radiative transfer in the presence of PRD effects (applicable to multilevels atoms, and accounting for the fundamental role of quantum coherences) remains since many years a fundamental unsolved problem. We are aware of the difficulties that the formulation of such a theory implies, and it is reasonable to expect that its development will proceed rather slowly, through a series of small, demanding steps. The strong possibility of not achieving novel and outstanding results in short or medium periods might explain why not many people (in particular among the new generations) decide to cope with this difficult challenge, definitely necessary for a new substantial acceleration in this research field.

The theory developed by Landi Degl’Innocenti at the beginning of the 1980s, and explained in great detail in LL04, keeps representing the most solid, advanced and complete theoretical approach to the physics of spectral line polarization developed so far. More than twenty years after its first development, after a countless number of successful applications, its soundness for the interpretation of all those signals which can be treated within the limit of complete frequency redistribution appears to be well established.

Acknowledgments. Financial support by the Spanish Ministry of Science through project AYA2010-18029 (Solar Magnetism and Astrophysical Spectropolarimetry) is gratefully acknowledged.
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