A LATTICE GAS MODEL FOR TWODIMENSIONAL BOUSSINESQ CONVECTION

W. SCHAFFENBERGER\textsuperscript{1}, A. HANSSLMEIER\textsuperscript{1}, M. MESSEROTTI\textsuperscript{2}

\textsuperscript{1}Institut für Geophysik, Astrophysik und Meteorologie, Universität Graz, Universitätsplatz 5, A-8010 Graz, Austria
\textsuperscript{2}Osservatorio Astronomico di Trieste, Via G.P. Tiepolo 11, I-34131 Trieste, Italy

UDC 52-464-17
Original scientific paper
(Received October 19, 2001; accepted November 19, 2001)

Abstract. In this paper, we present a 2-D model for simulating the convection of an incompressible fluid between two walls of different temperatures. In particular, a bidimensional cellular automaton (CA) was developed to study the evolution of a discrete particle system, which represents a modified Frisch-Hasslacher-Pomeau (FHP) lattice gas. The derivation of the model equations and some relevant diagnostics, such as the Rayleigh, Prandtl and Nusselt numbers, are briefly outlined. The diagnostics computed for test runs indicate the consistency of the model as well as the preliminary simulation performed with a CA.

Key words: the Sun - solar convection - lattice gases - cellular automata

1. Introduction

In a fluid in a gravitational field convection occurs when the temperature gradient has the same direction as the gravitational field and its absolute value exceeds some threshold. Hot gas is streaming upwards and cool gas downwards. Convection occurs, for example, in the core of early-type stars and in the envelope of late-type stars (Kippenhahn and Weigert, 1990). Hence it is a key factor in the theory of stellar structure and evolution and a variety of models have been conceived.
to give at least a parametric description of the process. Several authors have made numerical simulations, such as tridimensional ones solving the full set of the hydrodynamic equations (Brummel et al., 1996; Cattaneo et al., 1991; Chan and Sofia, 1989; Hurlburt et al., 1996; Porter and Woodward, 1994; Stein and Nordlund, 1989, 1998, 2000).

We present a convection model which simulates a fluid as a discrete system of many interacting particles in two dimensions. Although the interaction between the particles is very simple, the macroscopic behaviour is similar to that of a real fluid. The model we constructed is a bidimensional cellular automaton (CA) to represent a lattice gas which behaves like an incompressible fluid.

The paper is organized as follows. In Section 2 we introduce the concept of CA and lattice gases. Our model for twodimensional Boussinesq convection is described in Section 3 including the model equations. In Section 4 we present the results of our simulations. Finally, in Section 5 we summarize our results and discuss the limitations of the model.

2. Cellular automata as lattice gases

A CA consists of a set of identical cells, which are set out to form a regular pattern (lattice) along a line (1-D CA), on a plane (2-D CA) or on more planes in the tridimensional space (3-D CA).

All possible states for each cell constitute a set of finite discrete states and also the temporal evolution occurs in discrete time steps. The state of a cell at time $t_{i+1}$ is determined by a local rule, which is based exclusively on the states of the neighbouring cells at time $t_i$ and is the same for all the cells in the CA. The time evolution of the CA is then obtained by applying the evolutionary rule to all cells at each time step, resulting in successive configurations of the system.

CA can be used as an alternative approach in modelling physical systems (Vichniac 1984), such as gases. These models are called lattice gases (LGA), as the particles move from one cell to a neighbouring one in the CA lattice. According to the number of the neighbouring cells, each particle can have only a finite set of different velocities and a
finite number of particles with different velocities can occupy a cell. The dynamics of a lattice gas is determined by the evolutionary rule, which combines particle transport and collision rules. The collisions conserve mass and momentum. A detailed theory of lattice gases has been developed (Frisch et al., 1987; Wolfram 1986). Using this theory, one can derive the hydrodynamic equations and transport coefficients, such as the viscosity. For instance, the FHP model, named after Frisch, Hasslacher and Pomeau (Frisch et al., 1986) is based on an hexagonal cell neighbourhood, which allows six possible velocities.

3. The 2-D CA model for the convection

3.1. Basic concepts

To construct a model for the convection in 2-D, one must introduce temperature and buoyancy forces which drive the convection. For the original FHP model, no temperature is defined, because for most hydrodynamic problems, temperature is not important (Landau and Lifshitz, 1959). However, some lattice gas models have been developed for the simulation of thermal systems such as heat conduction in a fluid (Chopard and Droz, 1988). In these models, the number of possible velocities for the particles was increased so that the particles have different kinetic energies. Then, the temperature can be defined as the mean kinetic energy per particle (Fahner 1991; Chen et al., 1989).

In our model, we introduce the temperature in a different way. Instead of increasing the number of possible velocities, we introduce additional temperature states independent of the velocities. Each particle can have three different temperature states with the discrete values -1, 0 and 1. The temperature $T$ is now defined as the mean value of the temperature level per particle. Therefore the temperature can have values between -1 and 1. The temperature levels of the particles can also be interpreted as some kind of internal energy of the particles. The heat capacity can be set equal to one.

External forces can be simulated by velocity flippings (Frisch et al., 1987). To get a buoyancy force which is proportional to the temperature and acts in the Y direction, these velocity flippings must depend
on the temperature state of the particles and only the Y component of the velocity is changed. Particles at temperature state 1 have the Y component of the velocity changed to the positive Y direction, whereas particles at temperature state -1 have the velocity changed to the negative Y direction and particles at temperature state 0 are unaffected.

We adopted a 2-D FHP gas with the proper modifications to allow for the temperature and the buoyancy forces, which are required to simulate convective regimes. We used the collision rule for the FHP-I model (Frisch et al., 1987). It includes two particle collisions and symmetric three particle collisions (see Fig. 1). The sum of the temperature states is also conserved at each collision and it is redistributed randomly among the particles if only 2 particles are on a lattice node.

3.1.1. Boundary conditions

In our model, we use a rectangular simulation box (see Fig. 2). A hot wall \( T = 1 \) is located at the bottom and a cold wall \( T = -1 \)
at the top of the simulation box. The no-slip boundary condition for the velocity is implemented using the simple bounce-back rule, i.e. the velocity of each particle is inverted, when arriving at the wall. The temperature states of the reflected particles are \( j = 1 \) and \( j = -1 \) for the hot wall and the cold wall respectively.

For the two other edges of the CA, periodic boundary conditions are assumed. This means that if a particle reaches one edge of the lattice, it is reinjected from the opposite one. Therefore, the total number of particles is conserved.

3.2. Model equations

To derive the macroscopic equations at time \( t \) and position \( \vec{r} \), one has to introduce the macroscopic quantities like density and flow speed by averaging over several cells and time steps. These quantities can be derived from the single particle distribution function \( n_{i,j} \):

\[
\rho(\vec{r}, t) = \sum_{i,j} n_{i,j}(\vec{r}, t)
\]  

\( Hvar Obs. Bull. 25 \) (2001) 1, 49–60
\begin{align*}
\rho(\vec{r}, t) \vec{v}(\vec{r}, t) &= \sum_{i,j} \vec{v}_i n_{i,j}(\vec{r}, t) \quad (2) \\
\rho(\vec{r}, t) T(\vec{r}, t) &= \sum_{i,j} j n_{i,j}(\vec{r}, t) \quad (3)
\end{align*}

where \( \rho(\vec{r}, t) \) is the density, \( \vec{v}(\vec{r}, t) \) is the velocity and \( T(\vec{r}, t) \) is the temperature of the gas. The \( \vec{v}_i \), where \( 1 \leq i \leq 6 \), are the possible velocities of the particles and \( j = -1, 0, 1 \) are the temperature levels. The particle distribution function \( n_{i,j} \) gives the number of particles with velocity \( v_i \) and temperature state \( j \).

For the particle distribution function, the lattice Boltzmann equation holds:

\begin{equation}
n_{i,j}(\vec{r} + \vec{v}_i, t + 1) - n_{i,j}(\vec{r}, t) = \Delta^{\text{Boltz}}_{i,j}(n) + F_{i,j}(n) \quad (4)
\end{equation}

where \( \Delta^{\text{Boltz}}_{i,j}(n) \) is the collision term. It includes all possible collisions and redistributions of the temperature levels. \( F_{i,j}(n) \) includes velocity flippings due to the buoyancy forces. This forces conserve particle number and temperature states. The hydrodynamic equations of the model can be derived using the so-called Chapman–Enskog expansion (Frisch et al., 1987):

\begin{align*}
\vec{\nabla} \vec{v} &= 0 \quad (5) \\
\frac{\partial \vec{v}}{\partial t} + \mu(\vec{v} \vec{\nabla}) \vec{v} &= -\frac{\vec{\nabla} p}{\rho} + \nu \Delta \vec{v} + \frac{w}{2} \vec{v} \vec{\nabla} \alpha + \beta \vec{v} \vec{\nabla} \gamma \quad (6) \\
\frac{\partial T}{\partial t} + \vec{v} \vec{\nabla} T &= \chi \Delta T \quad (7)
\end{align*}

\begin{align*}
\nu &= \frac{1}{12 f (1-f)[(1-f)^2 + f^2] - \frac{1}{8}} \quad (8) \\
\chi &= \frac{1}{f (6 - 5f)(1-f)^3} - \frac{1}{4} \quad (9) \\
\alpha &= \frac{4(4 - \sqrt{4 - 3T^2})}{9} \quad \beta = 1 - f \quad (10)
\end{align*}
\[ \mu = \frac{11 - 2f}{2 - f} \quad f = \frac{\rho}{6} \]  

(11)

where \( p = \rho/2 \) is the pressure, \( \nu \) is the shear viscosity and \( \chi \) is the thermal diffusivity, independent of the temperature; \( w \) is the probability for velocity flipping and \( f \) is the particle number per velocity state. This set of equations is valid for small velocities and the density is assumed to be constant. Actually, LGA are compressible including the propagation of sound waves. For the FHP model, the speed of sound is \( 1/\sqrt{2} \) in units of the particle speeds. If the flow speed is small compared to the speed of sound, density variations are small and the gas can be treated as incompressible. In our model, density changes due to the buoyancy force must also be small, leading to the following condition for the probability for velocity changing \( w \):

\[ w < \frac{12}{(1 - f)\Delta TH} \]  

(12)

where \( H \) is the distance between the two walls and \( \Delta T = 2 \) is the temperature difference between the two walls at the top and the bottom of the simulation box.

Eqs. 5-11 are similar to the Boussinesq approximation of the equations for the convection (Landau and Lifshitz, 1959).

However, there are some deviations from the Boussinesq equations. The coefficient \( \mu \) of the convective term in the Navier-Stokes equation differs from one. This comes from the lack of Galilei-invariance. The buoyancy force includes an additional term which depends on the velocity. This term is the first in a series expansion of the buoyancy force in powers of the velocity. The terms of higher order are negligible, because we assume that the velocity is small compared to the velocities of the particles.

Two dimensionless parameters characterize the convection: the Rayleigh number \( Ra \) and the Prandtl number \( Pr \), defined as:

\[ Ra = \frac{\beta w \Delta TH^3}{2\nu\chi} \]  

(13)

\[ Pr = \frac{\nu}{\chi} \]  

(14)
The Rayleigh number depends on the density via $\beta$, $\nu$ and $\chi$. One can choose the density such that the Rayleigh number is maximal. This occurs for the optimal density $f_{opt} = 0.2$ and the resultant Prandtl number is $Pr = 0.4$. For $\Delta T = 2$, the maximal Rayleigh number is:

$$Ra = 0.682wH^3$$

(15)

A third dimensionless parameter comes from the additional term in the buoyancy force. It can be defined, for example, as:

$$W = \frac{\alpha g H^2}{\nu}$$

(16)

This new parameter is the ratio between the velocity-dependent term in the buoyancy force and the viscous force. It is zero in the standard Boussinesq-Oberbeck equations. The convection sets in when the Rayleigh number exceeds a limit and this critical Rayleigh number depends on the third parameter $W$. It rises from 1708 at $W = 0$ up to 8000 for $W = 400$. The corresponding wave number decreases with $W$. The convection cells become broader. The parameter $W$ for the optimal density is:

$$W = 0.645wH^2$$

(17)

The maximum value for $w$ is 1, but the probability for velocity flipping should be small compared to 1. The particle should collide with some particles between two velocity flippings. There are stronger conditions for the probability for velocity flipping. The changes in the density due to the buoyancy force should be small compared to the mean density of the lattice gas. Then one gets the following condition for $w$ for $\Delta T = 2$ and $f = f_{opt}$:

$$w < \frac{7.5}{H}$$

(18)

The maximal $w$ decreases with the size of the CA. The parameter $W$ becomes also smaller. The available Rayleigh number increases proportionally to the square of the size of the CA.

Another important dimensionless number is the Nusselt number, defined as:
\[ Nu = \frac{qH}{\rho c \chi \Delta T} \]  

(19)

where \( q \) is the energy flow and \( c \) is the heat capacity which is equal to one in our model. The Nusselt number characterizes the energy transport. If there is no convection, the Nusselt number is equal to one. Otherwise it is greater than one. The Nusselt number depends on the other dimensionless numbers.

Several test runs were made to verify the model equations. For example, we measured the heat conductivity for different densities. The results agree well with the theoretical predictions.

4. Results

Having tested the model, we present the results of our simulations for convection between two horizontal walls with different temperatures (see Table I). The initial state of the CA was the static conductive state with a linear temperature profile between the walls. For all simulations we used the optimal density \( f = 0.2 \) and therefore, the Prandtl number


\begin{table}
\centering
\caption{Parameters used in the simulation runs.}
\begin{tabular}{cccccccc}
\hline
run & size & w & Ra & W & Nu & Re \\
\hline
1 & 600*300 & 0.0002 & 3683 & 11.6 & 1.41 & 20.6 \\
2 & 600*300 & 0.0005 & 9207 & 29 & 1.95 & 36.1 \\
3 & 2000*1000 & 0.0001 & 68200 & 64.5 & 3.24 & 126 \\
\hline
\end{tabular}
\end{table}

for all simulations is $Pr = 0.4$. The simulation box was a rectangle with an aspect ratio of 2. To calculate the macroscopic quantities of the flow, we averaged over 30*30 cells and $10^2$ time steps. The time evolution was simulated over $10^5$ time steps.

For the first two runs, the size of the simulation box was 600*300 lattice units. The number of grid points was 207900. This is larger than 300*600 because we use a hexagonal grid. The Rayleigh number of the first run was $Ra = 3683$ and the parameter $W$ was $W = 11.6$. Convection occurred after $2 \cdot 10^4$ time steps and stationary convection developed. The Nusselt number was $Nu = 1.41$. The Reynolds number was $Re = 20.6$. We used the quadratic average of the flow speed as the characteristic velocity.

For the second simulation, the Rayleigh number was $Ra = 9207$ and the parameter $W$ was $W = 29$. Again stationary convection occurred. Because the Rayleigh number is larger, the convection begun after $10^4$ steps. The Nusselt number was $Nu = 1.95$ and the Reynolds number was $Re = 36.1$.

For the third simulation, the simulation box size was 2000*1000 cells with 2309000 grid points. The Rayleigh number was $Ra = 6.82 \cdot 10^4$ and $W = 64.5$. The convection begun after $10^4$ time steps. In contrast to the first two simulations with small Rayleigh number, the convection begun with double period in space. After $7 \cdot 10^4$ steps, the convection with double period disappeared and again stationary convection with single period occurred. The Nusselt number was $Nu = 3.24$ and the Reynolds number was $Re = 126$.

As an example, the temperature distribution and the velocity field of the third run after $8 \cdot 10^4$ time steps are shown in Fig. 3.
5. Conclusion

In this work, we developed a bidimensional CA model for a modified FHP lattice gas, which behaves like an incompressible fluid. The relevant equations were derived, having almost the same form as the hydrodynamic equations for the Boussinesq convection. To test the physical validity of the model, some diagnostics were performed in test runs, which demonstrate the consistency of the results.

The choice of such a simulation approach was suggested by the advantages intrinsic to CAs with respect to conventional numerical means such as the simple structure of the model which makes it easy to implement especially on parallel computers. Moreover no roundoff errors arise, because only integers are used. Anyway, CA have also some disadvantages. In fact, to calculate the macroscopic quantities such as density and velocity, a time consuming averaging procedure is necessary and statistical fluctuations become noticeable. Additionally CA models produce some artificial effects such as the velocity dependent term in the buoyancy force.

References

W. SCHAFFENBERGER ET AL.: 2-D SOLAR CONVECTION


"LATTICE" MODEL PLINA ZA DVODIMENZIONALNU BOUSSINESQ KONVEKCIJU

W. SCHAFFENBERGER¹, A. HANSLMEIER¹, M. MESSEROTTI²

¹Institut für Geophysik, Astrophysik und Meteorologie, Universität Graz, Universitätsplatz 5, A-8010 Graz, Austria
²Osservatorio Astronomico di Trieste, Via G.P. Tiepolo 11, I-34131 Trieste, Italy

UDK 52-464-17
Originalni znanstveni članak
(Primljeno 19. listopada 2001.; prihvaćeno 19. studenog 2001.)


Ključne riječi: Sunce - Sunčeva konvekcija - ”lattice” plinovi - ćelijski automaton