Phase Matrices for Higher Multipoles of Scattering in External Magnetic Fields

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Abstract. Scattering phase matrices for forbidden lines are derived, in the presence of an external magnetic field, using the quantum electrodynamical approach. The particular case of 0 → 2 → 0 M2 transitions are considered and Stokes profiles are shown in the strong field (Zeeman) and weak field (Hanle) limits, covering also the regime of intermediate field strengths.

1. Introduction

The so-called forbidden lines are found in the spectra of rarefied gases under special conditions, such as those found in nebulae, the solar corona, the extreme upper atmosphere of the earth and are as important as the allowed lines (Bransden & Jochain 1983; Osterbrock 1989; Casini & Lin 2000; Charro & Martín 2002; Lin & Casini 2002; Duric 2004; Caffau & Ludwig 2007; Aldenius & Johansson 2007). Forbidden lines are often unaffected by departures from LTE and are insensitive to atmospheric temperature uncertainties as well as micro-turbulence. As such they have been quite useful to measure abundances.

An atom makes a transition between two states by absorbing or emitting a photon, when a momentum ±k is exchanged between the photon and the atom, if k denotes the momentum of the photon. This results in \( \exp(\pm i k \cdot r) \), which is approximated by unity for an allowed transition. In general, one may expand the vector wave in terms of photon eigen states, referred to as ‘electric’ and ‘magnetic’ 2L-pole states, where L is the total angular momentum of the photon. Conservation of total angular momentum and parity gives rise to the selection rules. A transition is referred to as forbidden either when \( L > 1 \), or when \( L = 1 \) but with the atomic states having the same parity.

2. T - Matrix for Scattering

In scattering, a photon with momentum \( k' \) is absorbed leading to the emission of the scattered photon with momentum k. The initial and final energies of the atom are denoted by \( E_i \) and \( E_f \). The on-energy-shell transition matrix \( T \) for
scattering is
\[ T = \sum_n (J_f m_f |\mathcal{E}(k, \mu)|J_n m_n) \varphi_n (J_n m_n |A(k', \mu')|J_i m_i), \]
where the summation is with respect to intermediate states with energy \( E_n \) and width \( \Gamma_n \). The total angular momentum and magnetic quantum numbers are denoted by \( J \) and \( m \) with suffixes \( i, n, f \) denoting respectively the initial, intermediate and final states respectively. \( \mu, \mu' \) are circular polarization states which take values \( \pm 1 \). The line profile function \( \varphi_n = (E_i + k' - E_n - i\Gamma_n)^{-1} \), in the atomic rest frame. For astrophysical applications, one has to take into account the Doppler motion of the atoms (Stenflo 1998, see also Sampoorna, Nagendra, & Stenflo 2007).

If \((k, \theta, \phi)\) and \((k', \theta', \phi')\) denote the polar coordinates of \( k \) and \( k' \) respectively, the emission and absorption matrix elements are
\[ \langle J_f m_f |\mathcal{E}(k, \mu)|J_n m_n \rangle = \sum_L C(J_f L J_n; m_f M m_n) D^L_{M \mu}(\phi, \theta, 0)^* (-i \mu)^h(L) J_L(k), \]
\[ \langle J_n m_n |A(k', \mu')|J_i m_i \rangle = \sum_{L'} C(J_i L' J_n; m_i M' m_n) D^{L'}_{M' \mu'}(\phi', \theta', 0) \times (-i \mu')^h(L') J_{L'}(k'), \]
where \( J \) denote reduced matrix elements (Oo et al. 2007). The quantities
\[ h(L) = \frac{1}{2} [1 + \pi_f \pi_n (-1)^L] ; \quad h(L') = \frac{1}{2} [1 + \pi_i \pi_n (-1)^{L'}], \]
take two values 0, 1 depending on the parities of the initial \((\pi_i)\), intermediate \((\pi_n)\) and final \((\pi_f)\) atomic states.

When the quantization axis is parallel to the magnetic field \( B \), the interaction leads to a splitting of an energy level \( E \) into \((2J + 1)\) Zeeman sublevels with energies \( E_m = E + g B m \), where \( g \) is the Landé factor. For resonance scattering with \( J_n = J \), the summation over \( n \) in Eq. (1) may be replaced by a summation with respect to \( m \), when \( g B \leq \) the width of the levels. In Zeeman scattering, when the \((2J + 1)\) levels are distinct, the summation over \( n \) gets restricted to a single term with \( m \) specified through the relation \( E_i + k' = E_m = E_f + k \).

3. Scattering Phase Matrix

The \( 2 \times 2 \) density matrix of the scattered radiation is given by
\[ \rho(k) = T \rho(k') T^\dagger, \]
in terms of density matrix \( \rho(k') \) of the incident radiation and matrix \( T \), whose hermitian conjugate is \( T^\dagger \). The radiation density matrix \( \rho \) is related to the Stokes parameters \( S_p \) through
\[ \rho = \frac{1}{2} \sum_{p=0}^3 \sigma_p S_p ; \quad S_p = tr(\sigma_p \rho), \]
where $\sigma_0$ denotes the $2 \times 2$ unit matrix and $\sigma_{1,2,3} = \sigma$ denote Pauli matrices. $(S_0, S_1, S_2, S_3) \equiv (I, Q, U, V)$ constitute the Stokes vector $S$. The $4 \times 4$ scattering matrix $R$ is now defined through the relation $S(k) = R S(k')$. The matrix elements $R_{pp'}$ may be expressed, using Eqs. (1), (5) and (6), as

$$R_{pp'} = \frac{1}{2} tr(\sigma_p T \sigma_{p'} T^\dagger) = \frac{1}{2} \sum_{mm'} \varphi_m \varphi_{m'}^* p_{pp'}^{mm'},$$

where $p_{pp'}^{mm'}$ is referred to as the phase matrix for line scattering.

For a $0 \rightarrow J \rightarrow 0$ scattering, involving forbidden transitions ($J = L' = L > 1$), considered here, the elements of $p_{pp'}^{mm'}$ may be expressed in the form

$$p_{pp'}^{mm'} = g_p^{mm'}(k)^* g_{p'}^{mm'}(k'),$$

where

$$g_p^{mm'}(k) = |J_L(k)|^2 \sum_{\mu \mu'} \sigma_p^{\mu \mu'} (-1)^{m' - m} (\mu \mu')^h(L)$$

$$\times \sum_l C(LLl; m - m'm_l) C(LLl; \mu - \mu' \mu_l) D^l_{m_l \mu_l} (\phi, \theta, 0).$$

The product $\varphi_m \varphi_{m'}^*$ may be simplified using the conversion formula of Stenflo (1998, see also Bommier & Stenflo 1999). In the weak field limit, the splitting of the levels is of the same order as the radiative width of the spectral lines, leading to quantum interference between $m, m'$ and cause the Hanle effect. In the strong field limit, the levels with different $m$ are distinct, leading to Zeeman scattering with $m = m'$. For Rayleigh scattering, the levels with different $m$ are degenerate leading to a single scattering phase matrix $P$ independent of $m$.

4. Numerical Results for $L = 2$

We consider the example of single scattering involving a $0 \rightarrow 2 \rightarrow 0$ (M2) type forbidden transition. An unpolarized beam of radiation is incident on the atom. Therefore the four Stokes parameters $(I, Q, U, V)$ of the scattered radiation are $(R_{00}, R_{10}, R_{20}, R_{30})$ respectively. Fig. 1 corresponds to the choice of angles $\theta' = 45^\circ$, $\phi' = 0^\circ$ and $\theta = 90^\circ$, $\phi = 45^\circ$. The magnetic field is along the $Z$-axis of the atmospheric reference frame. The scattered Stokes parameters are

$$I = \frac{5}{8} [\varphi \varphi^*] ; \quad Q = 0 ; \quad U = 0 ; \quad V = 0,$$

for non-magnetic Rayleigh scattering case,

$$I = \frac{5}{4} \left[ 0.125 (\varphi_1 \varphi_1^* + \varphi_-1 \varphi_{-1}^*) + 0.1875 (\varphi_2 \varphi_2^* + \varphi_{-2} \varphi_{-2}^*) \right],$$

$$Q = \frac{5}{4} \left[ -0.125 (\varphi_1 \varphi_1^* + \varphi_-1 \varphi_{-1}^*) + 0.1875 (\varphi_2 \varphi_2^* + \varphi_{-2} \varphi_{-2}^*) \right],$$

with $U = V = 0$, for Zeeman scattering case and for the Hanle-Zeeman case:

$$I = \frac{5}{4} \left[ 0.125 (\varphi_1 \varphi_1^* + \varphi_-1 \varphi_{-1}^*) + 0.1875 (\varphi_2 \varphi_2^* + \varphi_{-2} \varphi_{-2}^*) \right]$$
Figure 1. Comparison of Zeeman (1st column) and Hanle-Zeeman scattering (2nd column) in a single scattering event: $\theta' = 45^\circ, \phi' = 0^\circ$ and $\theta = 90^\circ, \phi = 45^\circ$. The solid curves: $v_B = 0.004$, dotted curves: 0.02, dot-dashed curves: 0.1, dashed curves: 0.5, and long-dashed curves: 2.5.

The different curves in Fig. 1 represent the different values of magnetic field strength defined by a splitting parameter $v_B$. The damping parameter of the Voigt profile function $a = 0.004$. In the Zeeman and Hanle-Zeeman scattering,
the Stokes $I$ profile clearly shows splitting in the strong fields. When the fields are weak, these components are not resolved. For this choice of geometry, the Stokes $Q$ gets contributions not only from the energy states $|2, \pm 1\rangle$ but also from $|2, \pm 2\rangle$ in both the cases (see Eqs. (10) and (11)). Rayleigh scattered Stokes $Q$ is zero in the M2 transition, even for non-zero scattering angle (see Eq. (9)). The $I$ profile for Rayleigh scattering is similar to the solid curve of the Hanle-Zeeman case, and hence is not shown. From Eqs. (10) and (11) for Stokes $Q$, we note that they differ only in the interference term. The effect of this extra term in Eq. (11) is significant only for weak fields (compare solid, dotted and dot-dashed $Q/I$ curves in the 2nd row of Fig. 1). When fields are stronger (dashed and long dashed curves), the Zeeman and Hanle-Zeeman scattering give nearly same $Q/I$ profiles, showing that interference terms gradually become negligible.

The Stokes $U$ and $V$ are zero in Rayleigh and Zeeman cases but they appear in Hanle-Zeeman scattering cases in the weak field limit. The generation of $U$ and $V$ in the Hanle-Zeeman case is due to the interference between different $m$-states. As the magnetic field strength increases, $U/I$ decreases and approaches small values. The $V/I$ is extremely small in the weak field limit and develops anti-symmetric profiles as the field strength increases.

5. Conclusions

The phase matrices for forbidden line transitions are derived. The 3 regimes of Rayleigh, Zeeman, and Hanle-Zeeman scattering are studied. The Hanle-Zeeman scattering theory provides a complete and correct description of scattering in fields of arbitrary strengths.

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