NONLINEAR EVOLUTION AND INTERACTION OF PHASE-MIXED ALFVÉN WAVES IN SOLAR CORONA

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ABSTRACT

A significant input of energy into a plasma, as observed in the corona, may only be achieved if the amplitude of the perturbed magnetic field of the launched waves is 1-3% of background magnetic field $B_0$. Phase mixing of such finite-amplitude Alfvén waves creates short length-scales, giving rise not only to an effective interaction of the waves with the plasma and consequent plasma heating, but also to enhanced nonlinear interaction. We study the nonlinear dynamics of small-scale Alfvén waves in the framework of collisional two-fluid MHD. Parametric decay of the phase-mixed pump Alfvén wave into daughter Alfvén waves in the corona gives rise to a very effective spectral energy redistribution when the transversal length-scale of the phase-mixed wave becomes close to 10 km, well before the dissipative length-scale, 0.1 km, is produced by phase mixing.

1. INTRODUCTION

The phase mixing of Alfvén wave which leads to enhanced wave dissipation and consequent plasma heating, has been first proposed as a coronal heating mechanism by Heyaerts & Priest (1983). Since then, phase mixing has been studied both for open and closed magnetic configurations (Hood et al. 1997a, Hood et al. 1997b, Ruderman et al. 1998). The first conclusions on the asymptotic behavior of the wave amplitudes with height, $\sim \exp \left(-z^2/z_{e}^2\right)$, or time, $\sim \exp \left(-t^2/t_e^2\right)$, have been recovered for typical coronal conditions. These results were obtained in the one-fluid MHD approximation.

However, with the creation of short transversal length-scales in Alfvén waves, these waves become essentially two-dimensional in the sense that they have long-wavelengths along the magnetic field and short-wavelengths across it. In this situation the ion polariztion drift in the perpendicular direction creates a charge separation across $B_0$, while field-aligned electron flows tend to cancel this charge separation, and thus the motions of the ions and electrons decouple from each other. The finite temperature and/or electron inertia effects prevent complete charge cancellation, and a longitudinal wave electric field $E_z \parallel B_0$ arises. The presence of the longitudinal wave electric field, $E_z \parallel B_0$, and current, $j_z \parallel B_0$, bring about many new properties for short-scale Alfvén waves (SSAW), including the effective interaction of waves with plasma particles (either via kinetic effects at Cherenkov resonance, or via collisions), as well as the nonlinear interaction with other modes and among Alfvén waves themselves. Alfvén waves can balance the energy loss from the loop structures of active regions and from coronal holes, if the wave magnetic field amounts to 1-5% of background magnetic field $B_0$. In addition to linear kinetic effects, nonlinearity becomes important for waves of such amplitudes, and the ability of waves to participate in the different kinds of nonlinear interaction introduces a variety of new effects in Alfvén waves.

Linear and nonlinear properties of SSAWs have been extensively studied in the astrophysical and geophysical context (Voitenko et al. 1990, Hasegawa & Chen 1992, Voitenko 1994, De Azevedo et al. 1994, Voitenko 1998c). The efficiency of the energy exchange between waves and plasma particles caused by kinetic properties of SSAWs has been proven recently both by theory and by laboratory experiments (Jau et al. 1997).

It is thus surprising that almost no attention has been paid to these properties of the SSAWs created by the phase-mixing process in the non-uniform corona. Indeed, there are papers studying the creation of short transversal length-scales in Alfvén waves and the eventual collisional wave dissipation. There are also a few papers which concentrate on nonlinear effects in Alfvén waves introduced by phase mixing (Nakariakov 1997) and resonant absorption (Poedts & Goedbloed 1997). But these investigations have been carried out in the one-fluid MHD approximation, missing important properties of Alfvén waves.

As we shall show in the present article, the finite (ion) Larmor radius (FLR) effects in finite-amplitude Alfvén waves can come into play at length-scales which are not very short, even much longer than both the ion Larmor radius and the length-scale of collisional dissipation. We study the parametric decay of the phase-mixed SSAW into two secondary SSAWs, induced by the combined action of the finite wave amplitude and FLR effects. This is an important...
nonlinear process which can significantly modify the wave dynamics in phase mixing.

As for ideal MHD Alfvén waves, it is well known that they cannot interact among themselves. But we show that the three-wave coupling among SSAWs becomes important with the creation of small transversal length-scales in the phase-mixed Alfvén waves. In order to describe this process, we take into account that the motions of the electrons and the ions in AWs become decoupled at small length-scales: to describe SSAWs, one have to use at least two-fluid MHD. The nonlinear kinetic theory of SSAWs, including nonlinear three-wave resonant interaction and linear Cherenkov wave-particle resonant interactions, has been developed by Voitenko (1998a, b).

In this paper we use the much simpler two-fluid MHD model, which nevertheless allows us to study some properties of SSAWs, such as the thermal wave dispersion, dissipation, and nonlinear three-wave interaction. The collisional damping self-consistently appears from the electron-ion friction force, while kinetic Landau damping is taken from the kinetic theory in ad hoc manner.

2. BASIC EQUATIONS

The wave electromagnetic fields obey Maxwell’s equations
\[
\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t},
\]
\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},
\]
\[
\nabla \cdot \mathbf{E} = 4\pi \rho,
\]
where the current density \( \mathbf{j} = \sum q_i n_i \mathbf{v}_i \), and the charge density \( \rho = \sum q_i n_i \), have to be calculated using a suitable mathematical model of the plasma. The most popular models are based on the ideal MHD equations, the two-fluid MHD equations, and the kinetic Vlasov equations.

To take into account some important linear and nonlinear effects in the low-frequency SSAWs, we adopt the mathematical model of two-fluid MHD, where the electron and ion fluids are allowed to move in separate ways, but are coupled by collective electromagnetic fields and by the electron-ion friction force. The equations of motion for the electrons and ions are:
\[
\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \nabla) \mathbf{v}_e = -\frac{e}{m_e} \left( \mathbf{E} + \frac{1}{c} \mathbf{v}_e \times \mathbf{B} \right) - T_e \frac{n_e}{m_e} \mathbf{v}_e + \mathbf{R}_e,
\]
\[
\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \nabla) \mathbf{v}_i = \frac{e}{m_i} \left( \mathbf{E} + \frac{1}{c} \mathbf{v}_i \times \mathbf{B} \right) - T_i \frac{n_i}{m_i} \mathbf{v}_i + \frac{e}{m_i} \mathbf{R}_i,
\]
where \( p_{e,i} = n_e,i T_{e,i} \) is the ion/electron pressure, \( T_{e,i} = const \) (but we do not require \( T_e = T_i \)). The parallel friction force, \( \mathbf{R}_{\parallel} \parallel |\mathbf{B}_0| \), responsible for the (parallel) resistivity along \( \mathbf{B}_0 \), is
\[
\mathbf{R}_{\parallel} = -\nu (|\mathbf{v}_{\parallel} - \mathbf{v}_{\parallel}|),
\]
where \( \nu = 0.51 \nu_e \), and the electron collision frequency is
\[
\nu_e = \frac{4\sqrt{2}\pi \Lambda e^4 n_e}{3\sqrt{m_e} f_e^{3/2}},
\]
where \( \Lambda \) is the Coulomb logarithm. In addition, the particles number density of each species obeys the continuity equation
\[
\frac{\partial}{\partial t} n_s + \nabla (n_s \mathbf{v}_s) = 0,
\]
where the subscript \( s \) denotes electrons with \( s = e \) and ions with \( s = i \).

3. AFLVÉN WAVE EIGENMODE EQUATION

Taking into account the highly anisotropic character of the low-frequency SSAWs (the transversal gradients are much larger than the field-aligned ones, \( |\mathbf{n}_{\perp}| \gg |\mathbf{n}_{\parallel}| \)), we derived a second-order eigenmode equation for the effective potential \( \psi \) of SSAWs (\( \psi \) related to the field-aligned inductive (solenoidal) part of electric field, \( \mathbf{E}_{\parallel} = -\nabla_{\parallel} \psi \)) (see Voitenko & Goossens 1999):
\[
\{ \nabla_{\parallel}^2 V_{\perp} \}
\frac{\partial^2}{\partial t^2} \left( \frac{m_e}{m_i} \mathbf{v}_e - V_{\perp} \nabla_{\parallel} \right) - V_{\perp} \nabla_{\parallel}^2 \psi =
\frac{\partial^4}{\partial t^4} \left( \frac{m_e}{m_i} \mathbf{v}_e - V_{\perp} \nabla_{\parallel} \right) + \nabla_{\parallel}^2 \left( N_{ei} - N_e \right) - \nabla_{\parallel}^2 \nabla_{\parallel}^2 \frac{\partial^2}{\partial t^2} \left( \frac{m_e}{m_i} \mathbf{v}_e - V_{\perp} \nabla_{\parallel} \right) \}
\]
The coefficients \( N_{ei} \) here represent nonlinear terms:
\[
N_e = -\frac{\partial}{\partial t} \left( \mathbf{v}_e \cdot \nabla \phi_e \right) + \mathbf{v}_e \cdot \left( \frac{\partial}{\partial t} \frac{m_e}{m_i} \mathbf{v}_e \right) - \frac{1}{\Omega_{ei}^2} \frac{\partial}{\partial t} \left[ \mathbf{b}_0 \times \mathbf{f}_{\parallel} \right]
\]
and
\[
N_i = \frac{T_i}{T_e} \frac{V_{\parallel}}{V_{\perp}} - \frac{\partial}{\partial t} \left( \mathbf{v}_i \cdot \nabla \phi_i \right) + \mathbf{v}_i \cdot \left( \frac{\partial}{\partial t} \frac{m_i}{m_i} \mathbf{v}_i \right) - \frac{1}{\Omega_{ei}^2} \frac{\partial}{\partial t} \left[ \mathbf{b}_0 \times \mathbf{f}_{\parallel} \right]
\]
and
\[
N_{ei} = -\nabla_{\parallel}^2 \left( \frac{m_i}{m_i} \mathbf{v}_i \right) + \mathbf{v}_{\parallel} \cdot \left( \frac{\partial}{\partial t} \frac{m_e}{m_i} \mathbf{v}_e \right) - \frac{1}{\Omega_{ei}^2} \frac{\partial}{\partial t} \left[ \mathbf{b}_0 \times \mathbf{f}_{\parallel} \right]
\]
where the nonlinear force is
\[ \mathbf{f}_n = \frac{1}{c} \mathbf{v}_s \times \mathbf{B} - \frac{m_s}{q_s} (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s, \] (12)
and \( \phi_s = - (T_e/q_s) \ln \left( \frac{n_e}{n_0} \right). \) Only linear parts of the solutions of the perturbed velocities,
\[ \mathbf{v}_{s \parallel} = \frac{q_s}{m_s} \frac{1}{\Omega_s} \left[ \frac{\partial}{\partial t} - \mathbf{e}_{||} \times \left( \mathbf{E} - \frac{\nu e}{n_e} \mathbf{E}_{||} + \mathbf{v} \phi_{s \parallel} \right) \right], \] (13)
\[ \frac{\partial}{\partial t} \mathbf{v}_{s \parallel} = \frac{q_s}{m_s} \varepsilon \left( \mathbf{E} - \frac{\nu e}{n_e} \mathbf{E}_{||} + \mathbf{v} \phi_{s \parallel} \right), \] (14)
will be used in (9-12) (see Voitenko & Goossens (1999) for details).

The differential order of this equation may be reduced by dropping the term \( \sim V_{te}^2 \mathbf{E}_{||} \), which has a small effect on the Alfvén wave branch in a low-\( \beta \) plasma. In the above expressions \( \rho^2_\perp = V_{te}^2 / \Omega^2_s = (1 + T_e / T_s) V_{te}^2 / \Omega^2_s \), the electron skin length \( \delta^2_e = c^2 / \omega_{pe}^2 \), and ion skin-length \( \delta^2_i = (m_i / m_e) \delta^2_e \).

The eigenmode equation (8) contains the thermal (FLR) correction (\( \sim V_{te}^2 \mathbf{E} : \nabla \cdot \mathbf{v} \)), the electron inertia correction (\( \sim \delta^2_e \mathbf{v} \)), and the collisional dissipative term (\( \sim \nu \)) in the linear (left-hand side) part. The nonlinear (right-hand side) part contains terms due to nonlinear electron and ion density (\( \sim N_e, N_i \)) and due to nonlinear current density along background magnetic field, \( \sim N_{ei} \).

4. LINEAR DAMPING OF THE PHASE-MIXED ALFVÉN WAVE IN THE CORONA

4.1. Collisional dissipation in WKB-approximation

Consider the evolution of an AW, excited at magnetic field lines. The length scale of the transversal inhomogeneity of the equilibrium is \( L_\perp \) and the length scale of the field aligned inhomogeneity of the equilibrium is \( L_\parallel \). Both \( L_\perp \) and \( L_\parallel \) can vary over a wide range in the solar corona as we shall see below, but usually \( L_\parallel \gg L_\perp \). Let the wave frequency, which determines the parallel length-scale of the wave fields, \( \lambda_\parallel \), be so that \( \lambda_\parallel \ll L_\parallel \). Then, as the wave propagates, the perpendicular length-scale of the wave fields decreases in time as \( \lambda_\perp / L_\perp = \tau_A / t \) (\( \tau_A \) is the wave period). Therefore, even if the initially excited AW has a smooth distribution in the direction of the plasma inhomogeneity, phase mixing creates short wave length-scales \( \lambda_\perp < L_\perp \) in a few wave periods. In open magnetic configurations, where waves propagate upward from the footpoints, this corresponds to heights to the order of a few field-aligned wavelengths. From now on we shall study the dynamics of Alfvén waves, in which short perpendicular wavelengths, \( \lambda_\perp < L_\perp \), are either determined by a (unspecified) generator, or are created by phase-mixing.

In this situation we consider the evolution of the \( \omega \)-th harmonic of the wave field using a WKB-ansatz:
\[ A = A_0 \times \exp \left( -i \omega t + i \int \mathbf{k} \cdot d\mathbf{r} + \int \gamma dt \right). \] (15)

The wave vector \( \mathbf{k} \) and the growth/damping rate \( \gamma \) have to be calculated from the local dispersion relation, and the integration is along the path of the wave propagation. The effects of plasma inhomogeneity for these waves come about through the spatial dependence of the plasma parameters along the path of the wave propagation. This path is determined by the ray equations for the wave vector \( \mathbf{k} \) and position \( \mathbf{r} \):
\[ \frac{d\mathbf{k}}{dt} = - \frac{\partial \omega}{\partial \mathbf{r}} = - \omega \left( \frac{\partial V_A}{\partial \mathbf{r}} \right); \] (16)
\[ \frac{d\mathbf{r}}{dt} = \frac{\partial \omega}{\partial \mathbf{k}}. \] (17)

In accordance with the above, the wave frequency here is determined by the local dispersion relation found from the dynamic equation (8) for Alfvén waves. When we neglect the nonlinear terms in (8), we get the linear Alfvén wave frequency, \( \omega \), and the linear damping rate, \( \gamma_c \), which accounts for the thermal and electron inertia effects, and for electron-ion collisions. In a low-\( \beta \) plasma (\( \beta = V^2_A / V^2_e \)),
\[ \omega^2 = \frac{k^2 \gamma_e^2 \mu^2 + k^2 \gamma^2}{1 + k^2 \rho^2_\perp}; \] (18)
\[ \gamma_c = -0.5 \nu \frac{k^2 \rho^2_\perp}{1 + k^2 \rho^2_\perp}. \] (19)

When we neglect the electron inertia term, we find the SSAW dispersion to be close to the AW dispersion, known from the kinetic theory (Hasegawa & Chen 1976), for all values of the dispersion parameter \( k^2 \rho^2_\perp \). In the limit \( k^2 \rho^2_\perp \gg 1 \) they are identical.

It is convenient to introduce the dispersion function \( K (k_\perp) \):
\[ K^2 (k_\perp) = \frac{1 + k^2 \rho^2_\perp}{1 + k^2 \rho^2_\perp}; \] (20)
and to write the AW dispersion relation in the form
\[ \omega = k_\parallel V_A K (k_\perp). \]

The function \( K (k_\perp) \) gives the factor by which the parallel phase velocity of the AWs with perpendicular wavelength \( 2\pi / k_\perp \) exceeds \( V_A \).

In the case of weak AW dispersion \( (k^2 \rho^2_\perp, k^2 \rho^2_\parallel \ll 1) \), \( K (k_\perp) \approx 1 \), we recover the asymptotic result of Heyvaerts and Priest (1983) for the wave damping with height \( z \):
\[ A = A_0 \times \exp \left( \int \gamma_L \left( \frac{\partial \omega}{\partial z} \right)^{-1} dz \right) \]
\[ = A_0 \times \exp \left( -\frac{\rho^2_\parallel}{z^2} \right), \] (21)
with the characteristic height of the resistive (collisional) wave dissipation

\[ z_c = l_\omega \left( \frac{\nu_e}{\Omega_e} \right)^{-\frac{1}{3}}, \tag{22} \]

where

\[ l_\omega = \left( \frac{0.1 \omega^2}{\Omega_e^2 L_A^3 V_A} \right)^{-1/3}. \tag{23} \]

In a coronal hole the collisional dissipation distance can be estimated as \( z_c = (10^2 - 10^3) \times \lambda_c \). The transversal wavelengths \( A_\perp \sim 10^4 \) cm at these heights.

4.2. Collisional dissipation versus kinetic damping

Since the wave frequency and the other quantities that appear in (22) vary in the corona, and since the kinetic effect of Landau damping can come into play for short-scale AWs, let us examine the relative importance of resistive dissipation and Landau damping for the phase-mixed AWs.

The dissipation of an AW due to kinetic electron Landau damping is (Voitenko 1998c):

\[ \gamma_L = -\sqrt{\frac{\pi}{8}} \frac{\omega^2}{k L} \frac{V_A}{T_e} \frac{V_A}{\nu_e T_e}. \tag{24} \]

When we use Eq (19) for the dissipation of an AW due to e-i collisions, and compare it with (24), we find that collisional dissipation is stronger, \(|\gamma_c| > |\gamma_L|\), for the low-frequency part of AW spectrum:

\[ \omega_k < \omega_{cl}, \]  \tag{25} \]

where the marginal frequency \( \omega_c \) is

\[ \omega_{cl} = \sqrt{\frac{2}{\pi}} \frac{V_A}{T_e} \frac{K_k}{1 + k^2 \delta_e^2}. \tag{26} \]

In the limit of weak wave dispersion - which is likely for the coronal phase-mixing - the critical frequency does not depend on the wave parameters:

\[ \omega_{cl} = \sqrt{\frac{2}{\pi}} \frac{V_A}{T_e}. \tag{27} \]

This reflects the same dependence of the damping rate on the perpendicular wavenumber, \( \sim k^2 \), of both damping mechanisms for weakly dispersed AWs. The typical value in corona is \( \omega_{cl} \sim 1 \text{ s}^{-1} \). For the waves \( \omega_k > \omega_{cl} \) we find a stronger damping of the phase-mixed AWs due to kinetic Landau damping (cf. Eiffmov et al. 1994).

However, in addition to collisional dissipation and Landau damping, the finite-amplitude SSAW undergo also nonlinear interaction, described by the right-hand side of (8). Therefore, the amplitude of the pump AW undergoes changes not only due to collisional/kinetic dissipation, but also due to nonlinear interaction. Nonlinear AWs interaction and the question which process dominates for given plasma and wave parameters is investigated in the following sections.

5. NONLINEAR DAMPING OF THE PHASE-MIXED ALFVEN WAVE IN CORONA

5.1. Parametric decay instability of a pump SSAW

Parametric decay of SSAWs often plays a crucial role in wave propagation, nonlinear saturation of instabilities, and spectral dynamics (Voitenko 1998a,b). Let us consider a pump SSAW with frequency \( \omega_k \), wave vector \( \mathbf{k} \), and amplitude \( A_k \), propagating in the positive \( z \)-direction, i.e. \( k_z > 0 \). We shall proceed further in the local approximation and expand all perturbations entering the dynamic equation (8) into Fourier series. Two waves from the spectrum, \((\omega_1, k_1, A_1)\) and \((\omega_2, k_2, A_2)\), for which the resonant conditions \( k_1 + k_2 = k \) and \( \omega_1 + \omega_2 = \omega_k \) are satisfied, can effectively interact with the pump wave \((\omega_k, k, A_k)\).

Expressing all wave quantities through the parallel component of the electromagnetic potential \( A_\parallel \), we find from (8) the dynamical equations for waves 1 and 2, coupled via the parametric pump wave:

\[ \left( \frac{\partial}{\partial t^2} - \gamma_1 \right) A_1 = -U(k_1, k, -k_2)A_k A_k^*; \tag{28} \]

\[ \left( \frac{\partial}{\partial t} - \gamma_2 \right) A_2^* = -U(-k_2, -k, k_1)A_k^* A_1; \tag{29} \]

where the coupling coefficient

\[ U(k, k_1, k_2) = \frac{V_A}{4B_0} \frac{s_1 K_1 - s_2 K_2}{K_k} \frac{K_k}{\mu^2} \left( s_1 \frac{\mu_1^2}{K_1} + s_2 \frac{\mu_2^2}{K_2} + \frac{\mu^2}{K_k} \right) \frac{\sigma_1 \sigma_2 (k_1 \times k_2)}{s_3 \delta_e} \tag{30} \]

\[ s_3 = \mp 1 \text{ for } k_3 < \pm 0 (\alpha = 1, 2), k_z \text{ is chosen to be > 0}. \]

Apply operator \( \left( \frac{\partial^2}{\partial t^2} - (\gamma_1 + \gamma_2) \frac{\partial}{\partial t} + \gamma_1 \gamma_2 \right) \) to Eq. (28) and eliminating \( A_k^* \) by use of Eq. (29), to obtain the equation for the amplitude \( A_1 \) (the corresponding equation for the amplitude \( A_2 \) is obtained in the same way):

\[ \left( \frac{\partial^2}{\partial t^2} - (\gamma_1 + \gamma_2) \frac{\partial}{\partial t} + \gamma_1 \gamma_2 \right) A_1 = U(k_1, k, -k_2)U(-k_2, -k, k_1) |A_k|^2 A_1. \tag{31} \]

An exponential solution to (31), \( \sim e^{\delta t} \), has indices

\[ \delta = \frac{\gamma_1 + \gamma_2}{2} \pm \sqrt{\left( \frac{\gamma_1 - \gamma_2}{2} \right)^2 + \gamma_1 \gamma_2} \sqrt{\frac{s_3}{2} k^2 L^2}. \tag{32} \]

where the rate of non-linear interaction is

\[ \gamma_{NL}^2 = \frac{s_3}{2} \left( \frac{V_A}{4V_T} \right)^2 \frac{K_1 K_2 (K_2 - s_2 K_k) (s_1 K_k - K_1)}{\mu^2 \mu_1^2 \mu_2^2} \left( s_1 \frac{\mu_1^2}{K_1} + s_2 \frac{\mu_2^2}{K_2} + \frac{\mu^2}{K_k} \right)^2 |\mu_1 \times \mu_2|^2 \frac{B_k^2}{B_0^2}. \tag{33} \]

The usual condition for instability is \( \delta > 0 \). In the case of weak wave-particles interaction, \( \gamma_1, \gamma_2 \ll \gamma_{NL} \)
\( \gamma_{NL} \), the growth rate of the waves 1 and 2 is determined by the rate of the non-linear interaction:

\[ \delta \sim \gamma_{NL}. \]

For the rate of the non-linear interaction \( \gamma_{NL} \) to be real, i.e. \( \Re \gamma_{NL} > 0 \), the dispersion functions \( K_\alpha \) have to satisfy the following decay condition:

\[ (K_2 - s_2 K_1)(s_1 K_2 - K_1) > 0. \]  
(33)

This condition implies that at least one dispersion function of the decay product should be smaller than the dispersion function of the pump wave (\( \Re \gamma_{NL} < 0 \) if both \( K_1 \) and \( K_2 > K_4 \)). In what follows we choose \( K_1 < K_2 \), then the decay is possible into (1) parallel-propagating daughter SSARWs, \( k_{1z} > 0, k_{2z} > 0 \) (\( s_1 = s_2 = 1 \)), and into (2) counterstreaming daughter SSARWs, \( k_{1z} > 0, k_{2z} < 0 \) (\( s_1 = 1, s_2 = -1 \)) (here \( k_1 > 0 \) is chosen for the pump wave).

In what follows we will concentrate on the second decay channel, which is much stronger for SSARWs of a weak dispersion, \( \mu_1^2, \mu_2^2, \mu_3^2 \ll 1 \), likely for the AW phase-mixing in corona (Voitenko & Goossens 1999).

5.2. Decay into counterstreaming daughter SSARWs

\( (k_{1z} > 0, k_{2z} < 0) \)

For \( \mu_1^2, \mu_2^2, \mu_3^2 \ll 1 \), we use the asymptotic expression \( K(\alpha) \approx 1 + 0.5 k_1^2 \mu_1^2 \mu_0^2 \) in (32) with \( s_1 = 1, s_2 = -1 \), and maximize it, which results in a maximum growth rate of decay

\[ \gamma_{NL}^* = 0.3 \Omega_i V_A \frac{B_k}{B_0} \frac{V_L}{V_{Ti}} \mu_1^2, \]

achieved with \( \mu_1 = 0.776 \mu_1, \mu_2 = 0.442 \mu_2 \) (here we take \( T_s = T_i \) for simplicity).

Expression (34) shows the \( \sim k_1^2 \) dependence of the interaction rate on the perpendicular wavenumber of the pump wave, which is not so strong as in the case of parallel-propagating waves, \( \sim k_1 \) (Voitenko & Goossens 1999). This means that the decay into counterstreaming AWs is more effective for the weakly dispersing waves, created by phase mixing. Consequently, (34) gives a good approximation for the damping rate of the pump AW, and for the rate of the three-wave AWs interaction in k-space as well, if the decay into counterstreaming waves is not forbidden.

5.3. Collisionsal threshold of decay instability

Let us estimate the threshold amplitude for the decay instability of the phase-mixed pump in the corona, where the growth of the decay products due to nonlinear coupling via the pump wave may be balanced by their collisional dissipation. In a local approximation, the threshold amplitude for the pump wave to excite waves 1 and 2 is obtained from the marginal condition for decay \( \gamma_1 \gamma_2 = \gamma^2_{NL} \), where \( \gamma_1 \) and \( \gamma_2 \) are to be determined from (19).

In the coronal plasma, the collisional damping is

\[ \gamma_{c/(1,2)} = 0.5 \nu \delta_1^2 \delta_2^2, \]

and the marginal decay condition is

\[ \gamma^2_{NL} = 0.25 \nu^2 k_1^2 \mu^2 \delta_1^2. \]

For the strongest counterstreaming decay, the threshold appears to be independent of the perpendicular wavenumbers or other wave characteristics, and the threshold value depends only on plasma parameters:

\[ \frac{B_k}{B_0} = 0.57 \frac{m_e}{m_i} \frac{\nu V_A}{\Omega_i V_{Ti}}. \]

The actual value of the threshold with the typical coronal parameters is extremely low:

\[ \frac{B_k}{B_0} = 10^{-6} \sim 10^{-7}. \]

The decay condition \( \delta > 0 \) is certainly satisfied for the AWs with \( B_k/B_0 \sim 0.01 \), able to heat corona.

Equations (34–38) clearly demonstrate that the phase-mixed AWs with \( \mu \neq 0 \) can become nonlinearly unstable at very small amplitudes, \( B_k/B_0 \sim 10^{-6} \). With these amplitudes, the parametric decay becomes stronger than the collisional dissipation of the phase-mixed AW, \( \gamma_{NL} > \gamma_{c/L} \).

5.4. Nonuniformity thresholds

We consider three effects of nonuniformity on the parametric decay of phase-mixed AW.

1. It should be noted that although the decay of the \( \mu_1^2 \ll 1 \) pump wave into counterstreaming SSARWs is local in \( k_1 \)-space, it is non-local in \( k_2 \)-space because \( |k_{2z}/k_2| \ll \mu^2 \ll 1 \). This means that our results on this decay are applicable as long as the large parallel wavelength of counterstreaming (decay) SSARW is shorter than the inhomogeneity length scale, \( |k_{2z}^{-1} \gg L_{||} \). Although our results can be used for waves with \( \lambda_{2z} \sim L_{||} \) as order of magnitude estimates, for larger wavelengths they can be significantly affected by the non-uniformity. The corresponding marginal parallel wavelength of the pump wave and its parallel wavelength are related through \( \mu^2 \sim \lambda_{2z}/L_{||} \).

2. Also, the efficiency of the decay including counterstreaming AWs in a nonuniform plasma may be reduced by the finite correlation length along the field lines, \( L_{||} \). This correlation length may be estimated as \( L_{||} \approx V_A t_i / \epsilon \), where \( t_i \) is the duration of the quasi-periodic pulse which excites the pump wave. We can estimate the corresponding threshold for decay into counterstreaming SSARWs from the condition that the waves should overlap during growth time (\( \gamma^2_{NL} > V_A/L_{||} \)):

\[ \left| \frac{B_k}{B_0} \right| \approx 2.5 \sqrt{\beta} \left( k_1 \mu^2 L_c \right)^{-1} \frac{w_k}{\Omega_i}. \]

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which can be high for the waves with short correlation lengths.

3. The progressive detuning of the perpendicular wavenumbers of the interacting AWs triplet can limit the decay if the time of the nonlinear interaction is shorter than the time of the wavenumber's change. The physics of this process is as follows. In the course of time, the initially most effectively interacting triplet of resonant waves is removed from its top position in the wavenumber space, because the different temporal behaviour of the constant and evolving components of perpendicular wavevectors changes the wavenumber's ratios in the triplet. The corresponding limitation on the pump wave amplitude may be found from the condition $\gamma_{NL} / \omega < k_{\perp} L_A$.

6. LINEAR AND NONLINEAR DAMPING OF THE PHASE-MIXED ALFVÉN WAVE IN A CORONAL HOLE

The damping rate of a AW due to e-i collisions in coronal holes is:

$$\gamma_v \sim - 0.25 \nu_c k_{\perp}^2 \Delta \nu^2.$$  \hspace{1cm} (40)

The electron temperatures and densities in coronal hole are comparatively well determined (Wilhelm et al. 1998). Hence $\nu_c$ may be estimated as $\nu_c \sim 10^{12}$ s$^{-1}$ with typical $T_e \lesssim 10^6$ K and $n_e \lesssim 10^6$ cm$^{-3}$ in holes.

Collisional damping is stronger than Landau damping by $\omega_c \ll \omega_{NL}$, where the critical frequency, $\omega_{NL}$, is given by (26). The critical frequency $\omega_{NL} \approx 1$ s$^{-1}$ for typical coronal hole conditions, $V_A / V_{Te} = 1 / 2$, $\nu = 0.5 \nu_c = 4$ s$^{-1}$. Therefore, nonideal MHD can be applied only to the AWs with periods $\tau \gtrsim 10$ s. To describe the dissipation of AWs with shorter periods, $\tau \lesssim 10$ s, one has, in principle, to use the kinetic plasma model, accounting for the Landau damping. However, noting that the two-fluid MHD dispersion of AWs is close to the exact kinetic dispersion in the whole range of perpendicular wavenumbers and frequencies, we use the two-fluid MHD model and include Landau damping in a semi-empirically way.

As discussed in the section Discussion, we take the length-scale of the field-aligned inhomogeneity in a coronal hole $L_{||} \sim 10^6$ km, the transverse inhomogeneity length-scale in the range $1 \lesssim L_{\perp} \lesssim 10^5$ km, and the wave amplitudes in the range $0.01$ to $0.03$ for $B_0 / B_0$. For wave frequencies $1$ s$^{-1} \lesssim \omega \lesssim 10^4$ s$^{-1}$, the nonuniformity restrictions for the decay into counterstreaming waves may be satisfied. Then, using the damping rate (34) in (15), we find that the amplitude of initial wave decays with height as

$$A = A_0 \times \exp \left( - \frac{z^3}{z_N^3} \right), \hspace{1cm} (41)$$

where the distance of nonlinear damping

$$z_N \sim l_\omega \left( \frac{V_{Te}}{V_A} \right)^{1/3} \left( \frac{B_k}{B_0} \right) , \hspace{1cm} (42)$$

where $l_\omega$ is the same as (23).

Taking $\rho_i \sim 10^2$ cm, $B_k / B_0 \sim 10^{-2}$, and $\lambda_i \sim 10^5$ cm, we obtain a low damping height in the coronal holes, varying from $z_N \sim 10^4 \lambda_i$ with $L_{\perp} = 1$ km, up to $z_N = 10^3 \lambda_i$ with $L_{\perp} = 10^5$ km. The transversal wavelengths at these heights, $\lambda_i \sim 10^5$ cm, are still much longer than both the (collisional) dissipative length-scale and the ion gyroradius.

6.1. Collisional dissipation versus parametric decay into counterstreaming waves

To find the linear/nonlinear damping of the AW with height, we proceed further using a WKB-ansatz (15) for the AW amplitude. In the low-frequency range where collisional dissipation is stronger than Landau damping, $\omega > \omega_{NL}$, the damping rate includes linear collisional damping and nonlinear damping due to parametric decay:

$$\gamma = \gamma_v - \gamma_{NL} = -0.5 \nu k^2 \Delta \nu^2 - 0.3 \nu_c \frac{V_A}{V_{Te}} \left( \frac{B_k}{B_0} \right) .$$

Taking integrals in the exponent we find the same height behaviour of the amplitude as for collisional and nonlinear damping of the phase-mixed wave, but with the cumulative dissipation distance $z_{cN}$,

$$\frac{1}{z_{cN}^3} = \frac{1}{z_c^3} \left( 1 + \frac{z_c^3}{z_N^3} \right), \hspace{1cm} (43)$$

where the nonlinear damping height and the collisional damping height $z_c$ are given by (42) and (22).

We see that the characteristic height of the wave damping, $z_{cN}$, can be very different from that predicted by resistive MHD, $z_c$, if $z_c > z_{cN}$, which gives the condition for the nonlinearly-dominated phase-mixing:

$$\left| \frac{B_k}{B_0} \right| > \tilde{B}_{cN} = \sqrt{\frac{\nu_c}{\nu_{\Omega_i}}} \frac{V_A}{V_{Te}} . \hspace{1cm} (44)$$

The corresponding threshold amplitude $\tilde{B}_{cN}$ does not depend on the wave parameters and is almost the same as the collisional threshold for the parametric decay itself (37), which is a consequence of the local character of the AW parametric decay in $k_{\perp}$-space.

If the condition (44) is satisfied, the initial Alfven waves, excited at the base of the solar corona, are damped at heights $\sim z_N$. With typical coronal hole conditions discussed above, we obtain amplitudes of AWs, leading to the nonlinearity-dominated regime of phase-mixing:

$$\left| \frac{B_k}{B_0} \right| > 10^{-6} . \hspace{1cm} (45)$$

If the nonthermal broadening known from the spectral observations is due to AWs, then wave amplitudes $B_k / B_0 \sim 0.01 - 0.03$, and condition (45) is well satisfied. Then the damping height in this nonlinearly-dominated regime $z_{cN} \sim z_{cN}$.
6.2. Landau damping versus parametric decay into counterstreaming waves

In the frequency range where Landau damping is stronger than collisional one, \( \omega > \omega_{\perp L} \), we have to analyze the competition between Landau and nonlinear damping. Following the same procedure as in the previous section, we find the same height behaviour (41), but with cumulative dissipation distance

\[
\frac{1}{z_{L,N}^2} = \frac{1}{z_L^2} \left( 1 + \frac{z_L^3}{z_N^3} \right),
\]

where the Landau damping distance is given by

\[
z_L = \frac{1}{\omega} \left( \frac{\omega}{\Omega_i} \right)^{\frac{1}{2}}.
\]

The nonlinear damping dominates if \( z_L > z_N \), which gives the condition for the wave amplitude

\[
\frac{B_k}{B_0} > \tilde{B}_{LN} = \frac{V_{A}}{V_{Ti}} \frac{\omega}{\Omega_i}. \tag{47}
\]

The threshold amplitude \( \tilde{B}_{LN} \) in this case depends on the wave frequency and can be high for the high-frequency part of AW spectrum. Alternatively, taking the wave amplitude as known, we find the frequencies at which the nonlinear damping dominates:

\[
\frac{\omega}{\Omega_i} < \frac{\omega_{\perp N}}{\Omega_i} = \frac{V_{Ti}}{V_A} \left| \frac{B_k}{B_0} \right|. \tag{48}
\]

Taking \( \frac{B_k}{B_0} = 0.03 \), \( \frac{V_{Ti}}{V_A} = 0.1 \), and \( \Omega_i = 3 \times 10^5 \), we get \( \omega_{\perp N} = 10^3 \text{ s}^{-1} \) in holes.

The competition of the collisional dissipation and parametric decay into parallel-propagating waves is studied in the accompanying paper (Voitenko and Goossens 1999).

7. DISCUSSION

The parametric decay of the pump AW into two daughter AWs in a hot plasma has been first examined by Erokhin et al. (1978), followed by other authors (Volokitin & Dubinin 1989, Voitenko 1996c, Yukhimuk & Kucherenko 1993). The standard approach so far was to consider weak wave dispersion, \( k^2 \beta^2 < 1 \), and to assume that all three interacting AWs propagate in the same direction along the background magnetic field. However, the strongest decay of the weakly dispersing AW was missed in the previous investigations: the strongest decay occurs when the daughter waves propagate in opposite directions along the background magnetic field.

As a result of this nonlinear decay, the waves generated at the base of a coronal magnetic structure undergo a transition from laminar to turbulent propagation at certain height \( z = z_N \), determined by the plasma and wave parameters. Consequently, the whole picture of the AW dynamics in coronal plasma may be considerably modified, and not only at heights \( z > z_N \), but also at \( z < z_N \), because a part of the AWs can reverse their direction of propagation if the time of parametric decay into counterstreaming AWs is shorter than the time of wave dissipation. For a low-frequency short-scale Alfvén wave, we found that the collisional wave dissipation, represented by the Ohmic dissipation of the parallel wave current, is much weaker than the nonlinear damping due to parametric decay. The linear damping rate due to the ion-ion collisions (shear ion viscosity) can in some cases be comparable to Ohmic dissipation (Hasegawa & Chen 1976), but can hardly exceed it; under coronal hole conditions \( \gamma_i/\gamma_{hi} \sim 10 \).

In § 4 our results have been applied to the phase-mixed Alfvén waves in a coronal hole. The influence of the vertical inhomogeneity in coronal holes on the AW phase mixing has been investigated in detail in the linear approximation (Ruderman et al. 1998, De Moortel et al. 1999). The nonlinear generation of fast MHD waves by AWs has been considered by (Nakariakov et al. 1997), but this process seems to be less efficient for waves with amplitudes \( \sim 0.01 \), as suggested by observations. It was therefore important to check if the nonlinear process investigated in the present paper can influence phase mixed AWs in the corona, and in coronal holes in particular.

The length-scale of the field-aligned inhomogeneity in a coronal hole is primarily determined by the characteristic scale of the density variation with height, \( L_i \sim \rho_i \sim 10^8 \text{ km} \). However, the transversal inhomogeneity length-scale is uncertain. So, if \( L_i \) is determined by the visible plume sizes, then one can take \( L_i \sim 10^5 \text{ km} \). However, it is quite possible that the corona is much more structured in the horizontal direction, up to \( L_i \sim 1 \text{ km} \) (Woo 1996).

The wave frequency is not a well accurately determined parameter in the problem. From the point of view of the power present in the photospheric motions, and in the magnetic fields of chromosphere and corona, it can range from \( \omega \sim 10^{-3} \text{ s}^{-1} \), for the waves excited by granular motions, up to \( \omega \sim 10^5 \text{ s}^{-1} \), excited by the small-scale magnetic activity in the chromospheric network (Axford & McKenzie 1992). Persistent nonthermal plasma motions in the corona have been confidently detected by means of spectroscopic observations (Saba & Strong 1991, Tu et al. 1998, Harra & Ichimoto 1999). The wave amplitudes estimated from these data range from 0.01 to 0.03 for \( B/B_0 \), and energy requirements can be satisfied.

However, because of the lack of observational evidence that the low-frequency global modes supply the energy for the high-temperature corona, one can suppose that either the wave periods fall below the available time resolution, \( r < 1 \text{ s} \) (\( \omega > 1 \text{ s}^{-1} \)), or the wave coherence lengths can not yet be spatially resolved, i.e., at least, wavelength \( \lambda < 10^8 \text{ cm} \). This last condition in turn maps onto the frequency scale as \( \omega > 1 \text{ s}^{-1} \) for Alfvén waves if \( \lambda \) is the parallel wavelength. In principle, Alfvén waves can be highly localised in planes perpendicular to \( B_0 \), \( L_i < 10^8 \text{ cm} \), which makes them unresolved. But the direct footpoint excitation of short-scale waves requires too much power in the photosphere concentrated at small scales, and any alternative process involving an evolutionary structuring in the corona should be observed at the initial (large-scale!) stage if \( \omega < 1 \).
s$^{-1}$. In the closed magnetic configurations, like coronal loops, the global photospheric motions can excite short-scale standing AWs localised around resonant field lines, where the Alfvén travel time is equal to the typical photospheric timescale (Goossens 1994, De Groof et al. 1998).

As we restricted our analysis to the wavelengths $\lambda_2$ shorter than the vertical inhomogeneity length-scale, $L_{||} \sim 10^5$ km, our results can be applied to a part of the possible AW spectrum, corresponding to the frequency range $10^{-2} \, s^{-1} < \omega < 10^5 \, s^{-1}$. Moreover, in accordance to the above discussion, the lower part of this frequency range, $10^{-2} \, s^{-1} < \omega < 1 \, s^{-1}$, tends to be excluded from the observational point of view.

This consideration suggests that the waves responsible for coronal heating have frequencies $\omega > 1 \, s^{-1}$, or even higher (Tu et al. 1998, Wilhelm et al. 1998). Another kinetic dissipation mechanisms - ion-cyclotron damping - may become important at $\omega \lesssim \Omega_i$.

8. CONCLUSIONS

The results obtained in the present paper show that the process of phase-mixing, which is inevitable in the non-uniform solar corona, gives rise to the nonlinear coupling of finite-amplitude AWs with $B_f/B_s \gtrsim 0.01$, and the strongest coupling includes counter-propagating waves.

The parametric decay of the initial AWs into counter-propagating daughter waves is much stronger than the collisional damping and Landau damping in the wide frequency range $\omega < 10^4 \, s^{-1}$. The resulting spectral redistribution of the wave energy towards larger length-scales reduces the rate of the dissipation of wave flux. Only the high-frequency part of AW spectrum, $\omega > 10^4 \, s^{-1}$, is more effectively dissipated due to the Landau damping (or ion-cyclotron damping at $\omega \sim \Omega_i$).

The ordering of the characteristic frequencies is as follows: $\omega_{\text{min}} \sim (10^{-1} \, s^{-1}) < \omega_{\text{LL}} \sim (1 \, s^{-1}) < \nu_e \sim (10 - 100 \, s^{-1}) < \omega_{\text{WN}} \sim (10^5 - 10^4 \, s^{-1}) < \Omega_i \sim (10^5 - 10^5 \, s^{-1})$. The open question is: what are the frequencies of the waves?

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