Hall MHD on Block-Adaptive Grids

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Abstract. This proceedings paper is based on a much longer research paper that was published during the conference (Tóth et al. 2008). We present a conservative second order accurate finite volume discretization of the magnetohydrodynamics equations including the Hall term on three dimensional block adaptive grids with Cartesian or generalized coordinates. Both explicit and implicit time integration schemes are developed. The parallel scaling and robustness are demonstrated by three dimensional simulations of planetary magnetospheres.

1. Introduction

Ideal or resistive magnetohydrodynamics (MHD) provides a fairly accurate physical model for a wide range of laboratory and space plasma problems. In certain applications, however, some of the MHD approximations become invalid, and the simulations deviate from physical reality. One option to better describe the physics is using full or hybrid particle schemes (see e.g. Birn et al. (2001)). While particle and hybrid schemes can successfully model these problems, these algorithms are very costly, especially for three-dimensional (and possibly time-dependent) simulations. An alternative approach is to maintain the fluid description of the plasma but allow for the decoupling of the ion and electron fluids.

The Hall term is part of the generalized Ohm’s law and it describes the relative speed between ions and electrons. The magnetic field lines are frozen in the electron fluid, but not in the ion fluid due to the Hall effect. The Hall effect is important at length scales shorter than the ion skin depth. Including the Hall term modifies the induction equation and the total energy density equation of MHD.

We develop a parallel, explicit and/or implicit 3D Hall MHD scheme and implement it into the BATSRUS code (Powell et al. 1998). BATSRUS solves the MHD equations with second-order shock-capturing total variation diminishing
(TVD) schemes on a block-adaptive grid. Each block has a simple logically Cartesian geometry. The grid is refined by dividing a block into 8 smaller blocks, while coarsening is achieved by merging 8 smaller blocks into a larger one. The BATSRUS code achieves excellent scaling on distributed memory parallel supercomputers both with explicit and with implicit time-stepping. The implicit scheme is based on the Jacobian-free Newton-Krylov-Schwarz (NKS) method and it can be efficiently combined with the explicit scheme on a block-by-block basis (Toth et al. 2006). We also allow for solving for the Hall term in a limited region by introducing a spatially varying coefficient for the Hall term.

2. Equations

In Hall MHD the electric field is expressed as

\[
\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J} + \frac{1}{ne} \mathbf{J} \times \mathbf{B}
\]

(1)

where \( \mathbf{v} \), \( \mathbf{B} \), and \( \mathbf{J} \) are the velocity, magnetic field and current vectors, respectively, \( \eta \) is the resistivity, \( n \) is the number density and \( e \) is the electron charge. The last term is the Hall term. Note that the magnetic units are chosen such that the magnetic permeability of vacuum \( \mu_0 \) is unity. For sake of simplicity we neglect the electron pressure gradient term \( \nabla p_e/(ne) \), although it can be discretized the same way as the Hall term \( \mathbf{J} \times \mathbf{B}/(ne) \). The electron inertia is also neglected.

The electric field enters the induction and energy equations thus the MHD equations including the Hall term become

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})
\]

(2)

\[
\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \cdot (\mathbf{v} \rho \mathbf{v} + \mathbf{I} p + \mathbf{I} \frac{\mathbf{B}^2}{2} - \mathbf{BB})
\]

(3)

\[
\frac{\partial e}{\partial t} = -\nabla \cdot [\mathbf{v}(\varepsilon + p) + (\mathbf{v} + \mathbf{v}_H) \cdot (\mathbf{IB}^2 - \mathbf{BB} - \mathbf{B} \times \eta \mathbf{J})
\]

(4)

\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times [- (\mathbf{v} + \mathbf{v}_H) \times \mathbf{B} + \eta \mathbf{J}]
\]

(5)

where \( \mathbf{I} \) is the identity matrix, \( \rho \) and \( p \) are the mass density and the thermal pressure. The Hall velocity is defined as

\[
\mathbf{v}_H = -\frac{\mathbf{J}}{ne} = -\frac{\nabla \times \mathbf{B}}{ne}
\]

(6)

while the total energy density is \( e = \varepsilon + \mathbf{B}^2/2 = p/(\gamma - 1) + \rho \mathbf{v}^2/2 + \mathbf{B}^2/2 \) where \( \varepsilon \) is the hydrodynamic energy and \( \gamma \) is the adiabatic index.

There are only a few extra terms relative to classical MHD in the induction and energy density equations all proportional to \( \mathbf{v}_H \). This seemingly small modification of the MHD equations is quite challenging to implement in a conservative, accurate and efficient manner. There are two challenges:
• There is a second order spatial derivative of the magnetic field in the induction equation that cannot be rewritten into a simple Laplace operator. This makes second order accurate discretization complicated at resolution changes.

• The maximum wave speed increases from the fast magnetosonic wave to the whistler wave speed that is approximately inversely proportional to the wave length.

The second problem is important, because the shortest wave length that can be represented on the grid is twice the grid resolution \( \Delta x \), so the fastest wave speed can be estimated as (Huba 2003)

\[
c_w = |v| + c_f + \frac{|B| \pi}{en \Delta x}
\]

where \( c_f \) is the classical MHD fast magnetosonic speed. The maximum stable time step for an explicit time integration scheme is

\[
\Delta t < \frac{\Delta x}{c_w} \propto \Delta x^2
\]

for small enough \( \Delta x \). This can make the explicit time integration algorithms rather inefficient as the grid resolution is increased. An implicit scheme can potentially overcome this problem, since the time step is not limited by the CFL condition.

An additional difficulty arises, since the numerical dissipation required for the stability of the explicit numerical scheme is also proportional to the fastest wave speed of the hyperbolic system of equations, and this can reduce the spatial order of accuracy by one order. To see this, let us consider the usually first order accurate local Lax-Friedrichs scheme

\[
\frac{U^{n+1} - U^n}{\Delta t} = -\frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x} + \frac{c_{\text{max}}^{i+1/2}(U_{i+1}^n - U_i^n) - c_{\text{max}}^{i-1/2}(U_i^n - U_{i-1}^n)}{2\Delta x}
\]

where \( U \) is one of the conservative variables, \( F \) is the corresponding physical flux function, and \( c_{\text{max}} \) is the fastest wave speed. The superscripts denote the time level and the subscripts correspond to the spatial grid indexes. The index \( i + 1/2 \) identifies the cell face between the cell centers \( i \) and \( i + 1 \). The second term on the right hand side is the numerical dissipation and to lowest order in \( \Delta x \) it is proportional to

\[
\Delta x \frac{\partial}{\partial x} \left( c_{\text{max}} \frac{\partial U}{\partial x} \right)
\]

Since \( c_{\text{max}} \propto 1/\Delta x \) the local Lax-Friedrichs scheme is inconsistent for the Hall MHD equation.

In the following section we will describe our solutions to these challenging problems.
3. Numerical Discretization

A second order conservative scheme requires a second order accurate calculation of the physical fluxes at the center of the cell faces. Since the flux for the magnetic field $B$ contains the Hall term $J \times B / ne$ and $J = \nabla \times B$, we have to calculate the current with second order accuracy at the cell faces. For the $x$ face of a uniform Cartesian grid one can use simple central differencing and averaging as needed:

$$
J_{i+1/2,j,k}^x = \frac{B_{i,j+1,k}^z + B_{i+1,j+1,k}^z - B_{i,j-1,k}^z - B_{i+1,j-1,k}^z}{4\Delta y} - \frac{B_{i,j,k+1}^y + B_{i+1,j,k+1}^y - B_{i,j,k-1}^y - B_{i+1,j,k-1}^y}{4\Delta z} - \frac{B_{i+1,j,k}^z - B_{i,j,k}^z}{\Delta x}
$$

$$
J_{i+1/2,j,k}^y = \frac{B_{i+1,j,k+1}^x + B_{i+1,j,k+1}^x - B_{i,j,k-1}^x - B_{i+1,j,k-1}^x}{4\Delta z} - \frac{B_{i,j+1,k}^y - B_{i,j,k}^y}{\Delta x}
$$

$$
J_{i+1/2,j,k}^z = \frac{B_{i,j+1,k}^x + B_{i+1,j+1,k}^x - B_{i,j-1,k}^x - B_{i+1,j-1,k}^x}{4\Delta y} - \frac{B_{i,j,k+1}^y - B_{i,j,k}^y}{\Delta x}
$$

and similar formulas apply to the other faces. Note that the normal derivatives use the two closest cell centers only, while the tangential derivatives require 4 cell centers and averaging in the $i$ direction. Consequently one needs to use the first layer of ghost cells not only at the faces between the blocks but also along the 12 edges of the block.

The second order accurate central difference formulas (11) used to calculate the curl of $B$ on a uniform grid can be used at resolution changes as well if the ghost cells are filled in with at least 3rd order accurate magnetic field. An alternative is to use a one sided derivative. For example the $y$ gradient of $B^z$ at the cell center $j$ can be approximated as

$$
\left( \frac{\partial B^z}{\partial y} \right)_j = -\frac{3B^z_j + 4B^z_{j+1} - B^z_{j+2}}{2\Delta y}
$$

We use this latter approach to avoid the use of ghost cells in the tangential (relative to the face) derivatives on the coarse side of resolution changes. On the fine side we take the first approach, i.e. interpolate the magnetic field to the fine ghost cells with third order accuracy. This is not a unique choice, there are many other possible discretizations.

We use the Newton-Krylov-Schwarz (NKS) approach to formulate the implicit time integration scheme for Hall MHD. Thanks to the general implementation of the implicit scheme in the BATSRUS code (Toth et al. 2006), once the explicit Hall MHD scheme is implemented, the implicit time integration works as well. To make it efficient, however, the preconditioner needs to take into account the Hall term. This is not a simple task. For sake of simplicity and
efficiency the preconditioner is based on the Jacobian that uses the nearest 6 neighbor cells only on a 3D grid. Due to the higher order derivatives, the stencil of the Hall term is much more extended. We found that the preconditioner becomes much more effective if the Hall term is discretized on a stencil that is as compact as possible. In particular, when the current is calculated for the $X$ face at $i+1/2, j, k$, all $x$ derivatives should use only the $i+1$ and $i$ cell centers only.

4. Conclusions

We have developed a second order accurate Hall MHD scheme for block adaptive (Stout et al. 1997) Cartesian or general structured grids using both explicit and implicit time integration. For steady state solutions the explicit scheme can be used in combination with local time stepping. Care should be taken to use symmetric type limiters (like MC) instead of asymmetric limiters (like minmod or super-bee) to achieve second order accuracy in smooth regions. For time accurate runs the implicit scheme is much more efficient than the explicit scheme if the whistler wave speed is dominant. The preconditioner has to take into account the terms responsible for the whistler wave. Using the implicit scheme the Hall MHD simulation ran about 3-4 times slower than the classical MHD simulation, which is quite reasonable given the stiffness of the Hall MHD equations due to the whistler wave.

Figure 1. Scaling of the explicit, implicit and explicit/implicit time stepping schemes for a fixed size magnetosphere problem.
We present strong scaling results on an SGI Altix super computer (columbia at NASA Ames) for a time-dependent 3D magnetosphere simulation problem. Figure 1 shows the scalings for explicit, implicit and explicit/implicit time stepping, respectively. For the explicit scheme the stability condition required $\Delta t = 0.00295 \text{s}$ time steps, while the time step for the implicit scheme was $1 \text{s}$. The figure demonstrates a near perfect scaling of the explicit scheme from 32 to 1616 processors, which is quite remarkable for strong scaling with a fixed problem size. Note that there are only 3 blocks per processor when the code runs on 1616 processors.

The fully implicit and explicit/implicit algorithms are about 14 and 20 times faster on 32 CPU-s than the explicit scheme, respectively. The implicit time step is about 330 times larger than the explicit time step, and the implicit scheme requires about 25 Krylov iterations per step. As the figure shows the implicit schemes scale quite well up to 256 processors. In the explicit/implicit scheme most of the blocks (about 3400) require implicit time steps for the selected time step, so there is relatively little difference between the efficiency of the implicit and explicit/implicit algorithms.

The efficiency and good parallel scaling of our Hall MHD scheme enables us to do steady state and time accurate simulations in 3D. We have already used the Hall MHD code to simulate Titan’s interaction with the surrounding plasma (Ma et al. 2007). This steady state simulation uses a spherical grid with logarithmic stretching in the radial direction. As shown by Ma et al. (2007) the Hall MHD results match the values measured by the Cassini satellite significantly better than the results obtained with classical MHD simulations. We plan to use the Hall MHD code to study many space physics problems in the future.

References


