NONLINEAR RESONANT MHD WAVES IN THE ATMOSPHERE OF THE SUN

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ABSTRACT

We consider driven resonant nonlinear magnetohydrodynamic (MHD) waves in dissipative steady plasmas to study the effect of equilibrium flows on the nonlinear resonant heating behaviour of MHD waves. The nonlinear analogue of the connection formulae for slow MHD waves are derived. This nonlinear theory is then used to study the interaction of sound waves with one-dimensional isotropic steady plasmas modelling, e.g., the wave interaction in the magnetic canopy. We find that a steady equilibrium shear flow can significantly influence the nonlinear resonant absorption in the limits of thin inhomogeneous layer and weak nonlinearity. The presence of an equilibrium flow may therefore be important for the nonlinear resonant MHD wave phenomena. A parametric analysis also shows the nonlinear part of resonant absorption can strongly be enhanced by the equilibrium flow.

Key words: nonlinear MHD waves; resonant absorption

can transfer their energy to each other. The transfer plasma-wave is related to instabilities which are not subject of this study. The transfer wave-plasma is related to the fact that in inhomogeneous plasmas external driving waves can resonantly interact with the local oscillation modes. The transferred energy can be converted into heat with the aid of dissipation which is one of the basic feature of the solar plasma. A similar phenomenon is used to explain the observed loss of power of acoustic oscillations in sunspots.

Usually resonant absorption is investigated in a static equilibrium. However, Hollweg et al. (1990) showed that, for an incompressible linear plasma in planar geometry, the damping or excitation of surface waves can be strongly influenced by an equilibrium shear flow. Erdélyi & Goossens (1996) studied numerically the loss of acoustic power by resonant absorption in cylindrical magnetic flux tubes in linear compressible dissipative plasmas with steady equilibrium state. They showed that resonant absorption can be significantly influenced by velocity shears. They also found negative absorption of wave power which apparently can be attributed to the resonant instability found by Hollweg et al. (1990).

The general theory of nonlinearity relies on the idea that linearized equations provide a satisfactory first approximation for finite-amplitude sufficiently small perturbations. Successive approximations may then be developed by expansion in ascending powers of a characteristic dimensionless amplitude of the waves. This is known as weakly nonlinear theory. In the present work we apply this theory to study the resonant absorption of slow MHD waves in steady dissipative layers.

The process of resonant absorption was suggested first by Ionson (1978) as a non-thermal heating mechanism of the corona and since then it was studied by many authors in the linear regime. Wave dynamics, however, becomes nonlinear if amplitudes reach sufficiently large value. Linear studies of velocity scaling laws indicate that, with classical solar dissipation parameters, the velocities at the resonant layer are several orders of magnitude larger than the...
observed non-thermal velocities indicating nonlinear theory might be needed.

Nonlinearity gives rise to unexpected effects in solar plasmas which have not seen yet in linear approximations. Such an effect is, e.g., the apparition of solitons. A soliton arises as a result of competing effects, typically the tendency of nonlinearity to steepen a wave and the dispersion which tends to broaden waves. When dispersion balances the nonlinear steepening, a wave can propagate indefinitely without changing its form. Solitons are of interest in many areas of physics including, e.g., solar physics (see, e.g., Roberts & Mangeney 1982, Roberts 1984, Zhugzhda & Nakariakov 1999). Another important effect of nonlinearity appears in problems related to the nonlinear coupling of waves and phase-mixing. This coupling can be a source of wave generation with double frequencies which can provoke shock waves. At the same time a perpendicular inhomogeneity to the magnetic field can generate phase-mixing; another important nonlinear effect (see e.g. Oraevsky 1983, Wentzel 1977, Nakariakov & Oraevsky 1995, Nakariakov, Roberts & Murawski 1997). A third effect of the nonlinearity in the context of solar plasmas is emerging from the resonant absorption of MHD waves in magnetic structures. Linear theory predicts that the amplitude of perturbations in the vicinity of the ideal resonant position can be very large. The implication is that linear theory can break down due to this resonant behaviour of the waves in these regions (e.g., Ofman & Davila 1994, Ruderman et al. 1997b, Ballai et al. 1998).

The aim of the present study is to apply the nonlinear theory of slow dissipative layers in isotropic plasmas with steady equilibrium state to assess the effect of equilibrium flow on the efficiency of resonant energy absorption. We show that the nonlinear decrease of absorption coefficient found by Ruderman et al. (1997a) can be balanced or override into enhancement with equilibrium flow.

2. THEORY

It is known from linear MHD theory that the dissipation can remove singularities in the solutions of the MHD system. Dissipation acts normally only in a thin layer embracing the ideal singular position. The thickness of this layer is inversely proportional to the Reynolds number, \( R \). This means that the plasma outside the dissipative layer can be safely described by the ideal MHD equations. Another key quantities are the dimensionless amplitude of the waves, \( \epsilon \ll 1 \), and the characteristic scale of the inhomogeneity and dissipative layer, \( l_{in} \) and \( l_{dis} \), respectively. When deriving the governing equations for waves in the dissipative layers, we assume that that the largest nonlinear terms and the dissipative terms are of the same order, i.e., \( \epsilon \sim R^{-3/2} \).

The equilibrium quantities depend only on \( x \) and the background magnetic field lies in the \( yz \) plane in Cartesian geometry. The equilibrium bulk velocity \( C_0 \) is chosen to be parallel with the magnetic field lines. In contrast to the linear theory, now we cannot Fourier-analyze the perturbations. A new variable, \( \theta = x - Vt \), where \( V \) is the phase velocity is introduced. To be as close as possible to linear theory, we also suppose that waves are plane periodic propagating waves with a permanent shape. This also means that all perturbed quantities are periodic and depend on \( \theta \).

2.1. The Outer Region

In order to derive the equation that describes the wave motion in the dissipative layer we use the method of matched asymptotic expansions (MMAE, see, e.g., Nayfeh 1981). Ruderman et al. (1997b) applied this method to obtain the governing equation that describes resonant slow MHD waves in a static isotropic dissipative plasma.

Here we use only a simplified version of this method earlier applied in linear theory by Goossens et al. (1995), and in nonlinear MHD by Ballai et al. (1998). We do not repeat the whole cumbersome mathematical procedure of MMAE. For readers interested in the method and its previous applications they are well described in Ruderman et al. (1997b). The validity and applicability of the simplified version is also discussed in Ballai et al. (1998).

Because the dissipative layer is very narrow and amplitudes of perturbations are small, we are able to use the ideal MHD equations outside this region. Using the linear ideal MHD equations, all but two variables can be eliminated and we obtain a system of two equations for \( u \) (normal component of the velocity perturbation) and \( P \) (total pressure perturbation):

\[
\frac{\partial u}{\partial x} = \frac{V}{D} \frac{\partial P}{\partial \theta}, \quad \frac{\partial P}{\partial x} = \frac{\rho_0 D_A}{V} \frac{\partial u}{\partial \theta},
\]

(1)

where

\[
D = V^4 - V^2(C_A^2 + C_S^2) + C_A^2 C_S^2 \cos^2 \alpha,
\]

(2)

\[
D_A = V^2 - C_A^2 \cos^2 \alpha,
\]

\[
D_T = (C_A^2 + C_S^2)(V^2 - C_S^2 \cos^2 \alpha),
\]

(3)

where \( \alpha \) is the angle between the \( z \)-axis and the direction of magnetic field and \( V = V - C_0 \cos \alpha \) is the Doppler shifted phase velocity. All other variables can be expressed in terms of \( u \) and \( P \). In the above relations \( C_A, C_S \) and \( C_T \) are the Alfvén, sound and tube speed. In the vicinity of the ideal resonant position the solution to the system of equations (1) is expressed in form of a Fröbenius series with respect to \( x \). The coefficients of the expansions are different for \( x < 0 \) and \( x > 0 \) semi-planes. These expansions reveal in the vicinity of the resonant singularity, the total pressure, \( (P) \), perpendicular component of the velocity \( (u_\perp) \) and magnetic field \( (b_\perp) \) are regular; the normal component of velocity \( (u) \) and \( x \)-component of the magnetic field \( (b_x) \) have a logarithmic behaviour; and finally the pressure \( (p) \), density \( (\rho) \) and the parallel component of the velocity \( (u_\parallel) \) and magnetic field \( (b_\parallel) \) have an \( 1/x \) singularity.

2.2. The Internal Region

Let us now calculate the solutions for the dissipative layer. We look for solutions in the form of power
series of $\epsilon$. The most important result obtained in the
first order approximation (referred to by superscript 1) is

$$P^{(1)} = P^{(1)}(\theta),$$

(4)
i.e., the total pressure does not change across the
dissipative layer. This result is in agreement with
results obtained in linear theory (Sakurai et al. 1991,
Erdélyi 1968), and in nonlinear theory (see, e.g.,
Ruderman et al. 1997b and Ballai et al. 1998) of slow
resonant MHD waves in static equilibria. Unlike that
found in cylindrical geometry, Eq. (4) means that
an equilibrium flow does not change the continuity
of the total pressure across the dissipative layer.

In the next order approximation we obtain a linear
non-homogeneous system for variables with super-
script 2. The non-homogeneous part of these equa-
tions is expressed with the aid of terms with super-
script 1 (Deriving the compatibility codition of the
system we obtain an equation which describes the
slow wave motion in the dissipative layer

$$\text{sign}(\Delta)\sigma \frac{\partial^2 \theta}{\partial \theta^2} - \Delta \frac{\partial q}{\partial \theta} + k \frac{\partial^2 q}{\partial \sigma^2} = \frac{kq^4}{\rho_0 v_A^2 |\Delta|} \frac{dP}{d\theta},$$

(5)
where $\Delta$ is defined as

$$\Delta = \frac{d}{dx}(\nu^2 - C_T^2 \cos^2 \alpha).$$

(6)
The thickness of the dissipative layer is defined by
the relation

$$\delta_C = \left[ \frac{\nu}{k|\Delta|} \left( \nu + \frac{C_T^2}{C_A^2} \theta \right) \right]^{1/3}.$$  

(7)
We can see that the thickness of the dissipative layer
is now strongly modified by the equilibrium flow.
The modifications are two-fold: (i) a Doppler shift
in the phase speed; (ii) the change of $\Delta$.

In Eq. (5), the ratio of the second (i.e. non-linear
term) to the third (i.e. dissipative term) is of the order

$$\lambda \approx O(\epsilon(R_k l_n n)^{1/3})$$

(8)
The parameter $\lambda$ can be considered as a parameter of
nonlinearity, i.e., nonlinearity is important if $\lambda > 1$;
when $\lambda \ll 1$ the nonlinear term in equation (5) can
be neglected. Next, we connect the solution in the
two overlap region with the aid of the so-called jump
conditions. One of the jump condition is related to
the normal component of the velocity which has a
jump across the dissipative layer. After some algebra
(Ballai & Erdélyi 1998) this jump is

$$[u] = \left[ -\frac{\nu}{k \cos^2 \alpha} P \int_{\infty}^{\infty} \frac{\partial q}{\partial \theta} d\sigma, \right.$$

(9)
where square brackets indicate the jump of the quan-
tity across the dissipative layer considered as a sur-
facing of discontinuity. In this equation we take the
Cauchy principal part $P$ because the integral is di-
vergent at infinity. This equation is the nonlinear
connection formula obtained assuming steady equi-
libration motion in the inhomogeneous region. In con-
trast to the linear theory where the jump in $u$ was
obtained in terms of the total pressure $P$ and some

equilibrium quantities, the non-linear connection for-
ma contains an integral form of an unknown func-
tion $q$. The second connection formula is already
given by Eq. (4), i.e.,

$$[P] = 0.$$  

(10)
A simple physical interpretation of the continuity of the
perturbation of total pressure $P$ across the dissi-
pative layer is that there has to be a balance of forces
at both sides of the dissipative layer. Since the dissi-
pative layer is very thin, it has small inertia and
cannot alter drastically the pressure force by cross-
ing the dissipative layer from one side to the other.
The system of equations (1) (obtained for the outer
region) and the two non-linear jump conditions (9)
and (10), respectively, constitute a complete system
of equations and boundary conditions. Note though,
that we have to solve these simultaneously with Eq. (5).

3. APPLICATION: RESONANT INTERACTION OF ACOUSTIC WAVES WITH SLOW DISSIPATIVE LAYERS

Nonlinear theory of driven MHD waves in the slow
dissipative layer in isotropic steady plasmas de-
veloped by Ballai & Erdélyi (1998) is used to study
the nonlinear interaction of sound waves with one-
dimensional isotropic steady plasmas. For simplicity
we take the magnetic field parallel with the OZ-axis.

We adopt a three-layer model where monochromatic
sound waves impinge from the $x < 0$ unmagnetised
half-space (Region I) and penetrate into an inho-
mogeneous region $0 < x < x_0$ (Region II) where
the equilibrium magnetic field is parallel with the
OZ-axis. The third region, $x > x_0$, (Region III) is
penetrated by a homogeneous magnetic field. The
equilibrium flow is present in the magnetic regions.
The nonlinearity parameter, introduced by Ballai &
Erdélyi (1998), is assumed to be small and a regu-
lar perturbation method is used to obtain analytical
wave solutions. The presence of an Alfvén resonance
would complicate the analysis. To remove it, we as-
sume that the wave propagations and perturbations
of all quantities are independent of $y$. To achieve this,
we rotate the coordinate system around the z-axis
transforming the wave vector of an incoming sound
wave to be entirely in the $x$-plane.

The ideal slow resonant position(s) can be deter-
bined by the condition

$$\Omega^2 = k^2 C_T^2(x_c), \ \Omega = \omega - k \cdot C_0.$$  

(11)
For fixed (e.g., positive) driving frequency $\omega$,
Eq. (11) can have two solutions. This is different
from the static case because the introduced equi-
libration flow can couple the driving acoustic oscilla-
tions not only to the locally generated 'forward' propa-
gating slow modes but also to the 'backward' propa-
gating slow modes. In what follows we assume that
$C_T(x)$ is a monotonic function, and the equilibrium
flow is proportional to it, namely,

$$C_0(x) = f_{C_0} \times C_T(x),$$

(12)
where $f_{C_0} \ll 1$ is the plasma flow strength. This
equilibrium mass-flow is introduced in order to make
sure that there is a smooth transition to the boundary of the inhomogeneous region, so that there is no need to worry about a boundary layer. The main attempt here is to obtain a first insight into the effect of steady mass flows on the nonlinear resonant interaction of sound waves and slow continua.

### 3.1. Solutions For Weak Nonlinearity

The derivation of the equation of wave energy absorption is very cumbersome. The procedure is fully described in Ruderman et al. (1977) and Erdélyi & Ballai (1999). Applying successive orders of approximations one can show that nonlinearity generates higher harmonics in the outgoing wave in addition to the fundamental one. Gathering all the monochromatic components we find the coefficient of energy absorption can be written of the form

\[ \alpha = \alpha_{\text{lin}} + \lambda^2 \alpha_{\text{nonlin}}. \]  

(13)

In contrast to the static case, where nonlinearity decreases the coefficient of the wave energy absorption, in steady plasmas we will see that nonlinearity can increase the absolute value of the coefficient of absorption.

### 3.2. Inhomogeneous Bulk Motion

Results are applied to study the absorption of 5-min (p-mode) oscillations in magnetic canopy in the solar atmosphere. Strictly speaking p-modes (at least with low frequency) are evanescent in the photosphere and above, because their frequency is less than the acoustic cut-off frequency in this region. We approximate the (high-frequency) p-modes as running acoustic waves. The magnetic canopy is modelled by a horizontal magnetic field with subsonic field-aligned plasma motions. This plasma motion could be, e.g., associated with the chromospheric downflow observed very extensively in the nearby past (Dosecek et al. 1976) or recently by SOHO (Brekke et al. 1997, Chae et al. 1998, Warren et al. 1997). We suppose that the motion is inhomogeneous and simply proportional to the local cusp speed. Upward propagating acoustic waves are impinging from the non-magnetic photospheric side (Region I) into the magnetic canopy (Region II and III) where they are resonantly coupled to the inhomogeneous slow resonant layer. Figs. 1 and 2 show the effect of the field aligned plasma flow on the nonlinear part of coefficient of absorption and on the full (linear plus nonlinear) coefficient.

One can see that nonlinearity indeed decreases the absorption coefficient as found by Ruderman et al. (1997a) and Ballai & Erdélyi (1998) though this effect is ceased with increasing flow as predicted by Ballai & Erdélyi (1998). Note also, that the total absorption (linear plus nonlinear) is not so high indicating that only maximum around 15% of the incident energy flux can be converted into heat. At present we cannot comment the significance of this amount of energy to the global chromospheric and/or coronal heating. Figures 3 and 4 are similar to Figs. 1 and 2 but the flow is anti-parallel with the equilibrium magnetic field. Anti-parallel flows can have stronger effect on the resonant interaction causing more efficient energy change.

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**Figure 1.** Nonlinear correction of coefficient of energy absorption as a function of the plasma strength \( f_{\text{Cp}} \). The inhomogeneous flow and magnetic field direction coincide.

**Figure 2.** Total coefficient of energy absorption as a function of plasma flow strength \( f_{\text{Cp}} \). The inhomogeneous flow and magnetic field direction coincide.

**Figure 3.** The same situation as in Figure 1, but the flow is in anti-parallel to the direction of magnetic field lines.
3.3. Homogeneous Bulk Motion

Another possibility could be of modelling the plasma motion as a homogeneous flow in Region II and III. In this case at the interface of the external region and the inhomogeneous region an extra boundary condition is needed for the normal component of the velocity, i.e.,

\[ \frac{u_{z\to 0}}{V} = \frac{u_{z\to +0}}{V}. \]  

(14)

Figs. 5-6 shows the nonlinear correction and the total resonant absorption rate as a function of the flow strength parameter \( f_{C_0} \) in the case of parallel field-aligned plasma flow, while Figs. 7-8 show refer to anti-parallel flows. As a result, one can see that the equilibrium flow can strongly amplify the resonant interaction.

A parallel constant flow ceases, again, the effect caused by nonlinearity (Figs. 5-6). In this special cases a strong flow can even switch off entirely the resonant interaction (Fig. 6). An anti-parallel flow suppresses the nonlinear effect though it does not really have too much effect on the whole dissipative mechanism (Fig. 7-8). This is in agreement with results found numerically for resonant Alfvén waves in coronal loops (Erdélyi 1998).
4. CONCLUSIONS

The present study applies the nonlinear theory of resonant slow MHD waves in steady plasmas. With the use of the method of matched asymptotic expansions (Ruderman et al. 1997b) Ballai & Erdélyi (1998) obtained the nonlinear counterparts of the governing equation for slow wave motion inside a steady-dissipative layer. The jump conditions across the dissipative layer depend on the equilibrium flow in a more complicated way than just Doppler shifts. The effects of an equilibrium flow are not easy to predict in general terms, because to the best of our knowledge we cannot solve generally the governing equation (Eq. 5).

The nonlinear jump conditions together with the governing equations outside the dissipative layer and the simultaneous solution of Eq. (5) makes it possible to study the resonant absorption of nonlinear slow MHD waves without solving the full set of dissipative MHD equations. We applied the obtained analytical results to cases appropriate in solar physics. The model studied here can be considered as an approximate model of interaction of, e.g., 5-min sound waves with the magnetic canopy in the chromosphere, or interaction of acoustic waves with hot magnetic fibrils.

The magnetic canopy/hot magnetic fibril is modelled by a unidirectional magnetic field with subsonic field-aligned plasma motions. Propagating acoustic waves are impinging from the non-magnetic region, i.e., photospheric side (Region I) into the magnetic region, i.e., canopy/hot fibril (Region II and III). We have assumed that (i) the thickness of the slab containing the inhomogeneous plasma (Region II) is small in comparison with the wavelength of the incoming sound wave (i.e., $kz_0 \ll 1$); and (ii) the nonlinearity in the dissipative layer is weak; the nonlinear term in the equation describing the plasma motion in the dissipative layer can be considered as a perturbation.

We found that nonlinearity in the steady slow dissipative layer generates higher harmonic contributions in the outgoing sound wave in addition to the fundamental harmonic. This result is similar to the static case found by Ruderman et al. (1997b). The equilibrium flow in the slow dissipative layer can either increase or decrease the coefficient of the wave energy absorption depending on the wave characteristics, and the strength and direction of the field-aligned flow. Thus, a field-aligned flow has an important effect on the resonant interaction of acoustic (e.g., $p$-mode) waves and nonlinear slow resonant layers.

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