ABSTRACT

We will discuss the observed, heavily damped transversal oscillations of coronal loops. These oscillations are often modeled as transversal kink oscillations in a cylinder. Several features are added to the classical cylindrical model. In our models we include loop curvature, longitudinal density stratification and highly inhomogeneous radial density profiles.

We investigate the effect of vertical density stratification both numerically and analytically. We can conclude that longitudinal stratification establishes a coupling between different longitudinal mode numbers. On the other hand, the observational parameter \( \gamma \) changing Period is not influenced.

However, since the ratio of periods of different longitudinal numbers is different from 2, we can estimate the density stratification in a loop if two longitudinal mode numbers are observed. We can conclude that the inclusion of vertical density stratification provides an excellent tool for coronal seismology.

Additionally, we analyze a theoretical model for curved coronal loops. Because of the curvature, poloidal mode numbers are coupled. In the small curvature limit, however, only coupling to the two neighboring poloidal mode numbers occurs. Expressions for the (complex) frequency are obtained.

Key words: MHD, Plasmas, Sun: corona, Sun: oscillations.

1. INTRODUCTION

Coronal loop oscillations were first observed with TRACE by Schrijver et al. (1999), Aschwanden et al. (1999) and Nakariakov et al. (1999). A more extensive study was performed by Schrijver et al. (2002) and Aschwanden et al. (2002), who published the observational properties of 11 such events. Recently, Verwichte et al. (2004) detected both fundamental and overtone oscillations in a coronal loop arcade. While all these oscillations are perpendicular to the coronal loop plane, vertical oscillations have recently been reported by Wang & Solanki (2004).

The nature of these oscillations is believed to be a kink \( (m = 1) \) quasi-mode oscillation. These quasi-modes are damped by resonant absorption, which actually rapidly converts the fast kink mode to torsional Alfvén waves. These Alfvén waves are then available to heat the coronal loop (Moriyasu et al. 2004) to observed temperature profiles (Landi & Landini 2004).

Analytical calculations in cylindrical configurations of the frequency and damping times of such quasi-modes were published by Hollweg & Yang (1988), Goossens et al. (1992) and Goossens et al. (1995). In these analytical derivations, it is assumed that the radial inhomogeneity is quasi-discontinuous. Numerical calculations concerning more realistic radial inhomogeneity profiles were done by Hollweg (1990) and Van Doorsselaere et al. (2004a).

The observed oscillations also provide an excellent tool for coronal seismology. Nakariakov & Ofman (2001) used them to obtain an estimate for the value of the magnetic field in coronal loops and Goossens et al. (2002) and Aschwanden et al. (2003) exploited these oscillations to calculate the density ratio (internal/external) for a set of loops and compared it with the observed values. They found an agreement between the observed and calculated values within a factor 2, which is excellent, given the observational constraints.

2. CLASSICAL MODEL & IMPROVEMENTS

An often used model applies linearized MHD to a static equilibrium and neglects gravitation and gas pressure (\( \text{plasma-} \beta=0 \)). Additionally, the curvature of the loop is neglected and cylindrical geometry is assumed. The equilibrium magnetic field is taken in the \( z \)-direction only and is assumed to be uniform everywhere.

\[
\vec{B} = B \hat{z}.
\]

Consequently, the equilibrium density profile can be chosen freely. The coronal loop is then modeled by a den-
3. CURVATURE

Because we want to curve our initially straight cylindrical coronal loop, we have to employ another coordinate system. Following Van Doorsselaere et al. (2004b), we use toroidal coordinates. They are described by the transformation formulae:

\[
\begin{align*}
  x &= \frac{a \sinh u \cos \phi}{
    \cosh u - \cos v}, \\
  y &= \frac{a \sinh u \sin \phi}{
    \cosh u - \cos v}, \\
  z &= \frac{a \sin v}{
    \cosh u - \cos v},
\end{align*}
\]

where \((x, y, z)\) are the original Cartesian coordinates, \((u, v, \phi)\) are the new toroidal coordinates and \(a\) the (large) radius of the torus. As can be seen from Fig. 2, \(u\) is the new (dimensionless) radial coordinate, \(v\) the poloidal coordinate and \(\phi\) the toroidal coordinate, equivalent with \(z\) in the cylindrical model.

When taking a constant \(u\)-surface, a torus is outlined, thus making a good model for a curved coronal loop.

Because, in this geometry, a uniform magnetic field is not force-free, a better equilibrium magnetic field has to be found. Solving the current-free condition together with the solenoidal constraint for the magnetic field yields a unique form:

\[
\tilde{B} = B(u, v) e_\phi = B_a \frac{(\cosh u - \cos v)}{\sinh u} e_\phi.
\]

Here \(B_a\) is the magnetic field at \(u \to \infty\), i.e. the axis of the torus.

The governing equations for the linear perturbations are

\[
\begin{align*}
  \rho \frac{\partial \tilde{V}}{\partial t} &= -\frac{1}{\mu} (\hat{\nabla} \times \tilde{b}) \times \tilde{B}, \\
  \frac{\partial \tilde{b}}{\partial t} &= \hat{\nabla} \times (\tilde{V} \times \tilde{B}),
\end{align*}
\]

where \(\tilde{b} = (b_u, b_v, b_\phi)\) and \(\tilde{V} = (V_u, V_v, V_\phi)\) are the perturbations of the magnetic field and the velocity, respectively. Using the toroidal coordinate system and Fourier analyzing in the \(\phi\)-direction (wavenumber \(k\)), these can
be expanded to

$$\frac{\partial^2 \tilde{V}}{\partial t^2} + \omega_A^2 \tilde{V} = V_A^2 \phi \left( \frac{1}{B} \frac{\partial b_\phi}{\partial t} \right),$$  \hspace{1cm} (1)$$

$$\frac{\partial b_\phi}{\partial t} = - \frac{B}{a} \sinh u (\cosh u - \cos v)$$

$$\times \left\{ \frac{\partial}{\partial u} \left( \frac{V_u}{\sinh u} \right) + \frac{\partial}{\partial v} \left( \frac{V_v}{\sinh u} \right) \right\}.$$  \hspace{1cm} (2)$$

Here we defined the Alfvén speed by $V_A^2 = B^2/\mu_0$ and the Alfvén frequency as $\omega_A^2 = V_A^2 k^2 (\cosh u - \cos v)^2/a^2 \sinh^2 u$.

We also have $\tilde{V}_\perp = \tilde{V} - \tilde{b}_\phi \left( \frac{a^2 \sinh^2 u}{\cosh u - \cos v} \frac{\partial}{\partial u} \frac{\partial b_\phi}{\partial \phi} \right)$, which is the gradient in the cross-section of the coronal loop perpendicular to the axis.

Now the density may be chosen so that two cavities with constant Alfvén frequency are obtained:

$$\rho(u, v) = \begin{cases} \rho_{a,c} \left( \frac{\cosh u - \cos v}{\sinh u} \right)^4 & \text{for } u < u_0, \\ \rho_{a,i} \left( \frac{\cosh u - \cos v}{\sinh u} \right)^4 & \text{for } u_0 \leq u \leq u_0 + d, \\ \rho_{a,1} \left( \frac{\cosh u - \cos v}{\sinh u} \right)^4 & \text{for } u > u_0 + d. \end{cases}$$

$\rho_{a,1}$ is the outer boundary of the coronal loop, and $u_0 + d$ the inner boundary. $\rho_{a,i}$ and $\rho_{a,c}$ are the densities at the axis of the coronal loop, respectively for the inner part (subscript i) and the outer corona (subscript c). $\rho_{a}(u)$ is arbitrary function continuously connecting the internal and external region.

Using this density profile, Eqs. 1-2 can be reduced to a single equation for $b_\phi$ in both the internal and external region:

$$\nabla^2 \left( \frac{b_\phi}{B} \right) = - \left( \frac{k^2 \omega_A^2 (\cosh u - \cos v)^2}{\omega_A^2 \sinh^2 u} \right) \frac{b_\phi}{B}.$$

The general solution to this equation can be written as:

$$\sum_{m=-\infty}^{\infty} C_{i/e,m} \sqrt{\frac{\cosh u - \cos v}{\sinh u}} \times F_{i/e,m}(u) \exp \left( im(v-v_0) \right).$$  \hspace{1cm} (3)$$

In this solution, $C_{i/e,m}$ are constant coefficients, $F_{i/e,m}(u)$ gives the radial dependency (which can be expressed in terms of hypergeometric functions, see Van Doorselaere et al. 2004b) and $v_0$ is an arbitrary phase. This phase is determined by the form and direction of the incoming shock wave, which triggers the coronal loop oscillation. Thus, every phase can be attained, and vertical as well as horizontal oscillations are allowed. Apart from the intensity enhancement, the vertical coronal loop oscillations (observed by Wang & Solanki 2004) can thus be modeled by kink quasi-mode oscillations. Since horizontal and vertical oscillations are actually identical waves, it is not necessary to differentiate between them.

It is also clear from Eq. 3 that different poloidal mode numbers $m$ are now coupled (through the $v$-dependency under the square root).

By using the appropriate connection formulae (see e.g. Sakurai et al. 1991) between the internal and external region, a dispersion relation may be found. Expanding this dispersion relation in the small curvature limit $(\varepsilon = R/a \ll 1)$, shows that the frequency is modified to:

$$\omega = \omega_{\text{br}} - i \omega_{\text{r}} (1 + \varepsilon \pi (\frac{3}{4} \frac{1}{m^2 + \frac{1}{2}})),$$  \hspace{1cm} (4)$$

where

$$\omega_{\text{br}} = \frac{2 \omega_A^2 \lambda \omega_{\text{br}}}{\lambda^2 + \omega_A^2},$$

$$\omega_{\text{r}} = - \frac{\pi \omega_A^2}{\lambda^2} \frac{1}{\omega_A^2 - 1} \left( \frac{\omega_A^2}{\omega_A^2 - 1} (m^2 + \frac{1}{2}) \right),$$

which are actually the real and imaginary part of the oscillation frequency in the cylindrical case. Interesting to note is that only the imaginary part of the frequency is influenced by the curvature. The damping of the coronal loop oscillation is only slightly stronger in a curved configuration than in a straight flux tube. Since the influence of the curvature on the frequency is so small (only 12% more damping), it is unlikely that this effect can be observed with the current instruments.

Additionally, it can be observed that the frequencies do not depend on $v_0$. We can thus conclude that a vertical and horizontal oscillation must have the same frequency and damping rate in the limit of small curvature.

The connection formulae can also be expanded in $\varepsilon$ and show that, in the small curvature limit, a quasi-mode oscillation with poloidal mode number $m$ only couples to neighbouring poloidal mode numbers $m \pm 1$.

### 4. DENSITY STRATIFICATION

In this section, we return to the cylindrical model of Sect. 2. Instead of adding curvature, a density stratification along the loop is assumed. We now assume a density profile of the form:

$$\rho(r, z) = \rho(r) (1 - \alpha \sin \left( \frac{\pi}{L} z \right)).$$

$0 \leq \alpha < 1$ models the stratification in the $z$-direction. For $\alpha = 0$, we have an unstratified coronal loop equivalent to the model in Sect. 2. For $\alpha = 1$, we have an unphysical coronal loop with an empty loop top.

Analytical results can be obtained for $\alpha \ll 1$ (see Andries et al. 2005b). Assuming that the frequency has only a first order (in $\alpha$) correction, it can be shown that

$$\omega = \omega_0 + \alpha \omega_1 = \omega_0 - \frac{\alpha}{2} \omega_0 \bar{S}_{k,k}$$

$$= \omega_0 \left( 1 - \frac{\alpha}{\pi} \left[ 1 - \frac{1}{1 - 4k^2} \right] \right).$$
The density scale heights inferred from the observed ratio of the periods of the fundamental kink mode and the first overtone. The values in the first column are taken from Verwichte et al. (2004)

<table>
<thead>
<tr>
<th>$P_1/P_2$</th>
<th>$H$ (in $Mm$)</th>
<th>confidence interval (in $Mm$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.81 \pm 0.25$</td>
<td>65</td>
<td>$-\infty, -190 \cup [27, +\infty]$</td>
</tr>
<tr>
<td>$1.64 \pm 0.23$</td>
<td>36</td>
<td>$[20, 99]$</td>
</tr>
</tbody>
</table>

Table 1. Density scale heights inferred from the observed ratio of the periods of the fundamental kink mode and the first overtone. The values in the first column are taken from Verwichte et al. (2004)

When more realistic coronal loop models, i.e. with smoother density profiles, are considered, a numerical code has to be used. Arregui et al. (2005) used POL-LUX, a two dimensional numerical code with finite elements in the radial direction and a spectral discretization in the z-direction. The code is an eigenvalue code and was introduced by Van Der Linden (1991). The code uses the Jacobi-Davidson shooting method to find eigenvalues close to a given target eigenvalue.

A large parameter scan is performed and the dependency on $\alpha$ is investigated. The resulting plots can be seen in Fig. 4. These figures exhibit the dependence of $q_{TTTB}$ on the longitudinal wave number $k_z$, the density contrast $\zeta$, and the width of the inhomogeneous layer $l/R$. $q_{TTTB}$ is defined by:

$$
-\frac{\omega_1}{\omega_r} = -q_{TTTB} \frac{1}{l} \frac{\zeta - 1}{R \zeta + 1}
$$

i.e., it is $\omega_1/\omega_r$ normalized to the analytical value obtained with the model in Sect. 2. $q_{TTTB} = 1$ means that the analytical formula is perfectly true. From these figures it is clear that the analytical prediction for small stratification remains true for larger values of $\alpha$ and thick inhomogeneous layers. The only variation in these figures is the same as found by Van Doorsselaere et al. (2004a).

The theory of longitudinally stratified coronal loops is excellent to do coronal loop seismology. Because of the stratification, the ratio of the periods of the fundamental kink mode $P_1$ and the first overtone $P_2$ will significantly differ from 2. When calculating this ratio for different density scale heights $H$, a one to one relation is obtained (see Fig. 5). Consequently, when two periods are measured in the same coronal loop, the density scale height in this loop can be inferred. Andries et al. (2005a) used the values found by Verwichte et al. (2004) to find a density scale height in two coronal loops (see Fig. 6). Their results can be found in Table 1. In two cases they find a scale height of respectively 65 $Mm$ and 36 $Mm$, which is compatible with the observational value of 50 $Mm$.

5. CONCLUSIONS

We can conclude that we significantly improved the classical cylindrical model for a coronal loop. Curvature and
Figure 4. top left: $q_{TTTB}$ vs. $k_z$ and $\alpha$ for $l/R = 1$ and $\zeta = 5$; top right: $q_{TTTB}$ vs. $\zeta$ and $\alpha$ for $l/R = 1$ and $k_z = 0.04$; bottom: $q_{TTTB}$ vs. $l/R$ and $\alpha$ for $\zeta = 4$ and $k_z = 0.08$

Figure 5. The ratio of the periods of the fundamental mode and the first overtone versus the density scale height $H$.

Figure 6. The scale height versus $2 - P_1/P_2$ for the two cases studied by Verwichte et al. (2004). The observational estimate of the ratio of the periods is connected with the estimate of the density scale height by a dashed line. The dash-dotted lines connect the ends of the confidence interval, which is indicated by a thick grey line.
longitudinal density stratification were added. As a result of the curvature, coupling between different poloidal mode numbers occurs and quasi-mode oscillations are slightly (up to 12%) more damped. Additionally, it is shown that a poloidal phase may be chosen freely. Thus, curvature does not select preferential oscillation directions: vertical as well as horizontal oscillations are still possible, as observed recently.

Density stratification along the coronal loop, on the other hand, couples longitudinal mode numbers. The change of the frequency is proportional to the frequency itself. Consequently, the ratio of the imaginary and the real part of the frequency does not change.

Our model for longitudinally stratified coronal loops has proven to be a valuable tool for coronal loop seismology. Andries et al. (2005a) inferred values of $65 \pm m$ and $36 \pm m$ for the density scale height in two cases where the fundamental mode and an overtone were observed in the same loop. These values for the density scale height are compatible with the observed value of $50 \pm m$.

ACKNOWLEDGEMENTS

For his precious comments and help while parallelizing the POLLUX-code, the authors would like to thank Dries Kimpe. These results were obtained in the framework of the projects GOA 2004/01 and OT 02/57 (K.U.Leuven), G.0451.05 (FWO-Vlaanderen) and 90203 (ESA Prodex 8).

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