RANDOM FLOW EFFECTS ON SURFACE WAVES

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ABSTRACT

Studying the properties of surface waves is probably the simplest wave tool for diagnosing a medium. Surface waves are observed e.g. as the fundamental global oscillations (called the f-mode), and have also been detected at the boundaries of various solar structures (e.g. sunspot filaments, coronal loops, coronal funnels, solar wind tubes, etc.). SOHO and TRACE have demonstrated that the solar atmosphere and its magnetic structures are highly inhomogeneous at almost all spatial and time scales. The question naturally arises: does the random nature of the medium influence the propagation characteristics of the surface modes?

Murawski & Roberts (1993) investigated the effect of a random velocity field on the dispersion relation for f-modes propagating on the solar surface. Here we follow their general approach, which is a valuable one, but correct errors which appeared in that paper. We still find, as they did, that the simple model used gives a deviation of the f-modes from the theoretically predicted parabolic ridges which agrees qualitatively with observations. We find that turbulent background flows can reduce the eigenfrequencies of global solar f-modes by several per cent as found in observations at high spherical degree.

In the current paper we focus on the global solar global f-mode only. From a theoretical viewpoint the mode has a simple parabolic dispersion relation

\[ \omega^2 = g \kappa_h, \]

where \( \omega \) is the mode frequency; \( g \) is the gravitational acceleration at the surface of the Sun; and \( \kappa_h \) is the horizontal wave number. Observations of the f-mode, however, show (e.g. Libbrecht and Kaufman, 1988; Hindman et al. 2003; Dziembowski and Goode 2003; Antia et al. 2001) that at higher degree the frequency of the mode is considerably below the theoretically predicted parabolic ridge. Since this global mode is highly localised at the solar surface it is also expected that the f-mode is more sensitive to spatial and temporal changes taking place in the outer part of the Sun (i.e. at photospheric level).

The solar surface is very dynamic. For example, there are continuous granular motions, and there are continuously emerging sub-observational and observable magnetic fluxes; just to mention a few. All such dynamic properties may have some effect on the surface oscillations. Campbell and Roberts, 1989; Evans and Roberts, 1990; and Bogdan et al., 1992 carried out some pioneering studies on coherent atmospheric magnetic effects on solar global oscillations. On the other hand, Murawski and Roberts 1993 (henceforth referred to as MR) studied what frequency shifts may be caused by a random velocity field.

In the present paper we re-visit the general approach of MR, but correct errors which appeared in that paper. These errors include the omission of terms in the perturbation equation, leading to the neglect of some leading order terms in the correction to the dispersion relation; and the use of velocity correlations which do not satisfy the condition for an incompressible fluid.

We confirm their speculative results on the deviation of f-modes from the theoretically predicted parabolic ridges. In spite of the simplistic approach used in the current model, we find, that the obtained results are in good agreement with observations. We confirm that turbulent background flows shall indeed reduce, of the order of a few per cent, the eigenfrequencies of global solar f-modes in the limit of high spherical degree l.


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2. THE MODEL

We consider a simple, Cartesian model where the $z = 0$ surface corresponds to the solar surface and the $z$-axis is directed downwards so that the gravitational acceleration is in the positive $z$-direction.

The interior ($z > 0$) is taken to be an incompressible fluid where we assume that a horizontal, steady flow is present in the form

$$
U = [U_1(x, y), U_2(x, y), 0],
$$

and we also assume that the flow is irrotational

$$
\nabla \times U = 0.
$$

The condition for irrotational flow allows us to introduce a scalar potential $\Phi_0$ for the velocity field so that

$$
U = \nabla \Phi_0.
$$

The governing equations for the dynamics of the fluid are the simple hydrodynamic equations

$$
\nabla \cdot U = 0,
$$

$$
\frac{\partial U}{\partial t} + U \cdot \nabla U = -\frac{1}{\rho} \nabla p + g\hat{z},
$$

where $\rho$ is the constant density, $p$ is the fluid pressure and $g$ the local gravitational acceleration.

As the flow is irrotational and divergence-free the velocity potential $\Phi_0$ has to satisfy Laplace’s equation

$$
\nabla^2 \Phi_0 = 0.
$$

We assume that the atmospheric pressure varies horizontally

$$
p = p_A(x, y) \quad \text{in} \quad z < 0,
$$

so in the interior ($z > 0$) the pressure has the form

$$
p_I = \rho g z + p_A(x, y),
$$

and

$$
p_A + \frac{1}{2} \rho U^2 = \text{constant}.
$$

If $p_A$ is constant (as in MR where $p_A \equiv 0$) then $U$ must be constant (contrary to the assumption made by MR), and spatial correlations of $U$ must be independent of the separation distance. In a real fluid we expect correlations to decay with distance, so we assume that $p_A$ is random and non-constant, and that it gives the expected form of correlations for $U$. Thus we impose a horizontal pressure gradient as a surrogate for all the forces which would in reality contribute to producing such flows.

3. PERTURBATION EQUATIONS

We assume that in the presence of a wave motion the velocity field remains irrotational, so we have a velocity potential $\Phi$. The boundary conditions at the perturbed surface, $z = \eta(x, y, t)$, are the pressure continuity

$$
p_I = p_A,
$$

and the kinematic boundary condition

$$
\frac{d}{dt}(z - \eta) = 0.
$$

Let

$$
\Phi = \Phi_0 + \Phi',
$$

where $\Phi'$ denotes the surface perturbation. Far away from the surface we require that

$$
\Phi' \to 0 \quad \text{as} \quad z \to \infty.
$$

Bernoulli’s integral for irrotational flow

$$
\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla \Phi)^2 + \frac{p}{\rho} - gz = \frac{1}{2} U^2 + \frac{p_A}{\rho},
$$

together with the boundary conditions (10) - (11) in the linear approximation gives

$$
M \Phi' = (R + N + Q) \Phi',
$$

where

$$
M = \partial_t - g \partial_z,
$$

$$
R = -2(U_1 \partial_{xz} + U_2 \partial_{yz}),
$$

$$
N = -(U_1^2 \partial_{xx} + 2U_1 U_2 \partial_{xy} + U_2^2 \partial_{yy}),
$$

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\[ Q = -\frac{1}{2} \left( \mathbf{U}^2 \partial_x + \mathbf{U}^2 \partial_y \right) \]  

(19)

MR failed to include \( N \) or \( Q \). As we show below, this leads to an incorrect result.

4. RANDOM VELOCITY FLUCTUATIONS

We now take the basic velocity \( \mathbf{U} \) to be random. There are two comments to be made about the assumptions we are making about the flow.

Firstly, in this model the basic flow is steady, but varies randomly between realizations. Any dynamical interactions between the surface waves and the velocity fluctuations are ignored. This is acceptable if the timescale of the fluctuations is greater than that of the waves

\[ \frac{l}{u} \gg \frac{1}{\omega}, \]  

(20)

where \( l \) and \( u \) are the characteristic length and speed of fluctuations and \( \omega \) is the frequency of the surface wave.

Secondly, the flow in the model is irrotational, but real random flows are likely to be rotational. Assuming irrotational flow greatly simplifies the calculations, and ought to give qualitatively correct results.

We divide the perturbation field \( \Phi' \) into a mean and a fluctuating component

\[ \Phi' = \langle \Phi \rangle + \Phi''; \]  

(21)

where \( \langle \cdot \rangle \) denotes ensemble mean. We assume that

\[ \langle \mathbf{U} \rangle = 0, \]  

(22)

in which case (15) gives

\[ M \langle \Phi \rangle - \langle N \rangle \langle \Phi \rangle = \langle R \Phi'' \rangle + \langle N \Phi'' \rangle + \langle Q \Phi'' \rangle, \]  

(23)

\[ M \Phi'' - \langle R \Phi'' \rangle - \langle N \Phi'' \rangle - \langle Q \Phi'' \rangle \]

\[ = R \langle \Phi \rangle + N \langle \Phi \rangle - \langle N \rangle \langle \Phi \rangle + Q \langle \Phi \rangle. \]  

(24)

Here we have used the fact that \( \langle R \rangle = 0 \), and have also assumed that the random fluctuations are homogeneous, so \( \langle Q \rangle = 0 \). If the basic flow speed is small, then to leading order in \( U = |\mathbf{U}| \) (23) gives

\[ M \langle \Phi \rangle = 0 \quad \Rightarrow \quad \omega^2 - gk = 0, \]  

(25)

i.e. the usual deep water wave result. Similarly (24) gives

\[ M \Phi'' = R \langle \Phi \rangle \quad \Rightarrow \quad \Phi'' \sim U. \]  

(26)

Using this result we find from (23) and (24) that the leading order corrections to (25) are \( O(U^2) \), and that to this order we have

\[ M \langle \Phi \rangle - \langle N \rangle \langle \Phi \rangle = \langle R \Phi'' \rangle, \]  

(27)

\[ M \Phi'' = R \langle \Phi \rangle. \]  

(28)

In effect MR calculated their results to \( O(U^2) \). However, since they failed to include \( N \) their results are incorrect, even without going to higher order in \( U \).

5. DISPERSION RELATION

Applying Fourier transforms to (27) and (28), as in MR, gives the dispersion relation

\[ \omega^2 - gk = -k_1^2 R_{11} (0) - 2k_1 k_2 R_{12} (0) - k_2^2 R_{22} (0) \]

\[ + 4\omega^2 \int_{-\infty}^{\infty} \frac{dK_1 dK_2}{\omega^2 - gK} \Psi(k, K), \]  

(29)

where

\[ \Psi(k, K) = \]

\[ k_1 \left[ K_1 \hat{R}_{11} (K - k) + K_2 \hat{R}_{12} (K - k) \right] \]

\[ + k_2 \left[ K_1 \hat{R}_{21} (K - k) + K_2 \hat{R}_{22} (K - k) \right], \]  

(30)

and \( \hat{\cdot} \) indicates the Fourier transform. We have introduced the correlation function in the form

\[ R_{ij} (r) = \langle U_i (x) U_j (x + r) \rangle. \]  

(31)

If we assume that the velocity fluctuations are isotropic, then it is reasonable (see e.g. Batchelor (1953), § 3.3, 3.4) to take
\[ R_{ij}(r) = u^2 e^{-r^2/4l^2} \left\{ \frac{r_i r_j}{2l^2} + \left(1 - \frac{r^2}{2l^2}\right) \delta_{ij} \right\}. \] (32)

MR appeared to assume that \( R_{ij} \) was proportional to \( \delta_{ij} \), which is incorrect since it does not satisfy incompressibility.

Substituting (32) in (29) gives the dispersion relation for homogeneous isotropic velocity fluctuations, correct to \( O(u^2) \)

\[
\omega^2 - gk = -u^2 k^2 \\
+ \frac{8 \omega^2 u^2 \Omega^2}{\pi} \int_{-\infty}^{\infty} \frac{dK_1 dK_2}{\omega^2 - gK} e^{i(K-K_2)k} (K_1 k_2 - K_2 k_1)^2.
\] (33)

We assume, without loss of generality, that the wave propagates in the \( x \)-direction (i.e. \( k_1 = k, k_2 = 0 \)). It is straightforward to change to polar coordinates:

\[ K_1 = K \cos \theta, \quad K_2 = K \sin \theta, \]

and we non-dimensionalise by

\[ \xi = K l, \quad \kappa = kl, \]
\[ \Omega^2 = \frac{l \omega^2}{g}, \quad \alpha^2 = \frac{u^2}{gl}, \]

where \( \kappa \) and \( \xi \) are dimensionless wavenumbers and \( \Omega \) is the dimensionless frequency.

Substituting the above quantities in (33) we obtain

\[
\Omega^2 - \kappa = -\alpha^2 \kappa^2 \\
+ \alpha^2 \frac{8 \Omega^2 \xi^2}{\pi} e^{-\kappa^2} \int_0^{2\pi} \sin^2(\theta) d\theta \int_0^\infty f(\xi, \kappa) d\xi, \] (34)

where

\[ f(\xi, \kappa) = \frac{\xi^3}{\Omega^2 - \xi} e^{-\xi(\xi^2 - 2\xi \cos \theta)}, \]

and in the second integral \( \int \) denotes the Cauchy principal part of the integral.

Equation (34) is valid for small \( \alpha \), i.e. for velocity fluctuations which are small compared with the phase speed of waves whose wavelength is the spatial scale of the fluctuations.

For small wavenumber \( \kappa \), (34) gives

\[ \Omega \sim \kappa^{1/2} - \alpha^2 \left[ \frac{1}{2} \kappa^{3/2} + \sqrt{\pi} \kappa^{5/2} + 2 \kappa^{7/2} + O(\kappa^{9/2}) \right]. \] (35)

The equivalent result of MR was

\[ \Omega \sim \kappa^{1/2} - \alpha^2 \sqrt{\pi} \kappa^{5/2} + 2 \kappa^{7/2} + O(\kappa^{9/2})]. \] (36)

Figure 1. The blue (bottom) line is the present result, the red (middle) line is the result based on MR’s formula (36) and the green (top) line is \( \Omega^2 = \sqrt{\kappa} \) for \( \alpha^2 = 0.033 \)

At small \( \kappa \) the differences between our results and those of MR are small. It is clear that including turbulent background flows can give reductions in the \( f \)-mode frequencies as large as those observed. At larger \( \kappa \) we expect the differences between our results and those of MR to be more significant - the numerical calculation of the full result is work currently in progress.

6. ACKNOWLEDGMENTS

The authors acknowledge the financial support from NFS Hungary (OTKA, TO43741). RE acknowledges M. Káray for patient encouragement. AK also acknowledges the support received from LOC of the SOHO 13 conference, Palma de Mallorca, where this work was presented.

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