THE SEISMIC RADIUS OF THE SUN, AND STRUCTURE INVERSIONS

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ABSTRACT

It is known (Schou et al., 1997; Antia, 1998) that the effective radius of the Sun determined by f-mode frequencies is different by a few hundredths per cent from the photospheric radius determined by direct photometric measurement (Brown and Christensen-Dalsgaard 1998). It is fair to say that we still do not fully comprehend the implications of the difference, save that the two radii are rather different quantities: the radius inferred from f-mode frequencies is determined by the location of the maximum in the f-mode energy (Gough, 1993), whereas the photospheric radius is determined by extrapolation to some prescribed optical depth from a fiducial point in the the limb-darkening function using a theoretical solar model. Both depend in particular on the structure of the upper superadiabatic convective boundary layer, the physics of which is not well understood. In this report we attempt to shed some light on the difference by determining a seismic radius from f-mode frequencies; the outcome depends predominantly on the variation of sound speed, and it is consistent with the f-mode value (Takata and Gough 2001). By considering the mathematical structure of an inversion process that does not explicitly distinguish f modes from p modes, we offer an interpretation of the seismic radius. This interpretation has led us to revise the method by which we carry out structure inversions.

Key words: solar radius; inversion; p modes.

1. INTRODUCTION

It has been pointed out that the f-mode frequencies suggest a value of the solar radius which is slightly different from the photospheric radius (Schou et al., 1997; Antia, 1998). Based on these analyses, Basu (1998) has demonstrated the sensitivity of the inferred seismic radius on some details of the inversions, and Antia et al. (2000) and Dziembowski et al. (2001) have sought to determine a variation with the solar cycle. In addition, we have shown that p-mode frequencies can also be used to determine a seismic radius (Takata & Gough 2001), although the difference between that and the f-mode radius is not entirely clear.

It is important to be explicit about the meanings of the solar radius determined by different methods, for, in particular, the result of the direct measurement of the photospheric radius recorded by Allen (1973), and its recent revision by Brown and Christensen-Dalsgaard (1998), is not consistent with the f-mode radius. It is not surprising at all that there is such an inconsistency, because the photosphere, which is defined in terms of optical depth, is not a special place for surface gravity waves, nor for acoustic waves, which are the constituents of the seismic oscillations. Schou et al. (1997) and Dziembowski et al. (2001) base their discussions on an interpretation by Gough (1993) that the f-mode radius is determined by the location of the peak of the kinetic energy density of the f modes. From this interpretation, it is evident that a naively defined f-mode radius can differ from one f mode to another. To relate them requires knowledge of the stratification of the outer layers of the Sun. This raises an issue which motivates us to discuss an interpretation of the p-mode radius, and its difference from the f-mode radius.

The rest of this paper consists of three sections. In section 2, we interpret the p-mode radius by examining formulae of the structure inversion that take uncertainties in the solar radius and the gravitational constant into account. We demonstrate the importance of the difference between the radius of the Sun and that of the reference solar model, in the light of which we revise, in section 3, our inversions for the sound speed.

2. SEISMIC RADIUS INFERRED BY P-MODE FREQUENCIES

We base our discussion of structure inversions on optimally localized averaging (OLA) (Backus and Gilbert, 1968), partly because it is one of the widely
used methods in the field, and partly because in some ways it is the most readily interpretable.

Takata & Gough (2001) have extended the standard formula for the differences $\delta \nu_i$ between the oscillation frequencies of the Sun and the eigenfrequencies of a reference theoretical model to include possible differences, ignored in conventional structure inversions, in radius $R$ and in the gravitational constant $G$. We present here an essentially equivalent (but more compact) formula:

$$
\frac{\delta \nu_i}{\nu_i} = \int K_{c,\rho}^i \frac{\delta_x(c/r)}{c/r} \, dx + \int K_{\rho,c}^i \frac{\delta_x(G\rho)}{G\rho} \, dx \ . \tag{1}
$$

The meanings of the symbols in equation (1) are as followings: $\nu_i$ is the frequency of mode $i$, where the index $i$ refers to a set of the mode parameters $(n, l)$ ($n$ is the radial order and $l$ is the angular degree); $c$ is sound speed; $r$ is radius (the distance from the centre); $\rho$ is density; $x$ is the fractional radius $r/R$, where $R$ is a fiducial radius of the Sun whose meaning we discuss later; $\delta_x f$ means the difference in some quantity $f$ between the Sun and the reference model at the same fractional radius $x$, and $K_{c,\rho}^i$ and $K_{\rho,c}^i$ are corresponding kernels. Although a surface term is usually added to the right-hand side of equation (1) to take account of various uncertainties in the subsurface region of the Sun, we omit it here because it is not important in the discussions in this section. However, it is actually included in the structure inversions presented in section 3. We should point out that the radius difference $\delta R$ is implicitly included in equation (1) through the definitions of the quantities $\delta_x(c/r)$ and $\delta_x(G\rho)$; in fact, these quantities are not well defined without specifying the radius difference.

We note that Gough & Kosovichev (1993) have used a formula similar to equation (1) in the context of asteroseismology, where neither the radius nor the mass of the star is known.

Another basic equation used in structure inversions is the total mass constraint, which is usually included to ensure that the inversions are consistent with the observed value of the solar mass. Its extended version, which includes differences in the radius $R$ and the product $GM$ of the gravitational constant and the total mass, is given by

$$
\frac{\delta(GM)}{GM} = \int \frac{4\pi R x^2 \rho}{M} \frac{\delta_x(G\rho)}{G\rho} \, dx + 3 \frac{\delta R}{R} \ . \tag{2}
$$

Note that $\delta R$ and $\delta(GM)$ appear explicitly here.

Since the form of equation (2) is similar to that of equation (1), in conventional inversions for the solar structure these equations are usually treated equivalently with no caution. It will become evident later that such treatment is justified only if the fractional radius difference $\delta R/R$ is negligible.

So far we have not specified what $R$ means physically in equations (1) and (2). In fact it has been introduced simply as a scaling factor. To assist in thinking about it, we first draw attention to the following annihilator relation:

$$
\int K_{c,\rho}^i \frac{d\ln(c/r)}{d\ln r} \, dx + \int K_{\rho,c}^i \frac{d\ln \rho}{d\ln r} \, dx = 0 \ , \tag{3}
$$

reflecting the property that adiabatic eigenfrequencies of stars are invariant under uniform stretching of the radial coordinate provided that the profiles of all dependent variables are appropriately scaled homologously. It implies the important fact that there is a series of (isospectral) structures that cannot be distinguished by their adiabatic eigenfrequencies alone.

In practice, however, one obtains results from OLA with no apparent ambiguity. Therefore the stretching factor is determined implicitly by the procedure. We need to know which specific value is chosen.

In summary, we have two questions to answer here: (i) How can we know the stretching factor, $1+\delta R/R$, that is implicitly determined by the OLA method? (ii) What kind of principle is adopted in the process to pick up this specific value of the stretching factor? The total mass constraint (2), which is a physically different condition from the frequency equation (1), immediately gives us an answer to the first question. Once we have the density profile $\delta_x(G\rho)/(G\rho)$ without knowing what $R$, hence $x$, is, we can substitute this profile into equation (2) to get $\delta R/R$. Practically, we could perform another OLA inversion for $\delta R/R$ based on equations (1) and (2), which is what was actually done by Takata & Gough (2001). From a physical point of view, we can determine the size of the target star, which is otherwise ambiguous owing to the undetermined stretching of the radial coordinate, by constraining its total mass. Thus we propose here a procedure consisting of the following two steps: (step 1) carry out an inversion based on only equation (1), and (step 2) supplement this with an inversion for $\delta R$ based on both equation (1) and equation (2). This answers question (i); but we need to know more about the mathematics behind the OLA method before we can answer question (ii).

In the OLA method, we make inferences about quantities such as sound-speed differences (and/or density differences) in the vicinity of some specific point from integrals of the product of those quantities and appropriate averaging kernels, which are constructed as linear combinations of the kernels for each mode. Because relation (3) can be interpreted as the vanishing of the inner product of the kernel vector

$$
\begin{pmatrix}
K_{c,\rho}^i \\
K_{\rho,c}^i
\end{pmatrix}
$$

and the annihilator vector

$$
\begin{pmatrix}
\frac{d\ln(c/r)}{d\ln r} \\
\frac{d\ln \rho}{d\ln r}
\end{pmatrix}
$$

for all modes, we can say that the averaging kernels are orthogonal to the annihilator vector. This means that the inferences made by the OLA method without the total mass constraint (2) are never sensitive
to the annihilator component (5) of the target quantities \( \delta_z(c/r)/(c/r) \) and/or \( \delta_z(G\rho)/(G\rho) \). We can understand this property by confining attention to the projection of \( \delta_z(c/r)/(c/r) \) and/or \( \delta_z(G\rho)/(G\rho) \) into the function space spanned by the kernels \( K_{c,\rho} \) and \( K_{p,c} \). Then the inferences of the structure differences by the OLA method are essentially averages of the following expansion:

\[
\begin{pmatrix}
\delta_z(c/r) \\
\delta_z(G\rho)/(G\rho)
\end{pmatrix} = \sum_i A_i \begin{pmatrix}
K_{c,\rho}^i \\
K_{p,c}^i
\end{pmatrix},
\]

in which the coefficients \( A_i \) are constant. Equation (6) is known as the spectral decomposition. Substituting into this equation the identity

\[
\begin{pmatrix}
\delta_r C \\
\delta_r(G\rho)/(G\rho)
\end{pmatrix} = \begin{pmatrix}
\delta_z(c/r) \\
\delta_z(G\rho)/(G\rho)
\end{pmatrix} - \delta R \begin{pmatrix}
\frac{d\ln(c/r)}{d\ln r} \\
\frac{d\ln(G\rho)}{d\ln r}
\end{pmatrix},
\]

where \( \delta_r \) is the operator such that \( \delta_r f \) means the difference in quantity \( f \) between the target structure and the reference model at the same radius \( r \), we obtain the following relation:

\[
\begin{pmatrix}
\delta_r C \\
\delta_r(G\rho)/(G\rho)
\end{pmatrix} = \sum_i A_i \begin{pmatrix}
K_{c,\rho}^i \\
K_{p,c}^i
\end{pmatrix} - \delta R \begin{pmatrix}
\frac{d\ln(c/r)}{d\ln r} \\
\frac{d\ln(G\rho)}{d\ln r}
\end{pmatrix},
\]

the left-hand side of which is independent of \( \delta R/R \).

We can interpret the last term on the right-hand side of equation (8) as the annihilator term (5), which does not contribute to frequency differences at all. Taking the scalar product of both sides of equation (8) with the annihilator term (5), and integrating, yields, with the help of equation (3),

\[
\delta R = \lim_{x_0 \to x_s} \left( \int_0^{x_0} \frac{\delta_r C}{c} \frac{d\ln(c/r)}{d\ln r} + \frac{\delta_r(G\rho)}{G\rho} \frac{d\ln(G\rho)}{d\ln r} \right) dx,
\]

where \( x_s \) is the value of \( x \) at the ‘surface’ \( r = R \) of the star. If, as in a realistic stellar atmosphere, \( d\ln\rho/d\ln r \) almost diverges as \( x \to x_s \) whereas \( d\ln c/d\ln r \) stays finite, equation (9) reduces to

\[
\delta R \simeq \lim_{x_0 \to x_s} \left( \int_0^{x_0} \frac{\delta_r(G\rho)}{G\rho} \frac{d\ln\rho}{d\ln r} \right) dx = \lim_{x \to x_s} \frac{\delta_r(G\rho)}{G\rho} \frac{d\ln\rho}{d\ln r} \frac{d\ln\rho}{d\ln r}
\]

if \( \delta_r(G\rho)/G\rho \) is slowly varying. If, in addition, the differences in the structure are homologous, this expression tells us that \( 1 + \delta R/R \) is equal to the homology factor, and that the conventional OLA method (without the total mass constraint) should yield \( \delta_z(c/r)/(c/r) = \delta_z(G\rho)/(G\rho) = 0 \). Moreover, it is evident from equation (10) that the density profile near the surface is crucial in determining the p-mode radius, as indeed it is for the f-mode radius. Therefore the p-mode radius and the f-mode radius have something in common. However, we still have to think about what aspect of the surface density structure is most important. Because p modes are reflected at their upper turning points near the surface, we may naively think that the density profile in the vicinity of those turning points is the most important. If this interpretation is correct, then it must be that the p-mode radius is determined by the modes with the highest frequencies, because the upper turning point moves outward as frequency increases.

Now we consider the conventional method of structure inversion, in which the total mass constraint (2) is not distinguished from the frequency condition (1). By mixing those two kinds of constraint, we implicitly assume the following expansion of the seismically accessible component of the structure:

\[
\begin{pmatrix}
\delta_r C \\
\delta_r(G\rho)/(G\rho)
\end{pmatrix}_{\text{conv}} = \sum_i B_i \begin{pmatrix}
K_{c,\rho}^i \\
K_{p,c}^i
\end{pmatrix} + B_* \begin{pmatrix}
0 \\
\frac{4\pi R^2 \rho^2}{M}
\end{pmatrix},
\]

where the coefficients \( B_i \) and \( B_* \) are constants. Note that the last term on the right-hand side of this equation is relatively small at the surface, unlike the annihilator component (5), which is extremely large (possibly even divergent). Because the differences \( \delta_r C/c \) and \( \delta_r(G\rho)/(G\rho) \) generally contain an annihilator component, it is not a good idea to use equation (11) for representing such large differences at the surface. It will be seen in section 3 that so long as \( \delta R/R \) is comparable with the fractional sound-speed differences one is liable to generate large errors in localized averages constructed by the conventional method.

We can still reinterpret the conventional inversions if they are performed without explicit use of the total mass constraint. In that case, we should replace the labels of the profiles \( \delta_r C/c \) and \( \delta_r(G\rho)/(G\rho) \) by \( \delta_z(c/r)/(c/r) \) and \( \delta_z(G\rho)/(G\rho) \) respectively. Note that we cannot then know the radius difference, which is required for the operator \( \delta_z \) to be well defined, without carrying out the additional inversion for \( \delta R \) using the total mass constraint as well.

3. REVISED SOUND-SPEED INVERSIONS

As described in the previous section, we can divide the inversion process into two steps to take proper account of the radius difference. Here we demonstrate that such treatment is actually needed in practice. We concentrate on inversions for sound speed.

We first demonstrate how different from conventional inversions the new inversion are by performing inversions of eigenfrequencies of a known test model.

The test model is made by shrinking model S of Christensen-Dalsgaard et al. (1996) homologously.

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by 0.0667%. Eigenfrequencies of this model were calculated for all modes included in the SOHO/MDI 360-day data set (Schou 1999), to provide artificial data for test inversions using the raw model S as the reference. The results are shown in the left panel of Figure 1. One can see that with only a small homologous difference there can be substantial error in the inversions. The conventional inversion with the total mass constraint is inaccurate especially in the outer regions of the star (r/\(R_\odot\) > 0.7). On the other hand, without the total mass constraint the conventional inversion is in error by simply a constant, \(2\delta R/R\), as we anticipated at the end of section 2.

Next we invert real data, namely the SOHO/MDI 360-day data set (Schou 1999). The results are shown in the right panel of Figure 1. The constant difference between the revised inversions and the conventional inversions corresponds to \(2\delta R/R\), which has been estimated to be \(-8.6 \times 10^{-4}\) by Takata & Gough (2001). Note that we have plotted \(\delta(e_{c}^2/c^2)\) instead of \(\delta e_{c}^2/c^2\). The relation between these quantities is given by equation (7). This is why the conventional inversions with the total mass constraint look very different from others’ results in the literature. In fact, the \(\delta(e_{c}^2/c^2)\) profile inferred from conventional inversions carried out with the total mass constraint is quite close to the \(\delta(e_{c}^2/c^2)\) profile of the revised inversion. We stress, however, that we have to distinguish the meanings of \(\delta(e_{c}^2/c^2)\) and \(\delta e_{c}^2/c^2\).

4. CONCLUSION

We have discussed the inference of the seismic radius determined by p-mode and f-mode frequencies of the Sun. The fractional radius difference determined by OLA-type inversions can be interpreted in terms of an homologous component of the difference in the density profiles of the Sun and the reference solar model particularly near the surface. Conventional inversions should be revised to take account of the radius difference appropriately.

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