HELOSEISMIC MODES IN A MAGNETIC ATMOSPHERIC SOLAR MODEL

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ABSTRACT

The pulsation of the solar surface is caused by acoustic waves traveling in the solar interior. These f and p oscillation modes are not bounced back completely at the surface, but partially penetrate into the atmosphere. Here the atmospheric effects are investigated analytically on global oscillations in one-dimensional magnetohydrodynamic (MHD) models. We focus on the frequency spectrum of the f mode, hence an incompressible medium is considered. The global oscillation modes can be coupled resonantly to local atmospheric MHD modes (such as Alfvén and degenerated slow waves) causing frequency shifts and line broadening. An analytical approach is implemented for a first insight.

Key words: solar physics; helioseismology; magnetohydrodynamics; fundamental mode.

1. INTRODUCTION

The fundamental mode, or f mode, is a surface oscillation mode localised right beneath the photosphere. Its approximate frequency, as a simple surface mode, is \( \omega = \sqrt{g/k} \). The gravitational acceleration measured at the photosphere is g, the horizontal wave number of the oscillation mode is k. In more complex stratified media, the f mode frequency depends on the values of physical parameters (such as density, flow speed or magnetic field) taken at the surface (i.e. at the photosphere). Hence, the f mode is an effective diagnostic tool of the photospheric physical features. Among others, it provides an accurate measure of the solar radius, e.g. [1].

The f mode, since being a surface mode, is evanescent towards the solar core and into the photosphere. However, this does not mean that the mode is not present in the lower atmosphere! Though it is evanescent, it still will oscillate with an amplitude which decays exponentially with height. That small but finite amplitude may be significantly increased by resonant interaction with magnetohydrodynamic (MHD) slow and Alfvén waves in a magnetised atmosphere. Slow and Alfvén waves propagate basically along the magnetic field lines, with a locally varying frequency.

The idea of resonant coupling of global helioseismic modes to atmospheric MHD oscillation modes predicts that the presence of f and p (pressure) modes is observable in general in the lower atmosphere of the Sun. Atmospheric effects on p modes have been reported by, e.g., [2], [3] and [4].

Another important reason why it is worth of studying the resonant coupling of helioseismic modes to the atmosphere is that dissipation at the resonant location may also significantly contribute to the heating of the atmosphere.

The MHD model of the Sun is described in Section 2. The MHD equations are given in Section 3, which yields a compact analytical form of the dispersion relation, derived in Section 4. The dispersion relation is solved as an eigenvalue problem analytically in the thin-layer approximation in Section 5. The frequency spectrum and atmospheric effects on the f mode are presented in Section 6. The conclusions are drawn in Section 7.

2. EQUILIBRIUM

The model used to describe the solar interior and atmosphere is based on the one constructed by [7]. Its equilibrium profiles are simplified here so that the dispersion relation can be handled analytically without the loss of essential physics. The Sun is modeled with a plane parallel, three-layer plasma. The assumption of planar geometry is valid for small horizontal wavelength of perturbations compared to the solar radius. This restriction gives an estimate for the harmonic degree, l \( \gg 6 \) for which the global oscillations can be studied as plane waves in a planar equilibrium model.

The MHD equations are a set of coupled partial differential equations which describe the temporal and spatial variations of the plasma density, \( \rho \), plasma pressure, \( p \), plasma velocity, \( \mathbf{v} \) and magnetic induction, \( \mathbf{B} \). We use the ideal MHD equations – which are the continuity equation (1), the equation of momentum (2) and the induction equation with initial condition for the divergence of the magnetic induction (3) – and derive the dispersion relation for the eigenfrequencies of the system.

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \rho = 0, \quad (1)
\]
Figure 1. Equilibrium profile for a) plasma density, \( n_0(z) \), b) plasma pressure, \( p_0(z) \), c) magnetic induction, \( B_0(z) \) and d) Alfvén speed square, \( v_A^2(z) \) in the three-layer solar model with a magnetic atmosphere

\[
p \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \frac{1}{\mu} (\nabla \times \mathbf{B}) \times \mathbf{B} + g \rho, \quad (2)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad \nabla \cdot \mathbf{B} = 0. \quad (3)
\]

Notice that there is no energy equation coupled because of the incompressibility assumption of the plasma. The solar interior in the model is the semi-infinite layer below the photosphere, \( z < 0 \).

The equilibrium density is constant there, \( \rho_0(z < 0) \equiv \rho_i \), and the interior is free of magnetic field. (The index 0 refers to the equilibrium values of the physical quantities throughout the paper.) The plasma pressure decreases linearly with height with a gradient \(-g\rho_i\). The gravitational acceleration is taken constant, \( g = 273 \text{ m/s}^2 \) because the \( f \) mode is concentrated at and around the photosphere.

The square of the equilibrium magnetic field increases linearly from zero to its maximum, \( B_{0,L} \equiv B_0(z = L) \), in the intermediate transitional layer, thus the Alfvén speed square, \( v_A^2(0 \leq z \leq L) \equiv B_0^2(z)/(\mu \rho_0(z)) \), also increases linearly. Here \( L \) is the thickness of the transitional layer and \( \mu = 4\pi \times 10^{-7} \text{ N/A}^2 \) is the magnetic permeability. The density has a discontinuity at zero \( (z = 0) \) then it stays constant in the transitional layer: \( \rho(0 < z \leq L) = \rho_{tr} = \rho_i/N \) with \( N > 1 \). The total pressure, which is defined as the sum of the kinetic and magnetic pressures,

\[
P(z) \equiv p(z) + B(z)^2/(2\mu), \quad (4)
\]

decreases linearly with gradient \(-g\rho_{tr}\).

The upper semi-infinite layer represents the corona, where the Alfvén speed is constant, and the magnetic field strength and plasma density decrease exponentially with height. The plasma pressure also tends to zero towards the outer corona.

Hence, the equilibrium density, kinetic (plasma) pressure, magnetic field strength and Alfvén speed square are the following:

\[
\rho_0(z) = \begin{cases} 
\rho_i, & z < 0, \\
\rho_{tr}, & 0 \leq z \leq L, \\
\rho_{tr} \exp \left( \frac{L - z}{H} \right), & L \leq z, 
\end{cases} \quad (5)
\]

\[
p_0(z) = \begin{cases} 
p_0, & z < 0, \\
\rho_{tr} \rho_i \exp \left( \frac{L - z}{H} \right), & 0 \leq z \leq L, \\
p_L + gH \rho_{tr} \exp \left( \frac{L - z}{H} \right) - 1, & L \leq z, 
\end{cases} \quad (6)
\]

\[
B_0(z) = \begin{cases} 
0, & z < 0, \\
\sqrt{\mu \rho_{tr} v_A^2 z L}, & 0 \leq z \leq L, \\
\sqrt{\mu \rho_{tr} v_A^2 z L} \exp \left( \frac{L - z}{2H} \right), & L \leq z, 
\end{cases} \quad (7)
\]

\[
v_A^2(z) = \begin{cases} 
v_A^2(L), & 0 \leq z \leq L, \\
v_A^2(L) \exp \left( -\frac{L - z}{H} \right) - 1, & L \leq z. 
\end{cases} \quad (8)
\]

The photospheric equilibrium plasma pressure is denoted by \( p_{0,ph}(\equiv p_0(z = 0)) \). At the top of the transitional layer, the equilibrium plasma pressure takes \( p_L \equiv p(z = L) = p_{0,ph} - \rho_{tr} (gL + v_A^2 L^2/2) \). The Alfvén speed square at the top of the transitional layer, \( v_{A,L}^2 \equiv v_A^2(z = L) \) is a measure of the atmospheric magnetic field strength. The coronal scale height,

\[
H = \frac{p_L}{g \rho_{tr}}, \quad (9)
\]

is chosen so that the coronal plasma pressure tends to zero with increasing \( z \). The model is one dimensional in the sense that the physical quantities of the equilibrium depend only on the vertical coordinate, \( z \).

The initial profiles given in Equations (5) to (8) describe the equilibrium, from which a dispersion function for wave propagation will be derived in the following two sections. The dispersion relation can be solved analytically for several simplified though interesting limiting cases. We will study a thin-transitional-layer approximation for the general model, in which the resonant interaction of the \( f \) mode with MHD modes can be investigated analytically. Graphs of the equilibrium profiles for density, plasma
pressure, magnetic field strength and Alfvén speed can be seen in Figures 1a-d. 

The dimensional physical parameters specifying the equilibrium profiles are: \( \rho_i = 200 \text{ mg/m}^3, \rho_{tr} = 20 \text{ mg/m}^3, \)
\( H = 1.5 \text{ Mm} \) and \( v_{A,L} = 50 \text{ km/s} \).

3. GOVERNING EQUATIONS

3.1. Ideal MHD equations

It is a reasonable assumption to consider small perturbations around a static equilibrium, as the observed photospheric velocity oscillations (see, e.g., [5]) have amplitudes of only about 1 m/s, for oscillations of e.g. 75 m in amplitude, assuming 5 min period.

The perturbed quantities are then Fourier analysed with respect to the horizontal coordinates, \( x \) and \( y \). The temporal and the horizontal spatial dependencies of the perturbations are described by

\[
f_1(x, y, z, t) = f_1(z; k_x, k_y, \omega) \times \exp(i(k_xx + k_yy - \omega t)).
\]

The \( x \)-axis is chosen to be parallel to the horizontal magnetic field. We study global oscillations, of which the frequency does not vary either in space or in time: \( \omega \neq \omega(r, t) \). The linearized MHD equations in the Fourier space for an incompressible plasma with a horizontal magnetic field are

\[
\begin{align*}
\rho_1 + \frac{d\rho_0}{dz} \xi_z &= 0, \\
\rho_0 \omega^2 \xi_z &= \nabla P_1 - \frac{1}{\mu} \frac{d\vec{B}_0}{dz} \vec{B}_{1z} - \frac{i}{\mu} (\vec{k} \cdot \vec{B}_0) \vec{B}_1 - \vec{\alpha} \rho_1 \\
\vec{B}_1 &= i(\vec{k} \cdot \vec{B}_0) \xi_z - \frac{d\vec{B}_0}{dz} \xi_z.
\end{align*}
\]

The Lagrangian displacement vector, \( \xi \), of which the gradient is equal to the velocity of the plasma flow is defined as:

\[
\vec{v}(z; k_x, k_y, \omega) \equiv \frac{\partial \xi(x, y, z, t)}{\partial t} \equiv -i\omega \xi(z; k_x, k_y, \omega).
\]

The Eulerian total pressure perturbation is

\[
P_1(z; k_x, k_y, \omega) \equiv p_1(z; k_x, k_y, \omega) + \frac{1}{2\mu} \vec{B}_0(z) \vec{B}_1(z; k_x, k_y, \omega).
\]

Index 1 of the linear terms in the Fourier space will be omitted henceforth.

All but two perturbed quantities (i.e., \( \xi_z \) and \( P \)) can be eliminated from the linearized MHD equations (11). One will arrive at the following coupled first-order partial differential equations:

\[
\frac{\partial \xi_z}{\partial z} = \frac{dP(z)}{dz} = \frac{\rho_0(z) (\omega^2 - \omega_A^2(z))}{\xi_z} = k^2 P(z) \quad (14)
\]

\[
\frac{dP(z)}{dz} = \left( \frac{\rho_0(z) (\omega^2 - \omega_A^2(z))}{\xi_z} + g \frac{d\rho_0(z)}{dz} \right) \xi_z. \quad (15)
\]

The horizontal wave vector is defined as \( \vec{k} = (k_x, k_y) \). The local Alfvén frequency is

\[
\omega_A(z) \equiv \frac{(\vec{k} \cdot \vec{B}_0(z))/\sqrt{\mu \rho_0(z)}}{z} = k_x v_A(z).
\]

Equations (14) and (15) govern the perturbations about the equilibrium. The two governing equations have to be solved in each layer of the model. Once we have the solutions for \( \xi_z \) and \( P \), any other physical quantity can be obtained by their linear combinations.

3.2. Resonant absorption

In an incompressible atmosphere embedded in a horizontal magnetic field, incompressible MHD Alfvén waves can exist. They propagate along the magnetic field lines with the Alfvén speed, \( \vec{v}_A(z) = \vec{B}_0(z)/\sqrt{\mu \rho_0(z)} \). Their frequency is the local (i.e. height-dependent) Alfvén frequency, \( \omega_A(z) \).

Global photospheric oscillations leaking into the overlying atmosphere can interact resonantly with these local incompressible MHD Alfvén waves at height \( z_A \), where the local Alfvén frequency equals the frequency of the global oscillation.

The resonant term, \( \omega - \omega_A(z) \), in Equation (14) causes the governing equations to be singular at \( z = z_A \), as \( \xi_z(z \approx z_A) \sim \ln |z - z_A| \).

The Alfvén frequency for a given wave number varies continuously between 0 and \( \omega_{A,L} \equiv \omega_A(z = L) \) in the transitional layer (see Figure 1d). Hence, a height can be found in the transitional layer for any frequency between 0 and \( \omega_{A,L} \) where a local atmospheric Alfvén wave oscillates with that frequency. For this reason the region in the frequency spectrum below the characteristic frequency \( \omega_{A,L} \) is called Alfvén continuum. Global oscillations with a frequency lower than \( \omega_{A,L} \) interact resonantly with a local Alfvén wave at \( z_A \), while global oscillations with a frequency higher than \( \omega_{A,L} \) do not interact resonantly with a local Alfvén wave.

The amplitude of a global oscillation increases due to the resonant coupling so much that the linear ideal MHD equations fail to describe properly the atmosphere where resonant coupling occurs. Dissipative effects have to be taken into account thus, dissipative MHD equations have to be used in the vicinity of resonance. For that, we add a diffusive term to the induction equation (Equation (3)):

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{\eta}{\mu} \vec{v} \cdot \vec{B}.
\]

The coefficient \( \eta \) is the magnetic diffusivity.
Figure 2. Concept of resonant coupling of a global oscillation mode to a local Alfvén mode, that can occur in the magnetic model with a transitional layer.

The governing equations derived from Eqs. (1), (2) and (17) take the form

\[
\rho_0(z) \left[ \omega_r^2 - \omega_A^2(z) - i \eta \omega_r \omega_i \frac{d^2}{dz^2} \right] \times \\
\times \frac{d \xi(z)}{dz} = k^2 P(z),
\]

(18)

\[\frac{dP(z)}{dz} = \left( \rho_0(z) \left[ \omega_r^2 - \omega_A^2(z) - i \eta \omega_r \omega_i \frac{d^2}{dz^2} \right] + \\
+ g \frac{d\rho_0(z)}{dz} \right) \xi(z).
\]

(19)

It is reasonable to assume that the dissipative effects are negligible except in the vicinity of the resonant height. Hence the dissipative term in (19) can be ignored far away from the resonant location, as it would result in \( P \) being proportional to the square of the distance from \( z_A \), which is neglected.

By using these assumptions, \( \xi(z) \) and \( P(z) \) can be obtained, from which connection formulae can be derived in the vicinity of the resonant height, which connect the values of \( \xi \) and \( P \) below (\( z = z_A - \delta \)) and above (\( z = z_A + \delta \)) the resonant height:

\[\xi(z_A + \delta) = \xi(z_A - \delta) - \frac{i \pi k^2 L}{\omega_{A,L} \rho_{tr}} P(z_A - \delta),\]

\[P(z_A + \delta) = P(z_A - \delta).
\]

(20)

It can be shown that the distance from the resonant height, \( \delta \), can be chosen so that both the ideal MHD equations and the approximations used in the dissipative equations are valid at \( z_A - \delta \) and at \( z_A + \delta \).

Figure 2 summarises the concept of resonant coupling of a global mode with frequency within the Alfvén continuum (\( \nu_1^2 < \nu_{A,L}^2 \)). The resonant interaction takes place in the transitional layer at \( z = z_A \), where \( \nu_A^2(z) = \nu_1^2 \). Modes with frequency above the Alfvén continuum (\( \nu_2^2 > \nu_{A,L}^2 \)) do not interact resonantly with an Alfvén mode, as the local Alfvén frequency does not match the global frequency at any height.

Once we have the general ideal solution for the perturbations below the resonant height (with \( \lambda_1 \) and \( \lambda_2 \) integration constants, see Subsection 3.3) and also above the resonant height (with \( \lambda_3 \) and \( \lambda_4 \) integration constants), we can match \( \lambda_1 \) and \( \lambda_2 \) with \( \lambda_3 \) and \( \lambda_4 \) by using the connection formula (20). This yields the dispersion relation, \( f(\omega, k) = 0 \), which describes the global oscillations in the model. A more detailed derivation of connection formulae can be found in [6].

3.3. Spatial solutions for the ideal MHD equations

Substituting the equilibrium profiles of Eqs. (5) - (8) into Equations (14) and (15) we obtain the differential equations that govern the physical processes in the solar model.

Interior (\( z < 0 \)). The perturbations \( \xi \) and \( p \) in the field-free interior (\( z < 0 \)) with constant density (\( \rho_0(z) = \)}
\( \rho_i \) satisfy the following equations.

\[
\rho_i \omega_k \frac{d\xi_z}{dz} = k^2 p, \\
p \frac{dp}{dz} = \rho_i \omega_k^2 \xi_z.
\] (21)

We are looking for perturbations of which the kinetic energy is evanescent for \( z \to -\infty \), as it represents the solar core, toward which the energy of the perturbations should tend to zero. Hence, the evanescent solutions for Equations (21) are

\[
\xi_z(z < 0) = \xi_{z,PH} e^{kz}, \\
p(z < 0) = \xi_{z,PH} \rho_i \omega_k^2 e^{kz},
\] (22)

where the coefficient \( \xi_{z,PH} \) is integration constants.

**Transitional layer** (0 \( \leq z \leq L \)) – without resonance.

In the intermediate layer \( \rho_i(z) \equiv \rho_{ir} \) is constant and \( \omega_{\lambda,L}^2 \) increases linearly with \( z \). The following second order differential equation for \( \xi_z \) can be obtained by eliminating \( P \) from Eqs. (14):

\[
(z - z_A) \frac{d^2 \xi_z}{dz^2} + \frac{d\xi_z}{dz} - k^2 (z - z_A) \xi_z = 0
\] (23)

with \( z_A \equiv \frac{\omega_{\lambda,L}^2}{\omega_k^2} - L \).

By introducing \( \zeta \equiv k(z_A - z) \), Equation (23) becomes

\[
\zeta^2 \frac{d^2 \xi_z}{d\zeta^2} + \zeta \frac{d\xi_z}{d\zeta} - \zeta^2 \xi_z = 0,
\] (24)

which is the modified Bessel equation for index \( \nu = 0 \). The solutions are linear combinations of the modified Bessel functions \( I_0(\zeta) \) and \( K_0(\zeta) \):

\[
\xi_z(z) = \lambda_1 I_0(k(z_A - z)) + \lambda_2 K_0(k(z_A - z)).
\] (25)

For no resonance \( \omega^2 > \omega_{\lambda,L}^2 \), hence the variable of the Bessel functions, \( k(z_A - z) \) is positive in the transitional layer (compare to (23)).

The solution \( P(z) \) can be obtained from (14) by using \( dI_0(\zeta)/d\zeta \equiv I_1(\zeta) \) and \( dK_0(\zeta)/d\zeta \equiv -K_1(\zeta) \), where \( I_1 \) and \( K_1 \) are the modified Bessel functions of the first order:

\[
P(z) = \frac{\rho_{ir}}{k} \left( \omega_{\lambda,L}^2 \frac{z}{L} - \omega^2 \right) \times \\
\times \left( \lambda_1 I_1(k(z_A - z)) - \lambda_2 K_1(k(z_A - z)) \right). \] (26)

**Transitional layer** (0 \( \leq z \leq L \)) – with resonance.

The variable \( k(z_A - z) \) is positive only below the resonant height \( z < z_A \) in the transitional layer. The solutions of (25) and (26) for the non-resonant case are also solutions for the case of resonance only below the resonant height, because the modified Bessel functions are defined for the positive domain only. With the variable \( \zeta \equiv k(z - z_A) \) Equation (23) becomes the same Bessel equation as given in (24), so the solutions above the resonant height are also linear combinations of the modified Bessel functions of order zero:

\[
\xi_z(z) = \begin{cases} \\
\lambda_1 I_0(\zeta) + \lambda_2 K_0(\zeta), & 0 \leq z \leq z_A - \delta, \\
\lambda_3 I_0(\zeta) + \lambda_4 K_0(\zeta), & z_A + \delta \leq z \leq L.
\end{cases}
\] (27)

The total pressure perturbation can be obtained, again, from (14):

\[
P(z) = \begin{cases} \\
\frac{\rho_{ir}}{k} \left( \omega_{\lambda,L}^2 \frac{z}{L} - \omega^2 \right) \times \\
\times \left[ \lambda_1 I_1(\zeta) - \lambda_2 K_1(\zeta) \right], & 0 \leq z \leq z_A - \delta, \\
\frac{\rho_{ir}}{k} \left( \omega^2 - \omega_{\lambda,L}^2 \frac{z}{L} \right) \times \\
\times \left[ \lambda_3 I_1(\zeta) - \lambda_4 K_1(\zeta) \right], & z_A + \delta \leq z \leq L,
\end{cases}
\] (28)

where \( \zeta \equiv k(z_A - z) \) is positive for \( z < z_A \) and \( \zeta \equiv k(z - z_A) \) is positive for \( z > z_A \).

**Corona** (\( L < z \)). The coronal magnetic field and plasma density decrease exponentially in such a way that \( \omega_A(L \leq z) \equiv \omega_A(L) \). The governing equations for \( \xi_z \) and \( P \) are

\[
\rho_{ir} \exp \left( \frac{L - z}{H} \right) (\omega^2 - \omega_{\lambda,L}^2) \frac{d\xi_z}{dz} = k^2 P,
\]

\[
\frac{dP}{dz} = \rho_{ir} \exp \left( \frac{L - z}{H} \right) (\omega^2 - \omega_{\lambda,L}^2) \xi_z.
\] (29)

The evanescent solution for (29) is

\[
\xi_z(z) = \xi_{z,L} \exp \left( \frac{1 - \kappa}{2H} (z - L) \right),
\]

\[
P(z) = \xi_{z,L} \frac{\rho_{ir}}{k} \left( \omega^2 - \omega_{\lambda,L}^2 \right) \times \\
\times \left[ 1 - \frac{\kappa}{2H} \right] \exp \left( \frac{1 + \kappa}{2H} (z - L) \right),
\] (30)

where the coefficient \( \xi_{z,L} \) is integration constants and

\[
\kappa \equiv \sqrt{1 + 4k^2 H^2} \sqrt{\frac{\omega^2 - \omega_{\lambda,L}^2}{\omega^2 - \omega_{\lambda,L}^2}}.
\] (31)

with

\[
\omega_{\lambda,L}^2 \equiv \omega_{\lambda,L}^2 + \frac{4kH}{1 + 4k^2 H^2} g \kappa.
\] (32)

### 4. Dispersion Relation

Satisfying the boundary conditions at \( z = 0 \) and at \( z = L \) and for resonance also at \( z = z_A - \delta \) and at \( z = z_A + \delta \) yields to a dispersion relation, a function of the global frequency and the harmonic degree of the \( f \) mode. The eigensolution \( \nu(l) \) of the dispersion relation provides the spectrum of the solar model.
At the photosphere \((z = 0)\). The boundary conditions, that the vertical Lagrangian displacement, \(\xi_z(z)\), and the Lagrangian perturbation of total pressure, \(\delta P(z) \equiv P(z) - g\theta_0(z)\xi_z(z)\), have to be continuous functions of \(z\) at \(z = 0\) give us a connection between \(\xi_z, P_h\) and \(\lambda_1\) and \(\lambda_2\). Taking (22) at \(z = 0^-\) and (25) and (26) at \(z = 0^+\) yield

\[
\begin{align*}
\lambda_1 &= -\frac{C_{K0}}{C_{S0}}(\omega^2 - \omega_A^2)_{\xi_z, P_h}, \\
\lambda_2 &= \frac{C_{T0}}{C_{S0}}(\omega^2 - \omega_A^2)_{\xi_z, P_h} \\
C_{K0} &= NK_0(\zeta_0) - K_1(\zeta_0), \\
C_{T0} &= NI_0(\zeta_0) + I_1(\zeta_0), \\
\omega^2_K &= (N - 1)K_0(\zeta_0)gk/C_{K0}, \\
\omega^2_T &= (N - 1)I_0(\zeta_0)gk/C_{T0}, \\
C_{S0} &= I_0(\zeta_0)K_1(\zeta_0) + I_1(\zeta_0)K_0(\zeta_0)
\end{align*}
\]  

with

\[
\begin{align*}
A_{31} &= \delta_+ K_1^+ I_0^- - \delta_- K_1^- I_0^+ - \delta_+ \delta_- K_1^+ I_1^- \\
A_{32} &= \delta_+ K_1^+ K_0^- + \delta_- K_1^- K_0^+ + \delta_+ \delta_- K_1^+ K_1^- \\
A_{41} &= \delta_+ I_1^+ I_0^- + \delta_- I_1^- I_0^+ - \delta_+ \delta_- I_1^+ I_1^- \\
A_{42} &= \delta_+ I_1^+ K_0^- - \delta_- K_1^- I_0^+ + \delta_+ \delta_- I_1^+ K_1^- 
\end{align*}
\]

where

\[
\begin{align*}
I_j^\pm &= I_j \left( \frac{\pi i}{\pi} \delta_\pm \right) \\
K_j^\pm &= K_j \left( \frac{\pi i}{\pi} \delta_\pm \right), \quad j = 0, 1 \\
\delta_\pm &= \pi \left( \frac{2\omega_I k}{\omega^2_{\Lambda L}} \right) kL \pm i\delta \\
\omega &= \omega_r + i\omega_i 
\end{align*}
\]

Case of no resonance, \((z = L)\). For \(\omega > \omega_A\), the conditions, that \(\xi_z(z)\) and \(\delta P(z)\) are continuous functions of \(z\) at \(z = L\) connect \(\lambda_1\) and \(\lambda_2\) to \(\xi_z, L\). Taking (25), (26) and (30) at \(z = L\) yields

\[
\begin{align*}
C_{IL}\lambda_1 &= C_{KL}\lambda_2 \quad \text{with} \\
C_{IL} &= 2k\mathcal{H}I_1(\zeta_L) - (1 - \kappa)I_0(\zeta_L), \quad \text{and} \\
C_{KL} &= 2k\mathcal{H}K_1(\zeta_L) + (1 - \kappa)K_0(\zeta_L), \\
\zeta_L &= \frac{\omega^2 - \omega^2_{\Lambda L}}{\omega^2_{\Lambda L}} kL
\end{align*}
\]

Substituting (33) into (34) provides us an implicit dispersion equation for the non-resonant case, \(f(\omega^2, k) = 0\), of which the eigenvalues for \(\omega\) are the eigenfrequencies of the model atmosphere:

\[
C_{IL}C_{KL}(\omega^2 - \omega^2_K) + C_{K0}C_{IL}(\omega^2 - \omega^2_K) = 0. \tag{35}
\]

The coefficients above are functions of \(k\) and \(\omega^2\) through the modified Bessel functions.

Case of resonance, \((z = z_A \pm \delta)\). For \(|\omega| < \omega_A\), the connection formulae (20) give us the equations which connect \(\lambda_3\) and \(\lambda_2\) to \(\lambda_3\) and \(\lambda_4\):

\[
\begin{align*}
\lambda_3 &= \frac{A_{31}\lambda_1 + A_{32}\lambda_2}{\delta_+ [K_1^+ I_0^- + I_1^+ K_0^+]}, \quad \text{and} \\
\lambda_4 &= \frac{A_{41}\lambda_1 + A_{42}\lambda_2}{\delta_+ [K_1^+ I_0^- + I_1^+ K_0^+]}, \tag{36}
\end{align*}
\]

5. Thin-layer approximation

Assuming that the thickness of the transitional layer is much less than the wavelength of the global oscillation, \(kL \ll 1\), it is enough to keep the linear term in the Taylor expansion of the general dispersion relation around \(kL = 0\), \(f(\omega^2, k; kL)\) for without resonance and \(f(\omega, \omega_i, k; kL)\) for case of resonance.

Case of no resonance. It is reasonable to assume that a thin transitional layer causes small frequency shift from the eigenfrequency obtained in the two-layer model, i.e. in the model which has no transitional layer:

\[
\delta \omega \ll \omega_0, \quad \text{for} \quad \omega = \omega_0 + \delta \omega, \tag{41}
\]

where \(\omega_0\) is the eigensolution of the dispersion relation in the two-layer model. The eigenfrequency can be obtained as

\[
\omega(k) = \omega_0 - \frac{1}{2\omega_0} \left| \frac{\partial f(\omega^2, k, kL)}{\partial L} \right|_{L=0} \frac{\omega^2 - \omega_0^2}{\omega^2}, \tag{42}
\]
where $\omega_0$ is the eigenfrequency of the two-layer model, which can be obtained analytically.

**Case of resonance.** A global mode with frequency in the Alfven continuum ($\omega^2_p < \omega^2_{A,L}$) is resonantly coupled to a local Alfven wave at $z = L$. The resonant coupling modifies the eigenfrequencies $\omega_{0} \rightarrow \omega = \omega_f + i \omega_i$, where $\omega_f = \omega_0 + \delta \omega$. The dispersion function now is not a function of merely $\omega^2$ but of $\omega_f$ and $\omega_i$. Thus, the dispersion relation has a complex form:

\[ f(\omega_f, \omega_i, k; kL) + ig(\omega_f, \omega_i, k; kL) = 0. \]

It is enough to keep the first two terms of the Taylor series around $kL = 0$, because of the thin-layer approximation ($kL \ll 1$). It is reasonable to assume that the imaginary part of the global frequency, caused by the resonant coupling is much smaller than the frequency itself: $|\omega_i| \ll \omega_f$. This further simplifies the dispersion relation. Again, a thin transitional layer shifts the frequency slightly: $|\delta \omega| \ll \omega_f$. The complex dispersion relation linearized in $kL$, $\omega_i$ and $\delta \omega$ provides two real equations for the unknown variables. The frequency shift, $\delta \omega$, and the imaginary part of the frequency, $\omega_i$, can be expressed analytically, as two explicit functions of the harmonic degree, $l$.

### 6. RESULTS

The restrictions on page limit do not allow us to give a complete interpretation of the analytical form of the dispersion relation for the thin-layer model. The numerous aspects of the numerical results cannot be presented here in detail either. We restrict ourselves only to a brief introduction to analysing the results. A more detailed parametric study of the responses of the frequency and spatial behaviour of the $f$ mode to atmospheric changes is subject of an extended and detailed follow-up analysis.

The eigenfrequency spectrum as a function of harmonic degree, $l$, of the thin-layer approximation ($kL \ll 1$) for $l \leq 500$ is shown in Figure 3. The atmospheric magnetic field strength is fixed so that the Alfven speed at the top of the transitional layer, $v_{A,L}$, is 50 km/s. The frequencies $\omega_{C1}$ and $\omega_{C2}$ are characteristic frequencies of the two-layer ($L = 0$) model. The fundamental frequency $\nu_C = \sqrt{g k/(2 \pi)}$ is also plotted to see the change in the eigenfrequency, $\nu_f$, of the $f$ mode caused by the atmosphere. The characteristic frequencies $\nu_{C1}, \nu_{C2}$ and the Alfven frequency taken at $z = L$, $\nu_{A,L}$, divide the spectrum into regions for leaky-mode (there is no surface mode frequency in that region; shaded grey), real eigenfrequencies (white area) and the Alfven continuum with complex frequencies (yellow region).

The $f$-mode frequency is displayed for several values for $L$ between 0 and 1 Mm, but the difference is not recognisable, as the frequency shift caused by the presence of a thin transitional layer is much less than the order of mHz. The $f$ mode frequency is above the Alfven continuum for $l \leq l_2$, and it is equal to $\nu_C$ for that interval in the two-layer model. The $f$ mode does not exist for harmonic degrees between $l_2$ and $l_3$. The frequency of the mode falls into the Alfven continuum for $l > l_3$, hence it is coupled resonantly to local Alfven oscillations. The frequency of global oscillation modes becomes complex due to dissipation, if the modes are resonantly coupled to a local Alfven wave. The mode has a non-zero line width in the Fourier spectrum of the time series of the oscillation. The line width, $\Gamma \equiv -2 \omega_i$, measures the decay rate of the mode, i.e. how fast its amplitude decreases. $\Gamma$ for eigenmodes which are resonantly interact with local Alfven modes in the transitional layer is shown in Figure 4 for $L = 0.1, 0.5$ and 1 Mm. The line width increases with $l$ and with $L$. $\Gamma$ is about one $\mu$Hz for a 1 Mm thick transitional layer.
Figure 5. Frequency shift of the $f$ mode due to the presence of a transitional layer with $L = 0.1, 0.5$ and 1 Mm and $v_{A,L} = 50$ km/s for harmonic degrees, with which the $f$ mode frequency is above the Alfvén continuum.

Figure 6. Frequency shift of the $f$ mode due to the presence of a transitional layer with $L = 0.1, 0.5$ and 1 Mm and $v_{A,L} = 50$ km/s for harmonic degrees, with which the $f$ mode frequency is in the Alfvén continuum.

Figure 5 shows the frequency shift of the $f$ mode due to the presence of a thin transitional layer. This additional shift from the frequency for $L = 0$ is negative and increases in absolute value with increasing $l$, up to its minimum, then it rapidly increases to zero as $l$ approaches $l_2$. The frequency shift, $\delta \nu$, for modes which are coupled to an Alfvén wave are plotted in Figure 6. Those modes have a harmonic degree $l > l_3$. The frequency shift is negative, and its absolute value has a maximum for a harmonic degree right above $l_3$. That maximum is of the order of $\mu$Hz. Interestingly, the model is characterised by a threshold degree, $l_4$; for that value the $f$-mode frequency is not influenced by the transitional layer, it takes the same value for any $L$ as the frequency of the $f$ mode in the two-layer model. The frequency shift decreases again with $l$ for $l > l_4$.

7. CONCLUSION

The behaviour of the $f$ mode was studied in a plane parallel, one-dimensional solar model with incompressible plasma and with a magnetic atmosphere. The dispersion relation was derived analytically for the general three-layer model. Analytical solution for the dispersion relation was obtained for the thin-layer approximation. The presence of a transitional layer above the photosphere, where the magnetic field strength increases continuously reduces the $f$-mode frequency by a couple of hundred nHz. Dissipative effects at resonant coupling of the $f$ mode to a local Alfvén oscillation mode in the transitional layer produce non-zero line width in the spectral line, which is typically of the order of some hundred nHz. For an investigation of the $p$ modes compressibility has to be allowed in the model. Similarly, the atmospheric gravity modes are also missing from the model, as the Brunt-Väisälä frequency is zero in the interior and transitional layer and its square is negative in the corona. Understanding the behaviour of $p$ and $g$ modes is the aim of further investigation.

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