SWING COUPLING OF SOUND AND ALFVÉN WAVES

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ABSTRACT

The weakly nonlinear interaction of sound and linearly polarized Alfvén waves propagating along an applied magnetic field is studied in the case where the sound and Alfvén speeds are equal. The sound wave is coupled to an Alfvén wave with double period and wavelength. The Alfvén wave drives the sound wave due to the ponderomotive force, while the sound wave returns the energy back to the Alfvén wave through the swing wave-wave interaction (Zaqarashvili, 2001). As a result, the waves alternately exchange their energy during the propagation. The process of energy exchange is faster for waves with stronger amplitudes. The phenomenon can be of importance in the solar atmosphere, solar wind and other astrophysical situations.

Key words: plasma; MHD waves; mode coupling.

1. INTRODUCTION

Alfvén waves with varying magnetic field strength (e.g., linearly and elliptically polarized waves) generate density fluctuations due to the ponderomotive force. In the case where the Alfvén speed \(v_A\) is different from the sound speed \(v_s\), the frequency and spatial inhomogeneity of generated density fluctuations do not satisfy the dispersion relation of sound waves, and therefore cannot propagate as waves, but only affect the Alfvén speed. However, when \(v_A = v_s\), then the density perturbations may propagate as sound waves with double the frequency and wavenumber of Alfvén waves. The solution for forced sound waves undergoes a discontinuity in this particular case. Consequently, it has been suggested that the amplitude of sound waves may become extremely large (Hollweg, 1971). However, if one considers the swing (parametric) energy transmission from sound waves into Alfvén waves (Zaqarashvili, 2001), then wave coupling may become a complete process. Considering wave amplitudes as slowly varying in time, the analytical solution of nonlinear MHD equations can be found in the particular case of \(v_A = v_s\). Instead of discontinuity, here a long term modulation of wave amplitudes can be demonstrated. It indicates that the waves alternately exchange their energy during the propagation.

In the next section we present the mathematical formalism and physics of the coupling, and then briefly describe several possible astrophysical applications.

2. MATHEMATICAL FORMALISM AND PHYSICS OF WAVE COUPLING

Consider wave propagation in a homogeneous medium along a uniform magnetic field \(\mathbf{B} = (B_0,0,0)\) directed along the \(x\) axis of a cartesian coordinate system \(x,y,z\). The Alfvén wave is polarized in the \(y\) direction. Then the ideal magnetohydrodynamic (MHD) equations take the form:

\[
\frac{\partial b_y}{\partial t} + u_x \frac{\partial b_y}{\partial x} + b_y \frac{\partial u_x}{\partial x} - B_0 \frac{\partial b_y}{\partial x} = 0,
\]

\[
\frac{\partial u_y}{\partial t} + \frac{\rho u_x b_y}{4 \pi} \frac{\partial b_y}{\partial x} - \frac{B_0}{\pi} \frac{\partial b_y}{\partial x} = 0,
\]

\[
\frac{\partial b_y}{\partial t} + b_y \frac{\partial u_x}{\partial x} - \frac{B_0}{4 \pi} \frac{\partial b_x}{\partial x} = 0,
\]

\[
\frac{\partial u_z}{\partial t} + \frac{\rho u_x}{\partial x} = \gamma p - \frac{\partial u_z}{\partial x} = 0,
\]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial u_z}{\partial x} + \frac{\partial b_x}{\partial x} = 0,
\]

where \(p\) and \(\rho\) are the total pressure and density, respectively, \(u_x\) and \(u_z\) are velocity perturbations of Alfvén and sound waves respectively, and \(\gamma\) is the ratio of specific heats.

These equations describe the fully nonlinear behaviour of sound waves and linearly polarized Alfvén waves in a homogeneous medium. In the linear limit the waves are strictly different; the Alfvén waves are purely transversal with magnetic tension as the restoring force, while the sound waves are purely longitudinal with pressure gradients as the restoring force. Therefore coupling in a homogeneous medium can occur only through nonlinear interactions.
For a mechanical analogy of this process we recall the swing pendulum (Zaqarashvili & Roberts, 2002). There are two different oscillations of the pendulum: transversal oscillations due to gravity and longitudinal oscillations due to the stiffness of the spring. Transversal oscillations influence longitudinal ones due to the varying gravitational field component along the pendulum axis, while longitudinal oscillations influence transversal ones parametrically through the variation of the pendulum length. When the eigenfrequency of the transversal oscillation is half of the frequency of spring oscillations, then a resonant coupling may occur. Initial transversal oscillations directly drive longitudinal oscillations, which return their energy back to transversal oscillations through the parametric influence. The energy exchange between the oscillations occurs alternately, without dissipation.

A similar process occurs in the case of Alfvén waves and sound waves in a medium. The influence of the Alfvén waves is expressed through the ponderomotive force (the last term in equation (4)) and the influence (parametric) of sound waves is expressed through the nonlinear terms in equations (1) and (2).

2.1. Analytical solution

We express the total density and pressure as the sum of unperturbed and perturbed parts \( \rho_0 + \rho \) and \( \rho_0 + \rho_p \), respectively, and consider the weakly nonlinear process when the perturbations are much smaller than unperturbed values. Then the perturbed physical quantities can be represented as the product of a slowly varying amplitude \( C_j(t) \) and a rapidly oscillating term,

\[
\begin{align*}
\rho &= C_1(t) e^{i\phi_1(t,x)} + C_1^*(t) e^{-i\phi_1(t,x)}, \\
\rho_0 &= C_1(t) e^{i\phi_1(t,x)} + C_1^*(t) e^{-i\phi_1(t,x)}, \\
b_y &= C_2(t) e^{i\phi_2(t,x)} + C_2^*(t) e^{-i\phi_2(t,x)}, \\
b_y &= C_2(t) e^{i\phi_2(t,x)} + C_2^*(t) e^{-i\phi_2(t,x)}, \\
u_x &= C_3(t) e^{i\phi_3(t,x)} + C_3^*(t) e^{-i\phi_3(t,x)}, \\
u_x &= C_3(t) e^{i\phi_3(t,x)} + C_3^*(t) e^{-i\phi_3(t,x)}, \\
p &= C_4(t) e^{i\phi_4(t,x)} + C_4^*(t) e^{-i\phi_4(t,x)}. \\
p &= C_4(t) e^{i\phi_4(t,x)} + C_4^*(t) e^{-i\phi_4(t,x)}.
\end{align*}
\]

(6)

Here

\[
\phi_1 = \omega_A t - k_A x, \quad \phi_2 = \omega_s t - k_s x, \quad \phi_3 = \omega_s t - k_s x,
\]

(7)

where \( \omega_A, k_A, \omega_s, k_s \) are the frequencies and wave numbers of Alfvén and sound waves respectively.

Substitution of these expressions into equations (1)-(5) and averaging over rapid oscillations of phase leads to the cancelling of all exponential terms, so that only the first order and third order (with \( C_j \) terms) remain. In the first approximation (neglecting all third order terms), the sound and Alfvén waves are decoupled and the amplitudes \( C_j \) are constant. Third order terms with \( C_j \) are due to the advective terms in the momentum equation are significant only in the case of very large amplitudes and presumably lead to the steepening of wave fronts and consequently to the formation of shock waves. However, if

frequencies and wave numbers satisfy the conditions

\[
\omega_s = 2\omega_A, \quad k_s = 2k_A 
\]

(8)
or

\[
\omega_s = -2\omega_A, \quad k_s = -2k_A,
\]

then after averaging the second order terms (with \( C_j \) remain and wave amplitudes become time dependent. These conditions are fulfilled only in a medium with \( v_A = v_s \).

When condition (8) is fulfilled, then the averaging of equations (1)-(5) leads to the equations which govern the temporal behaviour of complex amplitudes (note that third order terms with \( C_j \) are neglected):

\[
\dot{C}_1 + i\omega_A C_1 + i k_A B_0 C_3 = 0, \quad (9)
\]

\[
\dot{C}_4 + i\omega_A C_4 + i \frac{B_0}{4\pi\rho_0} k_A C_3 + \frac{C_1}{\rho_0} C_4^* - i \omega_A C_2^* C_1 + i k_A C_2 C_3^* = 0, \quad (10)
\]

\[
\dot{C}_1 + 2i\omega_A C_1 - 2ik_A \rho_0 C_3 = 0, \quad (11)
\]

\[
\dot{C}_2 + 2i\omega_A C_2 - 2ik_A \frac{C_3}{\rho_0} - \frac{2ik_A C_3^2}{8\pi\rho_0} = 0, \quad (12)
\]

\[
\dot{C}_3 + 2i\omega_A C_3 - 2i\frac{\rho_0}{\rho_A} k_A C_2 = 0. \quad (13)
\]

The direct nonlinear influence of the Alfvén waves is expressed by the last term in equation (12) and the parametric influence of the sound waves is expressed through the last term in equation (9) and the last three terms in equation (10). Dropping second order terms in these equations leads to the linear system and thus to the decoupling of sound and Alfvén waves.

We now search for the solution of equations (9)-(13) in the form

\[
C_1 = (C_{10} + iC_{11}) e^{i\delta t}, \quad (9a)
\]

\[
C_2 = (C_{20} + iC_{21}) e^{i\delta t}, \quad (10a)
\]

\[
C_3 = (C_{30} + iC_{31}) e^{i\delta t}, \quad (11a)
\]

\[
C_4 = (C_{40} + iC_{41}) e^{i\delta t}, \quad (12a)
\]

\[
C_5 = (C_{50} + iC_{51}) e^{i\delta t}. \quad (13a)
\]

(14)

Note that the imaginary part of \( \delta \) indicates a growth (or decay), while the real part indicates a modulation of the amplitudes.

Substitution of expressions (14) into equations (9)-(13) and splitting into real and imaginary parts leads to the system

\[
(\delta + \omega_A) C_{31} + k_A B_0 C_{41} + k_A C_{20} C_{31} - k_A C_{21} C_{30} = 0, \quad (9c)
\]

\[
(\delta + \omega_A) C_{30} + k_A B_0 C_{40} - k_A C_{20} C_{30} - k_A C_{21} C_{31} = 0, \quad (10c)
\]

\[
(\delta + \omega_A) \left( 1 + \frac{C_{10}}{\rho_0} \right) C_{41} - \frac{1}{\rho_0} (\delta + \omega_A) C_{11} C_{40} + \frac{B_0}{4\pi\rho_0} C_{31} + k_A (C_{21} C_{40} - C_{20} C_{41}) = 0, \quad (13c)
\]
\[
(\delta + \omega_1) \left( 1 - \frac{C_{11}}{\rho_0} \right) C_{40} - \frac{1}{\rho_0} (\delta + \omega_1) C_{11} C_{41} + \\
\frac{B_0 k_A}{4\pi \rho_0} k_A (C_{20} C_{40} + C_{21} C_{41}) = 0,
\]
\[
(\delta + \omega_1) C_{11} - k_A C_{11} = 0,
\]
\[
(\delta + \omega_1) C_{10} - \rho_0 C_{20} = 0,
\]
\[
(\delta + \omega_1) C_{21} - \frac{C_{21}}{\rho_0} - \frac{k_A}{8\pi \rho_0} 2C_{30} C_{31} = 0,
\]
\[
(\delta + \omega_1) C_{30} - \frac{k_A}{8\pi \rho_0} (C_{30} - C_{31}^2) = 0,
\]
\[
(\delta + \omega_1) C_{31} - \gamma_0 k_A C_{21} = 0,
\]
\[
(\delta + \omega_1) C_{30} + \gamma_0 k_A C_{20} = 0.
\]

From these equations we find an expression for \(\delta\):
\[
\delta = -\omega_1 \left[ \frac{C_{11}}{\rho_0} \right] \left( 1 + \frac{1}{1 - \frac{C_{11}}{\rho_0}} \right). \quad (15)
\]

Note that \(\delta\) is always real (as \(\frac{C_{11}}{\rho_0} \ll 1\)) so there is no instability; the waves exchange energy

during propagation.

Because of the slowly varying amplitudes (i.e., \(\delta \ll \omega_1\)) only the solution with sign – is valid in expression (15). Therefore we finally obtain
\[
\delta \approx \frac{\omega_1}{2} \frac{|C_{11}|}{\rho_0}. \quad (16)
\]

Thus the amplitudes of the waves are modulated by the

low frequency \(\delta\). The ratio between low and high

frequencies, \(\delta/\omega_1\), depends on the initial amplitude

\(C_{11}\).

The approximate relation between the amplitudes of sound and Alfvén waves is
\[
\frac{|C_{11}|}{\rho_0} \approx \frac{|C_{30}|}{B_0}. \quad (17)
\]

### 2.2. Numerical results

Here we present the numerical simulation of the differential equation system (9)-(13). The long term evolution of the energies of sound \(E_s\) and Alfvén \(E_A\) waves is plotted on Figure 1. Initially, only an Alfvén wave is present. Consequently the energy transforms into the energy of sound waves. Then the sound waves return the energy back to the Alfvén wave.

Comparison of numerical results with the analytical expression for modulation reveals good agreement. The period of modulation depends on the initial amplitude: the process of energy exchange is faster for waves with larger amplitudes.

![Figure 1. The long term evolution of the energies of sound, \(E_s\), and Alfvén, \(E_A\), waves is shown. Initially all the energy resides in the Alfvén wave. An alternating energy exchange between the waves is clearly seen. Note that the process is averaged by rapid oscillations. The time is normalized by the frequency \(\omega_1\) of the Alfvén wave.](image)

### 3. DISCUSSION

The weakly nonlinear interaction between sound and linearly polarized Alfvén waves propagating along an applied magnetic field in a homogeneous medium is studied. Representing the wave solutions as a combination of slowly varying amplitudes and rapidly oscillating (in space and time) terms, we found complete wave coupling when \(v_A = v_s\). It is shown that an averaging of the nonlinear MHD equations by rapid oscillations in phase leads to time dependent amplitudes when the frequencies and wave numbers of the waves satisfy the conditions

\[
\omega_A = 2\omega_1, \quad k_A = 2k_A,
\]

where \(\omega_A, k_A\) and \(\omega_s, k_s\) are frequencies and wave numbers of Alfvén and sound waves respectively. The amplitudes of such waves are modulated by a longer period, indicating an alternating energy exchange between the waves (see Fig. 1). Waves with stronger amplitudes cause a faster process of energy exchange.

Note that only two waves take part in the process of coupling, the sound and linearly polarized Alfvén waves propagating in the same direction. Therefore the process can be called a two-wave interaction or wave interaction. The process is basically different from the well known three-wave interaction (Sadleev & Galsley, 1969). In the process of strong coupling the Alfvén wave drives the sound waves through the ponderomotive force, while the sound waves return their energy through the parametric action (Zaqarashvili, 2001; Zaqarashvili & Roberts, 2002).

Here we considered only the simplest case of wave coupling in the medium with \(\beta = v_s/v_A = 1\), i.e.,
when waves propagate with the same speed $v_A = v_s$ along the magnetic field. However, the same phenomenon may arise also in the case of $\beta \neq 1$, but between Alfvén and obliquely propagating magnetosonic waves.

The phenomenon can be of importance in the solar atmosphere, solar wind and various astrophysical and laboratory situations.

3.1. Damping of p-modes and consequent amplification of Alfvén waves in the solar atmosphere

The solar p-modes (sound waves) may be effectively damped in regions of the solar atmosphere with $\beta \approx 1$. The solar surface is covered with various magnetic flux tubes expanding outwards towards the corona. The sound waves may transform the energy into the Alfvén waves (or possibly into kink waves) at such regions due to swing coupling. Generated transversal waves may propagate through the transition region into the corona, causing the 5-minute intensity oscillations recently observed by TRACE (De Moortel et al., 2002). Due to the same mechanism, the 3-minute oscillations above active regions may penetrate into the corona in the form of Alfvén waves and cause the intensity oscillations to return their energy back into density perturbations.

3.2. Wave coupling in the solar wind

Swing wave coupling can occur in the solar wind. Alfvén waves propagating from the Sun may effectively transform their energy into density perturbations at the $\beta \approx 1$ region near the Earth. The coupling may also occur at $\beta \neq 1$ regions but now between obliquely propagating magnetosonic waves and Alfvén waves. However, this process requires future study.

REFERENCES