Time-Dependent Moment Equation Method for Supernova Lightcurves

Eric J. Lentz
Dept. of Physics and Astronomy & Center for Simulational Physics
University of Georgia, Athens, GA 30602, USA

E. Baron
Department of Physics and Astronomy, University of Oklahoma
440 W. Brooks St., Norman, OK 73019, USA

Peter H. Hauschildt
Dept. of Physics and Astronomy & Center for Simulational Physics
University of Georgia, Athens, GA 30602, USA

Abstract. We have developed a time-dependent method using the moment equations to solve for the temperature structure and radiation field of objects in which the temperature and structure change slowly relative to the radiation field, e.g., supernovae.

1. Introduction

There are many astrophysical objects which have important time variable properties. In many of these objects the radiation field is coupled to the matter and plays a significant role in the dynamics of the material properties. We have developed this technique to complement and extend our previous spectral modeling of supernovae. In supernovae, the gravitational and radiative forces are small and decrease rapidly as the ejecta expand. These effects are important during the first few days after the explosion when the supernova is faint but brightening and quickly become negligible. Without external forces, the ejecta expand ballistically and can be treated as a homologous expansion, \( r = vt \). The emphasis in calculating the energy flows and the emitted spectral energy distribution of supernovae is thus on the opacities and radiative transport.

A single time step in a light curve calculation differs primarily from a quasi-static atmosphere by the removal of two assumptions: energy balance and a specified luminosity. To include the effects of the flow of radiative energy through the atmosphere we have extended the Unsöld-Lucy to include the time effects. This method is a small refinement of the method developed by Lentz (2000). Additional modifications to the treatment of the source function and formal solution of the transport equation to include the effect of the time-derivative of the intensity will be reported in a later paper.
2. Time Dependent Temperature Correction

We begin with the first moment of the spherically symmetric radiative transfer equation

\[
\frac{\partial (f qr^2 J)}{\partial r} = -q \left[ r^2 \chi_F H + \frac{1}{c} \frac{D}{Dt} r^2 H \right],
\]

(1)

where we define the flux mean extinction, Lagrangian derivative,

\[
\chi_F \equiv \frac{1}{H} \int \chi_\nu H_\nu d\nu, \quad \frac{1}{c} \frac{D}{Dt} = \frac{1}{c} \frac{\partial}{\partial c} + \frac{v}{c} \frac{\partial}{\partial r},
\]

(2)

the Eddington factor \( f_\nu = K_\nu / J_\nu \), and the sphericity factor, \( q \) such that

\[
\ln(r^2 q_\nu) = \int_{r_{\text{core}}}^{r} \frac{3f_\nu - 1}{r' f_\nu} dr' + \ln(r^2_{\text{core}})
\]

(3)

and \( r_{\text{core}} \) is the inner radius of the model atmosphere. The zeroth moment equation of the RTE to \( O(\beta) \) is written

\[
\kappa_J r^2 J = \kappa_P r^2 S - \frac{\partial}{\partial r} r^2 H - \frac{1}{c} \frac{D}{Dt} r^2 J.
\]

(4)

Integrating Eqn. (1) and substituting Eqn. (4) gives

\[
\kappa_P r^2 S(r) = -\frac{\kappa_J}{f(r)q(r)} \int_{r}^{R} q(r') \left[ r'^2 \chi_F (r') H(r') + \frac{q(r')}{c} \frac{D}{Dt} (r'^2 H(r')) \right] dr' + 2\kappa_J R^2 H(R) + \frac{\partial}{\partial r} (r^2 H(r)) + \frac{1}{c} \frac{D}{Dt} (r^2 J(r)).
\]

(5)

We have made the approximation for the surface terms \( f(r)q(r) \approx f(R)q(R) \) and the second Eddington approximation, \( J(R) = 2H(R) \), where \( R \) is the outer radius. Now write the equation for the ideal, converged case where \( S(r) = B(r) + \Delta B(r) \), noting that the divergence of the luminosity plus the Lagrangian derivative of the mean intensity is the net heating, \((\partial/\partial r)r^2 H + (1/c)(\partial/\partial t)r^2 J = r^2 Q/4\pi \), where \( H_0 \) is the ‘target’ Eddington flux and \( Q \) is the net heating of the gas by radiation (see Sect. 3).

\[
\kappa_P r^2 [B(r) + \Delta B(r)] = -\frac{\kappa_J}{f(r)q(r)} \int_{R}^{r} q(r') \left[ r'^2 \chi_F (r') (H_0(r') - H(r')) \right. \\
+ \left. \frac{1}{c} \frac{D}{Dt} (r'^2 H_0(r')) \right] dr' + 2\kappa_J R^2 H_0(R) - \frac{r^2 Q}{4\pi}
\]

(6)

and subtract Eqn. (5) to get the approximate expression for \( \Delta B \),

\[
\kappa_P r^2 \Delta B(r) \approx -\frac{\kappa_J}{f(r)q(r)} \int_{R}^{r} q(r') \left[ r'^2 \chi_F (r') (H_0(r') - H(r')) \right. \\
+ \left. \frac{1}{c} \frac{D}{Dt} (r'^2 (H_0(r') - H(r'))) \right] dr' \\
+ 2\kappa_J R^2 (H_0(R) - H(R)) - \frac{\partial}{\partial r} (r^2 H(r)) - \frac{1}{c} \frac{D}{Dt} (r^2 J(r)) - \frac{r^2 Q}{4\pi}.
\]

(7)
Writing the equation for the converged case \((\partial/\partial r)\mathcal{H} + (1/c)(D/Dt)\mathcal{J} = Q/4\pi\). We have simplified the expression to use the "r^2" forms of the moments that include the \(r^2\) terms, i.e. \(r^2\mathcal{H} = \mathcal{H}\). The expression for the correction to the Planck function is

\[
\Delta B \approx \frac{1}{\kappa_P} \left( \kappa_J \mathcal{J} - \kappa_P B + \frac{\dot{S} - Q}{4\pi} \right) - \frac{\kappa_J}{\kappa_P} \left\{ 2(\mathcal{H}(R) - \mathcal{H}_0(R)) + \frac{1}{f_q} \int_{r}^{\infty} \left[ \chi_F(\mathcal{H} - \mathcal{H}_0) + \frac{1}{c} \frac{D}{Dt}(\mathcal{H} - \mathcal{H}_0) \right] dr' \right\},
\]

where \(\dot{S} = r^2 \dot{S}\) is the energy deposited by gamma-rays. This derivation follows the quasi-static method (c.f. Hauschildt & Baron 1999) adding only the \(Q\) and time-derivative of the deviation from the target flux.

3. The Missing Quantities

Unlike the static or equilibrium implementations of the Unsöld-Lucy method, we can no longer take energy balance as a constraint and the bolometric luminosity as an input parameter. The flow of energy between the radiation field and the matter is now another quantity that must be iterated with the other physical quantities.

The equation for matter/energy conservation is written as (from Mihalas & Mihalas 1984, Eqn. 96.7)

\[
\rho \left[ \frac{1}{c} \frac{De}{Dt} + \frac{p}{c} \frac{D}{Dt} \left( \frac{1}{\rho} \right) \right] = \dot{S} - Q,
\]

where \((1/c)(D/Dt) = (1/c)(\partial/\partial t) + \beta(\partial/\partial r)\) is the Lagrangian derivative, \(e\) is the matter energy per gram, \(p\) is the gas pressure, and \(\rho\) is the density, and the heating of the gas by radiation, \(Q\), is

\[
Q = \int (4\pi \eta_\nu - \chi_\nu J_\nu) d\nu.
\]

\(Q\) can also be expressed in terms of the matter quantities as

\[
Q = \dot{S} - \rho \left[ \frac{1}{c} \frac{De}{Dt} + \frac{p}{c} \frac{D}{Dt} \left( \frac{1}{\rho} \right) \right].
\]

The latter definition is likely to provide a more stable solution when far from convergence. For quasi-static models the target flux \(\mathcal{H}_0\), is transformed from the observers to the material frame. The observers frame flux is the fundamental result of a light curve calculation—the bolometric luminosity versus time. For the time-dependent case we numerically integrate the zeroth moment for the flux using the moments from the previous, converged, time-step and the current iteration of the current time-step.
4. Discussion and Conclusions

The time-dependent moment modification to the Unsöld-Lucy temperature correction procedure is similar to the static implementations of the Unsöld-Lucy temperature correction method. With the implicit method chosen, the time steps are limited by the changes in the relevant variables (radius, temperature, moments, etc.) so that the iterated solution at the forward time-step is physically meaningful. When the solution is approached, the corrections will naturally become small and the approximations made do not affect the converged result.

The integration of the emergent flux requires similar stability concerns as the energy balance in the modified Unsöld-Lucy method. As the model converges, the moment equation will be more closely balanced. If changes in the physical constants are too large, especially the moments, the flux specified may become computationally unstable with feedback into the temperature correction. These criteria are similar for the stability of the temperature correction equation without the effects on the flux. The most important criterion is the accurate computation of the energy terms as small errors in the energy equation are multiplied in the emergent luminosity which is the change in energy for the atmosphere. The method is being implemented into the PHOENIX general purpose model atmosphere and radiative transport code (Hauschildt & Baron 1999) and will require rigorous testing to determine the resolution required in time, velocity, density, space, and optical depth for accurate solutions of the energy equations.

References

Hauschildt, P. H., & Baron, E. 1999, J. Comp. Applied Math., 109, 41