Polarimetry from the Ground Up

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Abstract. Ground-based solar polarimetry has made great progress over the last decade. Nevertheless, polarimetry is still an afterthought in most telescope and instrument designs, and most polarimeters are designed based on experience and rules of thumb rather than using more formal systems engineering approaches as is common in standard optical design efforts. Here we present the first steps in creating a set of systems engineering approaches to the design of polarimeters that makes sure that the final telescope-instrument-polarimeter system is more than the sum of its parts.

1. Systems Engineering for Polarimetry

Systems engineering ensures that the total is more than the sum of its parts. Systems engineering is essential for the successful design of polarimeters where apparently unrelated effects such as instrumental polarization due to oblique reflections in the telescope and detector non-linearity couple to generate substantial systematic errors (Keller 1996). When designing a polarimeter, it is therefore crucial to optimize the entire system and not just the individual parts as has often been done in the past.

While systems engineering is common in standard optics, it has been largely absent in polarimetry because errors in polarization cannot easily be expressed as a scalar and optical elements such as polarizers and retarders have a major influence on the polarization. In the following we will present initial thoughts on establishing systems engineering approaches for polarimetry that will enable better polarimeter designs with predictable performance.

Systems engineering for polarimetry models and helps us understand the performance of polarimeter designs. It must address at least the following issues:

- a definition of the polarimeter performance to quantitatively compare different polarimeter designs with ideal components;
- a polarization error budget that can predict the performance based on known error statistics of real components;
- methods to maximize the performance of a polarimeter design.

While the first issue has been addressed in the literature, error budgets, a crucial systems engineering tool, have been lacking. In the following sections we present some initial ideas on systems engineering approaches for polarimeter designs.
2. Errors

Errors in polarization measurements fall into two classes: statistical (random) errors and systematic (instrumental) errors. Systems engineering must provide approaches to quantify and balance these two, very different, error sources.

2.1. Statistical Errors in Polarization Measurements

The influence of statistical errors on the final polarization measurement can be calculated using error propagation. These calculations become relatively simple if the following assumptions hold:

- there is a linear relation between the Stokes parameters of the incoming light and the signals that are measured;
- the noise does not depend on the position of a measurement in a sequence;
- the noise is independent of the signal, which is the case if
  - the noise is dominated by signal-independent detector noise (e.g. readout noise) or
  - the light that is analyzed is only slightly polarized;
- the noise statistic has a Gaussian distribution.

For the large number of photons, which are needed for accurate polarimetry, a Gaussian distribution is a good approximation to the Poisson distribution of photon noise. If the degree of polarization of the incoming beam is small, the measurements will all have very similar light levels, thereby justifying the assumption of independence of noise and signal.

2.2. Signal Matrix

The intensities measured by the detector can be combined into a signal vector \( \mathbf{S} \), which is related to the incoming Stokes vector, \( \mathbf{I} \) by the signal matrix \( \mathbf{X} \) (sometimes also called the synthesis matrix, e.g. Tyo 2002),

\[
\mathbf{S} = \mathbf{X} \mathbf{I}.
\]  

\( \mathbf{X} \) is a 4 by \( m \) matrix, where \( m \) is the number of intensity measurements that contribute to the polarization measurement. For example, \( m = 4 \) for most systems that use liquid crystals, while \( m = 8 \) for a rotating retarder. \( \mathbf{X} \) is a function of the free parameters of the polarimeter design. Since the polarimeter optics can be described by Mueller matrices, each row of \( \mathbf{X} \) corresponds to the first row of the Mueller matrix describing the particular intensity measurement as a function of the incoming Stokes vector.

To determine the Stokes vector \( \mathbf{I} \) from the measurements \( \mathbf{S} \), \( \mathbf{X} \) needs to be inverted. With

\[
\mathbf{Y} = \mathbf{X}^{-1}
\]  

the standard deviations of the Stokes parameters, \( \sigma_{I_i} \), are given by

\[
\sigma_{I_i} = \sqrt{\sum_{j=1}^{m} Y_{ij}^2 \sigma_{S_j}^2},
\]
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where $\sigma_{S_j}$ is the standard deviation of the intensity in measurement $j$. In many cases, the latter does not depend on the measurement number $j$ and one can rewrite the equation as

$$\sigma_{I_i} = \sigma_S \sqrt{\sum_{j=1}^{m} Y_{ij}^2}.$$  \hfill (4)

These errors can then subsequently be propagated into errors in the degree of circular, linear polarization and angle of linear polarization or polarization ellipse parameters, and then into, e.g., solar magnetic field parameters. The last step is important because the relations between Stokes parameters and magnetic field parameters or other physical quantities of interest are not linear, and one should optimize a polarimeter design according to what one wants to measure.

### 2.3. Systematic or Instrumental Errors

Because of the large photon flux from the Sun, solar polarimeters are largely limited by systematic (instrumental) errors rather than by statistical errors. A list and discussion of instrumental error sources can be found in Keller (2002). Our goal here is not to understand the individual error sources but to discuss ways to ‘add’ their influence such that we can predict the performance of a polarimeter under non-ideal conditions.

### 3. Polarimetric Efficiency

If one is able to translate the science requirements into requirements for the measurement of the Stokes parameters, then the concept of a **polarimetric efficiency** is useful. Any definition of the polarimetric efficiency must have the following properties:

- comparable between different polarimeter designs and measurement approaches;
- larger values should correspond to better designs;
- independent of the intensity throughput;
- consist of 4 quantities (‘Stokes efficiency’);
- the theoretical maximum efficiency shall be 1.

The polarimetric efficiency with which the component $i$ of the Stokes vector $(I, Q, U, V)^T$ is measured was defined by del Toro Iniesta and Collados (2000) as

$$\epsilon_i = \left( \frac{m}{m \sum_{j=1}^{m} Y_{ij}^2} \right)^{-\frac{1}{2}}.$$  \hfill (5)

Note that this definition is independent of the number of measurements that contribute to the measurement and fulfills all the requirements listed above if $X_{ii} = 1$. 
Since the polarimetric efficiency is independent of the intensity throughput, it is important to not directly compare polarimetric efficiencies of different designs but to also properly take into account the throughput of the corresponding design. For instance, a polarizing beam-splitter makes use of all photons, while a regular linear polarizer will not transmit more than 50% of all photons.

### 3.1. Analytic Optimization

When optimizing the design of a polarimeter, it is convenient to define a scalar function of the free design parameters such that the maximum or minimum of this merit function corresponds to the optimum polarimeter design. The polarimetric efficiency provides a well-defined merit function for optimizing a polarimeter design, in particular when one demands a fixed ratio (often unity) between the efficiencies for the different polarized Stokes parameters. Various merit functions can be found in the literature, but they are all closely related to the polarimetric efficiency.

For certain cases of merit functions, one can derive equations for the properties of the optimum polarimeter (e.g. del Toro Iniesta and Collados, 2000; Tyo 2002). In practice, one may often use numerical optimization schemes, but the analytic approach does reveal some of the basic properties of an optimum polarimeter design and allows us to derive the maximum performance. The maximum performance is the best performance of all polarimeter designs, while the optimum performance is the best performance that can be achieved with a given polarimeter design. For a polarimeter with maximum performance, $\epsilon_I = 1$ and $\epsilon_Q + \epsilon_U + \epsilon_V = 1$.

We extend Eq. 1 into the basic measurement equation

$$S = X(v)I + n.$$  \hspace{1cm} (6)

by explicitly showing the dependence on the free design parameters $v$ and the addition of random, zero-mean noise $n$. A minimum of $m = 4$ measurements is required to determine all four Stokes parameters.

To determine an estimate $I'$ of the Stokes vector $I$ from the measurements $S, X$ needs to be inverted. If $Y$ (sometimes called the analysis matrix or demodulation matrix) is the inverse of $X$, we obtain

$$I' = YS = Y(X(v)I + n) = YX(v)I + Yn.$$ \hspace{1cm} (7)

Our goal is to choose $X(v)$ and $Y$ such as to minimize the difference between $I'$ and $I$ given the standard deviations $\sigma_{S_j}$ of the measurement errors $n_j$. We tackle this problem in three steps: 1) derive an equation for the optimum $Y$ for a given $X(v)$; 2) derive an optimum $X$ for the optimum $Y$; and 3) choose optimum $X$ and $Y$ to obtain a polarimeter with maximum performance.

**Optimum Synthesis Matrix** For a given $X(v)$, $Y$ is not necessarily unique. It is obvious that apart from fulfilling $YY = 1$, $Y$ should minimize $Yn$, given the standard deviations $\sigma_j$. This can be written as a minimization problem, the solution of which is given by the generalized inverse (e.g. Albert 1972)

$$Y = (X^TX)^{-1}X^T.$$ \hspace{1cm} (8)
Among all possible \( Y \) that fulfill \( YX = 1 \), the generalized inverse minimizes the sum of squares of its rows. Therefore, the generalized inverse is the optimum synthesis matrix \( Y \), given a signal matrix \( X \). Since \( X^T X \) is symmetric and positive definite, it can be inverted with standard matrix inversion algorithms.

Realizing that the sum of squares of the rows of \( Y \) are the diagonal elements of \( YY^T \), the optimum polarimetric efficiencies for a given signal matrix \( X \) are derived (after some linear algebra) as

\[
\epsilon_{\text{opt},i} = \sqrt{\frac{1}{m} (X^T X)^{-1}_{ii}}. \tag{9}
\]

### Signal Matrix for Maximum Performance

We now derive the properties of the signal matrix \( X \) that provides maximum performance, i.e. minimizes the sum of squares of the rows of \( Y \) under the condition that \( XY = 1 \). Using Eq. 5, del Toro Iniesta and Collados (2000) showed that the squares of the maximum possible polarimetric efficiencies are given by

\[
\epsilon_{\text{max},i}^2 = \frac{1}{m} \sum_{j=1}^{m} X_{ji}^2 = \frac{1}{m} (X^T X)_{ii}. \tag{10}
\]

Hence the maximum possible efficiency is given by the sum of squares of the elements of the columns of the signal matrix \( X \), normalized with the number of measurements.

For \( YX = 1 \) to hold (remember that we have only considered the diagonal elements of this relation so far), we must also require that

\[
\sum_{j=1}^{m} Y_{ij} X_{jk} = \delta_{ik}, \tag{11}
\]

where \( \delta_{ik} \) is equal to 0 unless \( i = k \) where it takes on the value 1. Inserting

\[
Y_{ij} = \frac{X_{ji}}{\sum_{k=1}^{m} X_{ki}^2}. \tag{12}
\]

into the above equation and multiplying both sides of the equation with the denominator of the left side yields

\[
\sum_{j=1}^{m} X_{ji} X_{jk} = \delta_{ik} \sum_{j=1}^{m} X_{jk}^2. \tag{13}
\]

For \( i = k \), the equation is obviously correct. For \( i \neq k \), the right side is zero and we conclude that \( X^T X \) has to be diagonal for a polarimeter to achieve the maximum performance.

A polarimeter that achieves its maximum possible polarimetric efficiency therefore has a signal matrix \( X \) such that

\[
X^T X = m \begin{pmatrix}
\epsilon_{\text{max},1}^2 & 0 & 0 & 0 \\
0 & \epsilon_{\text{max},2}^2 & 0 & 0 \\
0 & 0 & \epsilon_{\text{max},3}^2 & 0 \\
0 & 0 & 0 & \epsilon_{\text{max},4}^2
\end{pmatrix}. \tag{14}
\]
Generation of Maximum Performance Signal Matrices  
We now go beyond the work of del Toro Iniesta and Collados (2000) to provide more insight into the properties of the signal matrix $X$ for a polarimeter with maximum efficiency. Since each row of the signal matrix $X$ is the first row of a Mueller matrix, it obeys the inequality
\[ X_{i1}^2 \geq \sum_{j=2}^{4} X_{ij}^2 , \quad (15) \]
where the equal sign applies if there are no depolarizing elements between the source and the detector. This is a direct consequence of the properties of a Mueller matrix.

The optimum polarimeter has an intensity efficiency of $\epsilon_1 = 1$. The other three efficiencies have to obey $\sum_{i=2}^{4} \epsilon_i^2 \leq 1$. This implies that the optimum efficiency for measuring polarized Stokes components with equal efficiencies is given by $\frac{1}{\sqrt{n}}$ where $n$ is the number of polarized Stokes parameters that are measured. If all Stokes parameters are measured, the maximum efficiency is $\frac{1}{\sqrt{3}} \approx 0.577$, if only two are measured, the maximum efficiency is $\frac{1}{\sqrt{2}} \approx 0.707$, and for a single polarized Stokes parameter, the maximum efficiency is obviously 1.

In a polarimeter with maximum efficiency, the elements of the first column of the signal matrix $X$ are equal to 1. Since the columns of the signal matrix need to be orthogonal, and the scalar product of the first column with any of the other columns corresponds to the sum over all elements of the other product, we conclude that
\[ \sum_{j=1}^{m} X_{jk} = 0, \quad k = 2...4 . \quad (16) \]

For a polarimeter reaching maximum efficiency, there will be no depolarizing elements, and we have
\[ \sum_{j=1}^{4} X_{jk}^2 = 1, \quad j = 1...m , \quad (17) \]
since the rows of $X$ correspond to the first rows of Mueller matrices. We can thus consider each row of the signal matrix as a point on the Poincaré sphere. In the following, we will show that by maximizing the average distance squared between these points (on the Poincaré sphere), we obtain a signal matrix that provides maximum polarimetric efficiency.

To maximize the separation between points on the Poincaré sphere, we maximize the $m$ functions
\[ \sum_{j \neq i}^{m} \sum_{k=2}^{4} (X_{ik} - X_{jk})^2 - \alpha_i \left( \sum_{k=2}^{4} X_{ik}^2 - 1 \right) , \quad (18) \]
where $\alpha_i$ are Lagrange multipliers. These equations are valid independent of the number of polarized components that are considered since the elements $X_{jk}$ are zero for polarized components of the Stokes vector that we do not want to measure.
By setting the derivatives of this function with respect to $X_{ik}$ to zero, we obtain

$$(m - \alpha_i) X_{ik} = \sum_{j=1}^{m} X_{jk}.$$  \hspace{1cm} (19)

By squaring both sides of this equation, summing over $k$, and remembering that $\sum_{k=2}^{4} X_{ik}^2 = 1$, we obtain

$$(m - \alpha_i)^2 = \sum_{k=2}^{4} \left( \sum_{j=1}^{m} X_{jk} \right)^2.$$  \hspace{1cm} (20)

Since the right side is independent of $i$, we conclude that all Lagrange multipliers take on the same value, i.e. $\alpha_i = \alpha$. Using this relation in Eq. 19 and summing over all $i$, we obtain

$$(m - \alpha) \sum_{i=1}^{m} X_{ik} = m \sum_{j=1}^{m} X_{jk}.$$  \hspace{1cm} (21)

Because $\alpha = 0$ is not a valid Lagrange multiplier, the only way to fulfill this equation is the requirement that the sums of columns of $X$ for $k = 2...4$ vanish, i.e.

$$\sum_{j=1}^{m} X_{jk} = 0.$$  \hspace{1cm} (22)

In other words, the points have to be distributed on the Poincaré sphere in such a way that their center of gravity is always at the origin of the sphere. This is the same requirement as for a polarimeter with maximum efficiency.

We conclude with calculating the distance $\Delta$ between points on the Poincaré sphere. The average distance squared between one point and all the other points is given by

$$\Delta^2 = \frac{1}{m - 1} \sum_{j \neq i}^{m} \sum_{k=2}^{4} (X_{ik} - X_{jk})^2,$$  \hspace{1cm} (23)

which reduces to

$$\Delta^2 = \frac{2(m - 1) + 2}{m - 1}.$$  \hspace{1cm} (24)

We finally obtain

$$\Delta = \sqrt{\frac{2 + \frac{2}{m - 1}}{m - 1}}.$$  \hspace{1cm} (25)

For $m = 2$ we obtain $\Delta = 2$, which corresponds to opposite sides of the Poincaré sphere. For $m = 3$ we obtain $\Delta = \sqrt{3}$, which corresponds to the side length of a triangle whose plane includes the origin of the Poincaré sphere. For $m = 4$ we obtain $\Delta = \sqrt[3]{8}$, which corresponds to the side length of a tetrahedron inside the Poincaré sphere. Note, that $\Delta$ only corresponds to the distance between points if the distance between all points is the same, which is the case for these examples. It should not be surprising that this average distance is independent
of the number of polarized Stokes components that are considered, since at no point in our derivation did we make any assumptions about the number of components that we want to measure.

Unfortunately, maximizing the average square distance of points is a necessary but not a sufficient criteria for generating a signal matrix corresponding to a polarimeter with maximum performance. The following signal matrix

\[ X = \begin{pmatrix} 1 & x & x & x \\ 1 & x & x & x \\ 1 & -x & -x & -x \\ 1 & -x & -x & -x \end{pmatrix} \]  

(26)

also has an average distance of \( \sqrt{\frac{8}{3}} \), but \( X^T X \) is obviously not diagonal.

4. Optimum Calibration

Once a polarization analysis system has been designed, we need to determine the optimum way to calibrate it, i.e. to experimentally determine the signal matrix \( X \). As it turns out, much of the previously derived results can also be applied to find the optimum calibration approach, i.e. based on the measured signal matrix \( X \), we need to determine the optimum matrix \( Y \). To determine all 16 elements of \( X \), we need to make (at least) 16 measurements of the signal \( S_i \), \( i = 1..m \), which corresponds to (at least) 4 different input Stokes vectors \( I_{ci} \), \( i = 1..m \) with (hopefully) known properties.

Based on the approach by Azzam, Elminyawi, & El-Saba (1988), we group the 4 calibration input Stokes vectors \( I_{ci} \) and the corresponding signal vectors \( S_i \) into 4 by \( m \) matrices. We can then write

\[ S = XI_c, \]  

(27)

with

\[ S = (S_1 S_2 ... S_m), \]  

(28)

\[ I_c = (I_{c1} I_{c2} ... I_{cm}). \]  

(29)

The signal matrix \( X \) is given by \( X = SJ \) with \( J I_c = 1 \). We need to choose \( I_c \) such as to minimize the error in \( X \), given errors in the measurements \( S \). This is the same problem that we faced when optimizing the polarimetric efficiency, and we can indeed apply the same reasoning. The calibration input Stokes vectors \( I_{ci} \) for maximum calibration accuracy should therefore obey the same relations as the rows of the signal matrix \( X \) of a polarimeter with maximum performance.

For a polarimeter that only measures one polarized component (e.g. a circular polarimeter), we would use the two corresponding orthogonal polarization states. For a system that measures two polarized components (e.g. a linear polarimeter), we would use three calibration Stokes vectors whose points are the corners of an equilateral triangle. For the linear polarimeter, we could use a rotating linear polarizer positioned 60° apart. For a vector-polarimeter, we would choose the corners of a tetrahedron as suggested by Azzam et al. (1988).

In practice, however, one has to take many more measurements so that the properties of the non-ideal polarization calibration optics can also be determined.
5. Polarization Error Budget

5.1. Classical Error Budgets

Error budgets are a classical tool in systems engineering to derive requirements for the individual parts of a system such that the system as a whole meets the requirements while minimizing the total complexity and/or cost. A typical example is the wavefront aberration in an optical system that contains many optical components. An academic example is shown in Tab. 1.

Table 1. Error budget example for the optical quality (80% encircled energy in arcsec) of a ground-based telescope with an instrument.

<table>
<thead>
<tr>
<th>level 1 item</th>
<th>level 2 item</th>
<th>level 3 item</th>
<th>level 1</th>
<th>level 2</th>
<th>level 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>atmosphere</td>
<td></td>
<td></td>
<td>0.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>telescope</td>
<td></td>
<td></td>
<td>0.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>primary mirror</td>
<td></td>
<td></td>
<td></td>
<td>0.17</td>
<td></td>
</tr>
<tr>
<td>mirror polishing</td>
<td></td>
<td></td>
<td></td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>mirror support</td>
<td></td>
<td></td>
<td></td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>thermal distortion</td>
<td></td>
<td></td>
<td></td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>secondary mirror</td>
<td></td>
<td></td>
<td></td>
<td>0.17</td>
<td></td>
</tr>
<tr>
<td>mirror polishing</td>
<td></td>
<td></td>
<td></td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>mirror support</td>
<td></td>
<td></td>
<td></td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>thermal distortion</td>
<td></td>
<td></td>
<td></td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>instrument</td>
<td></td>
<td></td>
<td></td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td></td>
<td></td>
<td></td>
<td>0.61</td>
<td></td>
</tr>
</tbody>
</table>

An error budget can have several levels since parts can again be looked at as a combination of smaller parts. Since the overall error is estimated from the combination of many sources, mistakes in the estimates of errors of individual components tend to average out. Furthermore, only the overall error has a requirement attached to it, and the individual errors of each level of the error budget can be allocated in different ways. This error allocation is an iterative process where one tends to minimize the complexity and cost of the system while focusing on the main contributors to the system error, which are quickly identifiable in the error budget.

For an error budget to make sense, one needs to figure out how to add errors. In the case of aberrations of an optical system, the wavefront errors of individual elements are not correlated, and one can simply assume that the final error is given by the square root of the sum of errors squared of the individual elements, the so-called root sum of squares (RSS). For a polarimeter, the issue is much more complicated since 1) polarization is a vector quantity and not a scalar; 2) retarders and polarizers affect the polarization in a major way and not in a minor way as required for RSS to make sense; and 3) errors of very different nature are combined non-linearly, such as the instrumental polarization of the optical system that feeds the polarimeter and the non-linearity of the light detector system (Keller 1996).
Another difficulty comes from the fact that a polarimeter is, in most cases, calibrated experimentally. This means that some (but not all) errors can be drastically reduced, and one should only include the error that remain after the calibration. This is somewhat analogous to wavefront aberration error budgets for optics that include active and/or adaptive optical elements.

Table 2. Schematic polarization error budget for a polarimeter.

<table>
<thead>
<tr>
<th>Level 1 item</th>
<th>Level 2 item</th>
<th>Level 3 item</th>
</tr>
</thead>
<tbody>
<tr>
<td>source variation</td>
<td>atmosphere</td>
<td>telescope</td>
</tr>
<tr>
<td>polarizer</td>
<td>detector system</td>
<td>undetected bias variation</td>
</tr>
<tr>
<td>calibration</td>
<td>polarizer</td>
<td>retarder</td>
</tr>
<tr>
<td>data reduction</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Because of all these difficulties, polarization error budgets have historically not been used in the design of polarimeters. First attempts have been described by Boger et al. (2003), but this does not go further than a list of potential errors. Table 2 shows several potential error sources in polarimeters, separated into three levels.

5.2. Errors in Mueller Matrices

If an optical element in a polarimeter can be described by a Mueller matrix, then any small error associated with this element alone can be approximated by a linear change in the Mueller matrix elements. For example, a Mueller matrix $M(\alpha, \beta)$ describes an optical element with two parameters $\alpha$ and $\beta$, e.g. a retarder with retardation and fast axis orientation. A Taylor approximation for the Mueller matrix with scalar errors $\delta\alpha$ and $\delta\beta$ in the respective scalar parameters $\alpha$ and $\beta$ can be written as

$$M(\alpha + \delta\alpha, \beta + \delta\beta) \approx M(\alpha, \beta) + m_\alpha \cdot \delta\alpha + m_\beta \cdot \delta\beta$$  \hspace{1cm} (30)

For some parameters of some elements it can be that the first-order Taylor expansion is insufficient and that the second order has to be considered. For simplicity, we will assume in the following that the first-order expansion is adequate. Instead of working with specific errors $\delta\alpha$ and $\delta\beta$, we need to look at Eq. 30 in a statistical sense such that $\delta\alpha$ and $\delta\beta$ correspond to characteristic values of the magnitude of the expected error, e.g. the standard deviation for a zero-mean Gaussian distribution or the extreme values of a uniform distribution.
The matrices $m_{\alpha,\beta}$ can then be interpreted as normalized standard deviations of the elements of the Mueller matrix $M$.

For a uniform distribution of errors in $\alpha$ over $\pm \Delta$ we can calculate the variance of the Mueller matrix with respect to errors in $\alpha$ as

$$M^2_\alpha (\Delta) = \int_{-\Delta}^{+\Delta} (M(\alpha + \epsilon, \beta) - M(\alpha, \beta))^2 \delta \epsilon ,$$

(31)

where the $^2$ does not indicate a matrix multiplication but the square of the individual matrix elements. The normalized standard deviation then becomes

$$m_\alpha = \lim_{\Delta \to 0} \frac{\partial}{\partial \Delta} M_\alpha (\Delta) .$$

(32)

Hence, the Mueller matrix with errors can be written as

$$M(\alpha, \beta) \pm m_\alpha \cdot \delta \alpha \pm m_\beta \cdot \delta \beta .$$

(33)

As an example, we show a linear retarder with fast axis angle $\theta$ and retardance $\phi$. The corresponding Mueller matrix is

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos^2(2\theta) + \cos(\phi) \sin^2(2\theta) & 0 & \cos(2\theta) \sin(\phi) \\
0 & \cos(2\theta) \sin(2\theta) - \cos(2\theta) \cos(\phi) \sin(2\theta) & \cos(2\theta) \cos(\phi) \sin(\phi) + \sin^2(2\phi) & 0 \\
0 & -\sin(2\phi) \sin(\phi) & 0 & \cos(2\theta) \cos(\phi) \sin(\phi) - \sin(2\theta) \cos(\phi) \\
\end{pmatrix} .
$$

(34)

For a uniform error distribution in retardance $\phi$ we obtain the normalized standard deviation matrix with respect to $\phi$ as

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sqrt{\sin^4(2\theta) \sin^2(\phi)} & 0 & \sqrt{\cos^4(2\theta) \sin^2(\phi)} \\
0 & \sqrt{\sin^2(2\theta) \sin^2(\phi)} & \sqrt{\cos^2(2\theta) \sin^2(\phi)} & 0 \\
0 & \sqrt{\cos^2(\phi) \sin^2(2\theta)} & \sqrt{\cos^2(\phi) \cos^2(2\theta)} & \sqrt{\sin^2(\phi)} \\
\end{pmatrix} .
$$

(35)

### 5.3. Statistical Distribution of Parameter Errors

In the best case scenario, the statistical distribution for the random errors is known from a large number of components with identical requirements. However, this is often not the case and one has to make assumptions. While a Gaussian distribution might be a natural choice for certain alignment errors, it is often too optimistic an assumption, in particular for manufacturing errors. When components are manufactured, the manufacturer often stops the processing once the component meets the specifications. It is therefore likely that the component will have properties close to the maximum errors, which is a very different distribution from a normal distribution where the parameter is most likely to be at the specified value. A uniform distribution of the errors is therefore often a better choice.
5.4. Combining Mueller Matrix Errors

Stenflo (1994) showed that weakly polarizing Mueller matrices $M_i$ can be written as $E + m_i$, where $E$ is the unity matrix and $m_{i,jk} \ll 1$ and that the product of such matrices can be approximated by their sum. Therefore, RSS can be applied to weakly polarizing and retarding elements.

This result can be generalized to strongly polarizing and retarding elements written as $M_i + m_i$ and again $m_{i,jk} \ll 1$. However, $M_i$ is not the unity matrix, and all matrix elements $M_{i,jk}$ can be of order unity. The product of such matrices can then be approximated according to

$$\prod_{i=1}^{n} (M_i + m_i) \approx \prod_{i=1}^{n} M_i + \sum_{i=1}^{n} \left( \prod_{j=1}^{i-1} M_j \right) m_i \left( \prod_{j=i+1}^{n} M_j \right)$$ (36)

The resulting product becomes the sum of the product of error-free matrices and all the Mueller matrix errors transformed using the ideal Mueller matrices of elements before and after the current element. Because the transformed errors are additive, we can RSS these transformed Mueller matrix errors and use a classic error budget approach to estimate the contribution of individual errors to the overall system.

6. Outlook

To make these tools useful, systematic errors that cannot be expressed in terms of Mueller matrices must be included, and calibration and data reduction errors must be added. A library of normalized standard deviation matrices for common polarimetric components and their respective parameters must be collected.

The polarimetry error budget described here neglects the coupled effect of simultaneous errors in all design parameters. A Monte Carlo simulation avoids this drawback by considering simultaneous errors. However, such end-to-end simulations require substantial efforts and do not provide a direct insight into the main error contributors and how their effects can be balanced.

Only the application of these proposed techniques to several real instruments will show whether this is indeed a useful tool for designing polarimeters.

References

Tyo, J. S., 2002, Appl.Optics, 41, 619