FRAME DEPENDENCE OF THE NEGATIVE ENERGY WAVE FORMULA!

Jesse Andries and Marcel Goossens
Centre for Plasma Astrophysics, K.U.Leuven,
Celestijnenlaan 200B, B-3001 Leuven, Belgium
jesse.andries@wis.kuleuven.ac.be

ABSTRACT

In this paper we show that the classical negative energy wave (N.E.W.) formula is frame dependent. By analogy with a simple mechanical problem, we show that the negligible second order perturbations become energetically important and function as an energy source for the linear waves.

1. INTRODUCTION

It has long been known that an inhomogeneous flow can cause waves to become overstable. Several recent studies deal with overstabilities and overreflection of waves on inhomogeneous parallel flows that seem to be caused by dissipation or wave leakage. Often these phenomena are explained in the context of negative energy waves (N.E.W.) [6, 14, 13, 8, 12, 15, 10, 11, 9, 5]. However many authors have raised questions concerning the use of the N.E.W. formulation [7, 16, 17, 2, 3, 1]. In a discussion by Cairns [6] it is argued that surface wave eigenmodes can be driven by dissipation if they are N.E.W.'s. The idea is that if the wave amplitude grows when energy is extracted (e.g. by dissipation) the wave has “negative energy”. However, the formula of wave energy derived by Cairns (used unaltered in these N.E.W. studies) exhibits a peculiar phenomenon which we have shown to be due to the confusion of a total and a partial time derivative [4]. The energy (in particular the sign of it) is different when the same wave is described in two different frames moving relative to each other at constant speeds (along the equilibrium flow). While adherents of the N.E.W. formulation claim that this does not lead to frame dependent results, which is certainly true when adapted correctly, the frame dependence does create confusion. Moreover we feel there is little advantage to such a frame dependent definition, especially because, as we show here, the frame dependence can be overcome by a minor adjustment, leaving the rest of the theory intact. In this paper we establish an analogy with a simple mechanical problem that shows how frame dependence of the energy can arise by neglecting higher order changes, that are, however, energetically important.

Walker [17] has discussed the problems arising with N.E.W.'s in inhomogeneous media. In particular, Walker treated the problem of reflection and transmission at a sharp boundary. His analysis was based on the linear (normal mode) wave energy, which is always positive. In his view the flow boundary is identified as an energy source. Clearly this viewpoint is very different from the “negative energy” formulation where the driver is the only energy source. In view of these results Andries and Goossens [3] reconsidered the reflection problem at a smooth boundary, which enables resonant mode coupling in the boundary layer. By treating the problem in the light of Walker’s discussion we were able to reveal and explain the crucial profile dependence of the resonant amplification mechanism.

Given the obvious relation between overreflection and instability of trapped waves, we have generalized Walker’s treatment to non-stationary eigenmodes, and discussed its relation to the “negative energy” eigenmode treatment [4]. The crucial point is that the background flow cannot be considered to be time-independent when the wave that is present on the background is changing in time. Some of the statements by Andries and Goossens [4] concerning the second order perturbations are made more clear here.

The paper is organized as follows. First we show clearly that the classical N.E.W. formula is frame dependent. Secondly, we show that a similar frame dependence can be (falsely) obtained for a simple mechanical problem. A proper analysis shows that the energy is independent of the reference frame. Thirdly, we obtain a frame independent version of the N.E.W. formula. Fourthly, we show that in this new formulation the flow is acting as an energy source. The difference between Walker’s and the N.E.W. approach is whether to include this additional work done by the flow in the wave energy or not. We then explain how by analogy with the mechanical problem the energetically important second order changes arise. Finally we state our conclusions.
2. FRAME DEPENDENCE OF THE N.E.W. FORMULA

In what follows we assume a configuration which is inhomogeneous in the $x$ direction only. The magnetic field and the equilibrium velocity have no $x$ components.

In negative energy wave literature the following formula is used to decide whether a wave has negative or positive energy:

$$ E \sim \omega \frac{\partial D(\omega, \vec{k})}{\partial \omega}, $$

where $\omega$ is the oscillation frequency of the wave, $\vec{k}$ is the wave vector in the $y$ and $z$ directions. And $D(\omega, \vec{k})$ is a function so that $D(\omega, \vec{k}) = 0$ is the dispersion relation. We want to point out that this formula is frame dependent when different reference frames are considered that move relative to each other in the $z$ direction. Consider a new reference frame (indicated by *):

$$ z^* = z - V_{frame} t. $$

In this reference frame an observer does not only see a different flow but also a different frequency for the same wave:

$$ v_{z}^* = v_z - V_{frame}, $$

$$ \omega^* = \omega - k_z V_{frame}. $$

Together this leads to the fact that the Doppler shifted frequency is independent of the frame:

$$ \Omega^* = \omega^* - k_z v_{z}^* = \omega - k_z v_z = \Omega. $$

Moreover $D(\omega, \vec{k})$ does only depend on the frequency and on the flow through the Doppler shifted frequency $\Omega$. Thus $\frac{\partial D}{\partial \omega}$ as well as $D$ is independent of the frame. But since $\omega$ is dependent on the frame, we conclude that the energy, as calculated in formula (1), is dependent on the reference frame.

3. AN ANALOGOUS MECHANICAL PROBLEM

To support the false statement that frame dependence of the energy is a natural phenomenon, one might set up an example as follows: When a car stops, an observer by the roadside will see it slow down, while an observer moving at the initial speed of the car will see it speed up.

This example was put forward by the referee of Ref. [4], and it is very instructive to show why it is wrong and misleading.

With the notation $m_c, m_e$ and $v_{ce}, v_{eo}$ for the mass and the initial speed of the car and the earth respectively, we get the following expression for the initial energy in the system:

$$ E_0 = \frac{1}{2} m_c v_{ce}^2 + \frac{1}{2} m_e v_{eo}^2. $$

Since there is no external force acting on the system, the momentum is conserved during the braking process:

$$ m_c v_{ce} + m_e v_{eo} = m_c v_c + m_e v_e. $$

The car has stopped when $v_c = v_e \equiv v$, and thus from the above formula:

$$ v = \frac{m_c v_{ce} + m_e v_{eo}}{m_c + m_e}. $$

Taking into consideration that $m_c \ll m_e$ we get to lowest ($0^{th}$) order:

$$ v \approx v_{eo}. $$

So far, so good. The problems now arise when we use this result to calculate the energy in the system after the car has stopped:

$$ E \approx \frac{1}{2} m_c v_{ce}^2 + \frac{1}{2} m_e v_{eo}^2, $$

in order to obtain the change in energy:

$$ \Delta E \approx \frac{1}{2} m_e (v_{eo}^2 - v_{ce}^2). $$

When this expression is rephrased in a different reference frame it is seen immediately that $V_{frame}$ does not cancel out, thus meaning that the energy is frame dependent.

However, the calculations have not been carried out carefully enough. Since we used the lowest order approximation for $v$ to calculate $E$, expression (2) is only valid to lowest order. But this isn’t good enough. Since $E_0$ already possesses both $0^{th}$ and $1^{st}$ order terms, we also need to consider both orders in $E$. Therefore we should calculate $v$ to first order:

$$ v \approx v_{eo} + \frac{m_c}{m_e} (v_{eo} - v_{eo}). $$

Which leads to first order:

$$ E \approx \frac{1}{2} m_e v_{eo}^2 + \frac{1}{2} m_e v_{eo}^2 + m_c (v_{eo} - v_{eo}) v_{eo}. $$

And thus to:

$$ \Delta E \approx \frac{1}{2} m_c v_{eo}^2 + m_c (v_{eo} - v_{eo}) v_{eo} - \frac{1}{2} m_e v_{eo}^2, $$

$$ = \frac{1}{2} m_c (v_{eo} - v_{eo}) (v_{eo} + v_{eo} - 2 v_{eo}), $$

$$ = \frac{1}{2} m_c (v_{eo} - v_{eo})^2. $$

This is clearly independent of the reference frame. From this straightforward calculation we remember that the deceleration of the car accelerates the earth. While this acceleration itself is a negligible effect, it leads to a non-negligible contribution to the energy. This is due to the fact that we need to multiply by $m_c$, which is large, to get the earth kinetic energy. Taking this into account removes the frame dependence of the energy.

This mechanical analogy shows that one should hope to find a frame-independent energy formula.
4. REMOVAL OF THE FRAME DEPENDENCE

A frame independent energy formula can be obtained straightforwardly when following the derivation of the formula by Cairns [6]. A wave is set up by driving it at a certain surface. Due to energy conservation the energy in the wave is then considered to be equal to the total amount of work done by the driver. This work can be calculated as the jump in energy flux over the driving surface:

\[ W_d = [F_x]^+ - [F_x]^-, \]

where \( p_T \) is the Eulerian perturbation of total pressure, and \( v_x \) the Eulerian perturbation of the x component of the velocity. This is connected to the Lagrangian displacement by:

\[ v_x = \frac{d\xi_x}{dt} = -i\Omega \xi_x. \]

However, Cairns used \( v_x = -i\omega \xi_x \). Notice that Cairns drives the wave at the flow boundary surface where the Doppler shifted frequency is undefined (for more comments on this see Andries and Goossens [4]). It is clear that using the correct relation (4), the energy formula becomes:

\[ E \sim \Omega_d \frac{\partial D(\omega, k)}{\partial \omega}, \]

where \( \Omega_d \) should be taken at the position of the driver.

Notice that \( \Omega \) varies with position because of the inhomogeneous flow. Thus although the frame dependence is removed, a position dependence is introduced. However, this should not disappoint us, since it exactly emphasizes that the result of an energy extraction from a wave (by dissipation, wave leakage or whatever) is dependent on where that energy extraction is operative.

Remark that the old and the new (Doppler shifted) energy formula are exactly the same when a frame is used which is fixed to the flow at the position of the driver. However, we think it is more transparent to include the position dependence directly in the formula by the Doppler shift, rather than hiding it in the choice of the reference frame. Moreover, in the case of resonant absorption, which is effectively functioning as a local dissipation mechanism, several resonant layers can occur simultaneously, so that in the old formula different reference frames would have to be used for the different resonances. This problem does not arise in the new (Doppler shifted) formulation.

5. THE FLOW AS ENERGY SOURCE

Using the formula (4) in (3) we obtain an additional wave energy input at the flow boundary:

\[ W_T = \frac{1}{2} \mathcal{R} (-i\Omega)^0 \mathcal{R}_{0 \xi_x p_T} = -\frac{1}{2} k [V]^0 \mathcal{R}_{0 \xi_x p_T} \]

where \( k \) is the wave number and \( \mathcal{R} \) the Rayleigh stress tensor. The second order term gives an additional contribution to the wave energy input:

\[ -k [V]^0 \mathcal{R}_{0 \xi_x p_T} = -\frac{k}{\Omega} \mathcal{R}_{0 \xi_x p_T} \]

However, the work calculated here is not work done by an external driver. It was shown by Walker [18] that the second order part of the equation of conservation of total energy falls apart in two separate equations. One of them involves only the second order perturbations of the energy due to products of the first order perturbations. This is the equation used to get to the expression of wave energy flux in (3). But it has to be stressed that this is not the full second order energy relation, which would involve not only terms due to the second order perturbations but also additional terms due to products of first order quantities. Thus the additional work by the flow boundary is just compensated by changes in the second order mean flow.

It is generally true that second order effects are harmonics of the first order effects. Therefore if the second order part of the total energy conservation equation is averaged over y or z direction we expect the terms due to second order perturbations to vanish. However, it can be easily shown that the second order effects also include mean second order changes. The second order changes are driven by products of first order changes (e.g. the second order equations in Walker [18]). If we consider the following relations:

\[ \sin \theta \cos \theta = \frac{1}{2} \sin(2\theta), \]

\[ \cos^2 \theta = \frac{1}{2} (1 + \cos(2\theta)), \]

\[ \sin^2 \theta = \frac{1}{2} (1 - \cos(2\theta)). \]

it is clearly seen that the second order effects are composed of a second order harmonic (2\theta) and mean changes. The mean changes are small but, like the small acceleration of the earth in the mechanical problem, they do interact energetically with the first order perturbations. Since these mean second order perturbations are uniform in y and z direction, it can be disputed whether they should be considered as part of the wave are rather as a part of the background that is changing due to the non-stationarity of the wave. If you consider them to be part of the wave, then wave energy is conserved. This is in agreement with the N.E.W. approach. If the mean second order changes are considered to be changes of the background, then they energetically interact with the linear waves and can represent a linear wave energy source/sink. This is in agreement with Walker's approach.

While the part of the work due to second order changes can always be put to zero at the driving surface (by means of the second order boundary conditions), the same cannot be done at the flow boundary. It is exactly at the flow boundaries that the mean flow exchanges energy with the linear wave. At the flow boundaries there is no energy input or extraction to the combined system of wave and mean flow. The additional work on the linear wave does not change anything to the conclusion that the wave can be build up by extracting energy, and thus that
the total resulting perturbation that is created has "negative energy". Expression (5) is important. The appearance of $F_x$ in the expression clarifies how the creation of a wave energy flux (e.g. by dissipation or wave leakage) makes the flow act as an energy source/sink for the wave.

6. CONCLUSIONS

We have seen that the classical N.E.W. formula is frame dependent. From a simple mechanical analogy we concluded that a frame independent formula would be more appropriate. By introducing the Doppler shift in the relation between the Eulerian velocity perturbation and the Lagrangian displacement, we have removed the frame dependence from the N.E.W. formula. However, this resulted in a position dependence which indicates that the effect of dissipation on a wave is dependent on where that dissipation is operative. The flow boundary is seen to be an additional energy source. However, the additional energy is cancelled by the energy in mean second order changes. The N.E.W. approach and Walker's normal mode approach differ in considering the mean second order perturbations as part of the wave or rather as part of the changing background. As in the mechanical analogy, the higher order perturbations are seen to interact energetically with the lower order perturbations.

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