SWING COUPLING BETWEEN FAST MAGNETOSONIC AND ALFVÉN WAVES

T.V. Zaqarashvili\(^1\) and B. Roberts\(^2\)

1. Abastumani Astrophysical Observatory, Al. Kazbegi ave. 2a, 380060 Tbilisi, Georgia
2. School of Mathematics and Statistics, University of St Andrews, St Andrews, Fife KY16 9SS, Scotland

We suggest a mechanism of energy transformation from fast magnetosonic waves propagating in homogeneous medium across a uniform magnetic field to Alfvén waves propagating along the field. The mechanism is based on swing wave-wave interaction [T.V. Zaqarashvili, Astrophys. J. Lett. 552, 107 (2001)]. The standing fast magnetosonic waves cause a periodical variation in the Alfvén speed, with the amplitude of an Alfvén wave being governed by Mathieu’s equation. Consequently, sub-harmonics of Alfvén waves with a frequency half that of magnetosonic waves grow exponentially in time. It is suggested that the energy of nonelectromagnetic forces, which are able to support the magnetosonic oscillations, may be transmitted into the energy of purely magnetic oscillations. Possible astrophysical applications of the mechanism are briefly discussed.

PACS numbers: 55.35.Mw

I. INTRODUCTION

Developments in plasma theory raise interest in the study of interactions between different waves. It is shown that nonlinear interaction leads to the generation of resonant triplets (or multiplets) in the plasma [1]. The nonlinear interaction between magnetohydrodynamic (MHD) waves has been studied in various astrophysical situations [2–4]. Additionally, MHD wave coupling due to inhomogeneity of the medium [5–8] or a background flow [9] has also been developed.

Recently, a new kind of interaction between sound and Alfvén waves has been discussed by Zaqarashvili [10]. The physical basis of this interaction is the parametric influence; sound waves cause a periodical variation in the medium’s parameters, which affects the velocity of transversal Alfvén waves and leads to a resonant energy transformation into certain harmonics. In a high \(\beta\) plasma, it is shown that periodical variations of the medium’s density, caused by the propagation of sound waves along an applied magnetic field, results in Alfvén waves being governed by Mathieu’s equation (here \(\beta = 8\pi p / B^2 \gg 1\), where \(p\) is the plasma pressure and \(B\) is the magnetic field). Consequently, harmonics with half the frequency of sound waves grow exponentially in time. The same phenomenon was developed in the case of standing sound waves [11]. The process of energy exchange between these different kinds of wave motion is called swing wave-wave interaction. This terminology arises from an analogy with a swinging pendulum [12].

In this paper we further develop the theory for interactions between fast magnetosonic waves propagating across an applied magnetic field and Alfvén waves propagating along the field. Finally, we briefly describe the applications of the theory to various astrophysical situations.

II. COUPLING BETWEEN FAST MAGNETOSONIC AND ALFVÉN WAVES

Consider motions of a homogeneous medium, with zero viscosity and infinite conductivity, as described by the ideal MHD equations:

\[
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u} - \mathbf{B} \cdot \nabla \mathbf{u}, \quad \nabla \cdot \mathbf{B} = 0, \tag{1}
\]

\[
\frac{\rho \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \left[ p + \frac{B^2}{8\pi} \right] + \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{4\pi}, \tag{2}
\]

\[
\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho + \rho \mathbf{u} \cdot \nabla \mathbf{u} = 0, \tag{3}
\]

where \(\mathbf{u}\) is the fluid velocity. We consider adiabatic processes, so the pressure \(p\) and density \(\rho\) are connected by the relation

\[
p = p_0 \left( \frac{\rho}{\rho_0} \right)^\gamma, \tag{4}
\]

where \(p_0\) and \(\rho_0\) are the unperturbed uniform pressure and density and \(\gamma\) is the ratio of specific heats. We neglect gravity, though it may be of importance under some astrophysical conditions.

Linear analysis of equations (1)-(4) show the existence of three kinds of MHD waves: Alfvén and magnetosonic (fast and slow) waves. The difference between these waves is that the restoring force of Alfvén waves is the tension of magnetic field lines, \((\mathbf{B} \cdot \nabla) \mathbf{B} / 4\pi\), acting alone, while the restoring force of magnetosonic waves is mainly the gradient of ordinary and magnetic pressures, \(-\nabla \left[ p + B^2 / 8\pi \right]\). The various waves can be distinguished by their different speeds and polarizations. The linear evolution of the waves in a homogeneous medium is governed by the usual linear wave equations.

Consider a uniform, unperturbed, magnetic field \(\mathbf{B}_0 = (0, 0, B_0)\) directed along the \(z\)-axis, and the case of magnetosonic wave propagation across the field in the \(x\) direction. Then there are only fast magnetosonic waves (the
slow wave is absent) which in the linear approximation is described by the equations:

\[
\frac{\partial b_z}{\partial t} = -B_0 \frac{\partial u_z}{\partial x},
\]

\[
\rho_0 \frac{\partial u_z}{\partial t} = -\frac{\partial}{\partial x} \left[ c_s^2 \rho + \frac{B_0 b_z}{4\pi} \right],
\]

\[
\frac{\partial \rho}{\partial t} = -\rho_0 \frac{\partial u_z}{\partial x},
\]

where \( b_z \) and \( u_z \) are the perturbations of magnetic field and velocity, respectively, and \( c_s = \sqrt{\gamma \rho_0 / \rho_0} \) is the sound speed. Here and afterwards \( \rho \) denotes the perturbation of density (in equations (1)-(4) \( \rho \) was the total density).

The wave equation for linear fast magnetosonic waves then follows,

\[
\frac{\partial^2 u_x}{\partial t^2} - V_f^2 \frac{\partial^2 u_x}{\partial z^2} = 0,
\]

where \( V_f = \sqrt{c_s^2 + V_A^2} \) is the phase velocity of fast waves and \( V_A = B_0 / \sqrt{4\pi \rho_0} \) is the Alfvén speed.

The solution of the wave equation can be either propagating or standing patterns. The boundedness of the medium leads to the formation of a discrete spectrum of harmonics which represent the normal modes (eigenmodes) of the system. We consider the standing fast magnetosonic waves, which have a straightforward extension to cylindrical (pulsating magnetic tube) and spherical (pulsating sphere with dipole-like magnetic field) geometries. The solutions for standing (plane) fast magnetosonic waves are:

\[
u_x = \alpha V_f \sin(\omega_n t) \sin(k_n x),
\]

\[
\rho = \alpha \rho_0 \cos(\omega_n t) \cos(k_n x),
\]

\[
b_z = \alpha B_0 \cos(\omega_n t) \cos(k_n x),
\]

where \( k_n = \frac{n \pi}{l} \) (\( n = 1, 2, \ldots \)) is the eigenvalue for a system of size \( l \) in the \( x \) direction, \( \omega_n \) is the corresponding eigenfrequency, and \( \alpha \) is the relative amplitude of the waves. Eigenvalues and eigenfrequencies are related by the dispersion relation \( \omega_n/k_n = V_f \).

It is seen from the expressions (9) that standing fast magnetosonic waves cause a local periodical variation in both the density and the magnetic field. This variation is maximal near the nodes of the velocity and approaches to zero near the antinodes. The amplitude of the variation is considered to be small (\( \alpha \ll 1 \)), and so does not affect the fast magnetosonic wave itself.

Consider now the influence of the density and the magnetic field variations (9) on Alfvén waves, considered to be polarised in the \( yz \) plane. Then the velocity fields of fast magnetosonic and Alfvén waves are decoupled. The linear equations for Alfvén waves are:

\[
\frac{\partial b_y}{\partial t} = B_0 \frac{\partial b_y}{\partial z},
\]

where \( b_y \) and \( u_y \) are small perturbations of the magnetic field and the velocity. These equations lead to the wave equation

\[
\frac{\partial^2 b_y}{\partial t^2} - V_A^2 \frac{\partial^2 b_y}{\partial z^2} = 0.
\]

The influence of the fast magnetosonic waves can be expressed by modifying equations (10) and (11), which now became

\[
\frac{\partial b_y}{\partial t} = (B_0 + b_z) \frac{\partial u_y}{\partial z} - \frac{\partial u_x}{\partial x} b_y,
\]

\[
(\rho_0 + \rho) \frac{\partial u_y}{\partial t} = \frac{\partial^2 b_y}{\partial t^2} - \frac{\partial^2 b_y}{\partial x^2} = 0.
\]

Here we have neglected the advective terms \( u_x \partial b_y / \partial x \) and \( (\rho_0 + \rho) u_x \partial u_y / \partial x \) for several reasons. At the initial stage, the perturbations \( b_y \) and \( u_y \) of Alfvén waves propagating along the \( z \) axis do not depend on the \( x \) coordinate; each magnetic surface across \( x \) evolves independently. The \( x \) dependence arises due to the action of the fast magnetosonic waves, and so the neglected terms are second order in \( \alpha^2 \). Moreover we can consider the Alfvén waves at the velocity node of standing fast magnetosonic waves, where these terms are zero. In principle, the coordinate \( x \) stands as a parameter in equations (13) and (14) of Alfvén waves.

Equations (13) and (14) lead to the Hill type second order differential equation

\[
\frac{\partial^2 b_y}{\partial t^2} - \frac{(2B_0 + b_z) b_z \partial b_y}{B_0(B_0 + b_z)} \frac{\partial b_y}{\partial t} - \frac{(B_0 + b_z) b_z - \dot{b}_z^2}{B_0(B_0 + b_z)} b_y - \frac{(B_0 + b_z)^2}{4(\rho_0 + \rho)} \frac{\partial b_y}{\partial z^2} = 0,
\]

where \( \dot{b}_z \) denotes the time derivative of the perturbing field. Introducing

\[
b_y = h_y(z, t) \exp \int \frac{(2B_0 + b_z) \dot{b}_z}{2B_0(B_0 + b_z)} dt
\]

and neglecting terms of order \( \alpha^2 \) leads to the equation

\[
\frac{\partial^2 h_y}{\partial t^2} - V_A^2 \left[ 1 + \alpha \cos(k_n x) \cos(\omega_n t) \right] \frac{\partial^2 h_y}{\partial z^2} = 0.
\]

Comparing equations (17) and (12) we can see that the influence of standing fast magnetosonic waves is expressed through a periodical variation of the Alfvén speed.

Performing a Fourier transform of \( h_y \) with \( h_y = \int \hat{h}_y(k_z, t) e^{ik_z x} dk_z \), equation (17) leads to Mathieu’s equation [13]

\[
\frac{\partial^2 \hat{h}_y}{\partial t^2} + \left[ V_A^2 k_z^2 + \delta \cos(\omega_n t) \right] \hat{h}_y = 0,
\]
where
\[ \delta = \alpha V_A^2 k_z^2 \cos(k_n x), \] (19)
with \( x \) playing the role of a parameter. Equation (21) has main resonant solution if
\[ \omega_A = \frac{B_0 k_z}{\sqrt{4\pi \rho_0}} = \frac{\omega_n}{2} \] (20)
and it can be expressed as
\[ \hat{h}_y = h_0 e^{i\frac{\omega_n t}{2}} \left[ \cos \left( \frac{\omega_n t}{2} \right) - \sin \left( \frac{\omega_n t}{2} \right) \right], \] (21)
where \( h_0 = h(0) \). The solution has a resonant character within the frequency interval
\[ \left| \omega_A - \frac{\omega_n}{2} \right| < \frac{\delta}{\omega_n}. \] (22)

Equation (21) shows that the harmonics of Alfvén waves with half the frequency of fast magnetosonic waves grow exponentially in time. The growth rate of Alfvén waves is maximal at the velocity nodes of fast magnetosonic waves and tends to zero at the antinodes (see equations (9) and (19)). The amplitude of the magnetic field component in Alfvén waves depends on the \( x \) coordinate, i.e. there is the periodical magnetic pressure gradient along this direction. Energy conservation implies that this gradient leads to the damping of initial fast magnetosonic waves, i.e. the energy transformed into Alfvén waves is extracted from fast magnetosonic waves. To show this, we consider the backreaction of amplified Alfvén waves on the initial fast magnetosonic waves.

The dependence of \( b_y \) on the \( x \) coordinate leads to an additional term in the equation of motion (6) for fast waves,
\[ \rho_0 \frac{\partial u_x}{\partial t} = - \frac{\partial}{\partial x} \left[ c_s^2 \rho + \frac{B_0 b_z}{4\pi} \right] - \frac{\partial}{\partial x} \left[ \frac{b_y^2}{8\pi} \right]. \] (23)

Therefore the wave equation (8) now becomes
\[ \frac{\partial^2 u_x}{\partial t^2} - V_f^2 \frac{\partial^2 u_x}{\partial x^2} = - \frac{\partial}{\partial \xi} \left[ \frac{b_y^2}{8\pi} \right]. \] (24)

The additional term has the frequency of the initial fast magnetosonic waves \( \omega_n \) (within the order of \( \alpha^2 \)) and can be considered as the external, periodic force. At the initial stage it can be neglected as of second order of smallness. However, it becomes significant because of the exponential growth of amplitudes (see equation (21)). It oscillates out of phase with respect to the initial fast waves (9), thus leading to their damping (as expected from physical considerations).

Swing coupling between fast magnetosonic and Alfvén waves may be generalised from rectangular geometry to other symmetries, though a detailed description is beyond the scope of this paper.

### III. DISCUSSION

The suggested mechanism of energy transformation from fast magnetosonic into Alfvén waves has important consequences. It can be noted that the Alfvén waves hardly undergo either excitation or damping processes, while magnetosonic waves can be easily excited by external, even nonelectromagnetic, forces. Then swing interaction leads to the intriguing but natural suggestion that the energy of the nonelectromagnetic force which supports the magnetosonic waves in the system can be transmitted into the energy of purely magnetic incompressible oscillations. This result has many astrophysical applications. We briefly describe several of them.

#### A. Swing absorption

Resonant interaction between MHD waves, due to the inhomogeneity of the medium, was proposed by Ionsons [8]. It arises where the frequency of an incoming wave local frequency of the medium. Then a resonant energy transformation may take place, known as resonant absorption.

Similar phenomenon may arise also due to swing wave interaction. In this case fast magnetosonic waves can transform their energy into Alfvén waves, even in a homogeneous medium. For given medium parameters (magnetic field, density) the energy of fast waves may be ‘absorbed’ by harmonics with wavelengths satisfying the resonant condition (20). Consequently, fast magnetosonic waves can transmit their energy into Alfvén waves in any spatial distribution of density or magnetic field. The process can be called swing absorption. The particular point of swing absorption is that energy ‘absorption’ occurs through the harmonics with half the frequency of incoming waves (see equation (20)). The process may be of importance in the Earth’s magnetosphere and in the solar atmosphere.

#### B. Torsional Alfvén waves in solar coronal loops

Swing wave interaction may play an important role in the excitation of torsional Alfvén waves in solar coronal loops. It may be suggested that any external action on the magnetic tube, anchored in the highly dynamical photosphere, causes a radial pulsation at the fundamental frequency, like a tuning fork (see also [14]). For a tube of radius \( r_0 \) the fundamental frequency of pulsation will be of order \( V_f/r_0 \), where \( V_f \) is the phase velocity of fast magnetosonic waves at the photospheric level. If we consider the Alfvén and sound speeds to be of order \( \sim 10 \) km/s and the radius of order \( \sim 10^2 \) km, then the period of fundamental mode of pulsation will be a few tens of seconds.

Radial pulsations of the tube may lead to the resonant (exponential) amplification of torsional Alfvén waves.
with half the frequency of the pulsations. These high frequency torsional Alfvén waves can propagate upward and carry energy from the photosphere into the magnetically controlled corona or they may be damped in chromospheric regions leading to the heating of the chromospheric magnetic network.

C. Coupling between stellar pulsations and torsional oscillations

Swing wave interaction may be of importance in stellar interiors. A radial pulsation of a spherically symmetric star with dipole-like magnetic field may lead to the amplification of torsional oscillations. There are a number of energy sources which can support pulsations: radiation, nuclear reactions, tidal forces in binary stars, convection, etc. (e.g., [15]). Then the transformation of pulsational energy into torsional oscillations may lead to new sources for stellar magnetic activity [16].

D. Enhanced chromospheric activity in tidally interacting binaries

Observations show enhanced chromospheric and coronal activity in relatively close binaries [17]. The considered binary systems are well separated so that mass transfer by Roche-lobe overflow does not occur. The chromospheric, transition-region and coronal emissions from the binaries are enhanced in comparison to single stars with the same mass, chemical composition, age and mean surface rotation rate. This is somehow a strange phenomenon, because in the framework of dynamo theory the magnetic activity does not depend on whether the star is single or component of a binary system.

It is supposed that the deviation of the primary star from spherical symmetry due to the tidal influence of the companion leads to stellar pulsation in its fundamental mode [18]. The stellar radial pulsation amplifies torsional Alfvén waves in a dipole-like magnetic field, buried in the interior, according to the swing wave-wave interaction. Then amplified Alfvén waves lead to the onset of large-scale torsional oscillations, and magnetic flux tubes arising towards the surface owing to magnetic buoyancy diffuse into the atmosphere producing enhanced chromospheric and coronal emission.

IV. CONCLUDING REMARKS

The swing wave-wave interaction [10] is developed here in the case of fast magnetosonic waves propagating across a magnetic field and Alfvén waves propagating along the field. In the case of oblique propagation, slow magnetosonic waves also exist and they may transmit their energy into Alfvén waves. In some cases the coupling between slow magnetosonic and Alfvén waves may be of importance. Also, the coupling in the case of different geometries (cylindrical, spherical) may be important in astrophysical situations. The most important result of swing wave interaction is that it reveals a new energy channel for Alfvén waves, permitting the transformation of energy of nonelectromagnetic origin into the energy of electromagnetic oscillations.

Acknowledgments

TZ is grateful to the organizing committee, particularly G. Tsiropoula, for financial support.