Symmetry Considerations in Stellar Dynamos

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Abstract. The dynamics of stellar magnetic fields is discussed. An ODE model describing the interactions of global (large-scale) magnetic fields with velocity fluctuations is presented. The model is derived using symmetry arguments and does not rely on assumptions of how the large-scale field may be generated from small scales. The model is compared with the results of mean field models described by PDEs. It is shown that aspects of the complicated behaviour found in nonlinear dynamos (including cycle modulation and parity ‘flipping’) can be understood with reference to the underlying symmetry properties of the equations.

1. Introduction

Solar and stellar magnetic activity is known from examining Ca\(^+\) emission to exhibit complicated temporal behaviour (Wilson 1994). Magnetic cycles in stars (with typical periods of 7 – 15 years) are modulated on a longer timescale, a modulation that often leads to periods of reduced activity (Grand Minima). Indeed there is even an example of a binary star system, UX Arietis (Massi \textit{et al.} 1998), that shows modulation of a basic magnetic cycle, with activity being followed by quiescent periods. The most famous example of a reduction in solar activity occurred in the seventeenth century (the Maunder Minimum) when few sunspots were observed. Proxy records such as terrestrial isotopes (e.g. \(^{14}\)C and \(^{10}\)Be) demonstrate (see e.g. Stuiver, Grootes & Braziunas 1995) that this minimum was not an isolated event, but that numerous similar episodes of reduced activity have occurred in the past with a characteristic timescale between events of 200 years.
Figure 1. Butterfly Diagrams: Plots of the toroidal field $B$ as a function of colatitude $\varphi$ and time $t$ from a two-dimensional mean field dynamo model (Tobias 1997, Beer et al. 1998). Four different solutions are shown. (a) Periodic dipole, (b) periodic quadrupole, (c) quasiperiodic dipole, (d) chaotic ‘flipping’ between different parities.

The azimuthally averaged spatial structure of the solar magnetic field can also be observed. Hale’s polarity law for sunspots indicates that the underlying toroidal field is usually antisymmetric about the equator. The poloidal field is derived from a potential that is usually symmetric about the equator and so the solar field is usually dipolar. However, as the sun emerged from the Maunder Minimum of over one hundred sunspots observed only four were in the northern hemisphere (Ribes & Nesme-Ribes 1993), indicating a substantial asymmetry between the two hemispheres. Thus the observed complicated temporal behaviour is associated with spatial asymmetry and consequent changes in the parity of the field.

It is now widely believed that stellar magnetic fields are generated and maintained by dynamo action. Stellar turbulence and large-scale flows (e.g. differential rotation) combine to produce large-scale magnetic fields. Because of
the complexity of the problem, however, only local models of turbulent dynamos can currently be studied. Much success has been achieved by looking at global models where the small-scale effects are parametrized – using mean field electrodynamics (MFE). Producing mean field models has become an industry in itself despite worries about the validity of the theory (see, for example, Hoyng 1999). Mean field models have been very successful in reproducing many aspects of stellar activity, from modification of the differential rotation profiles via $\Lambda$-quenching (Kitchatinov, Rüdiger & Küker 1994), to modulation of basic cycles in the nonlinear regime and even the production of solutions that show the characteristic symmetry-breaking of the magnetic field as solutions emerge from Grand Minima (Tobias 1996, 1997). Some examples of the range of solutions found in two-dimensional PDE models are given in Figure 1. Figure 2 demonstrates that in the PDEs two different types of modulation may be found. Type 1 modulation is characterised by a transfer of energy between the dipole and quadrupole modes, whereas Type 2 modulation involves a continuous transfer of energy between the magnetic field and the velocity perturbations. These two types of modulation found in the PDEs are therefore fundamentally different in their origin.

What is the future for mean-field theory? It is possible to produce more and more elaborate models with differing assumptions about the distributions of the physical inputs and the values of the parameters, but it is not clear that this is ultimately useful. What is necessary is to compare and contrast these models and to determine which features appear in all the models and are therefore generic, and which are model-dependent.

Some progress in this direction can be made by considering the underlying mathematical structure of the dynamo equations and analysing its consequences for the allowed solutions. For example, the fact that nonlinear dynamo waves always have a non-zero group velocity implies that such waves must interact with the equator in a nontrivial way; this interaction may result in complicated nonlinear behaviour (see Soward 1992, Tobias et al. 1997 or Proctor, this volume). In this paper we focus on a model that describes the interaction between modes with opposite parity with respect to the equator and the associated large scale flows. These interactions are constrained by the symmetries of the rotating sphere or spherical annulus in which the dynamo operates. The analysis of the resulting model, derived in detail in Knobloch et al. (1998; hereafter KTW98), tells us much about the nature of the solutions of the dynamo problem that are allowed by the symmetries.

2. Models that do not depend on MFE

A number of alternative approaches to simply integrating the (partial differential) mean field equations have been tried. A number of authors have studied low-order (ODE) models since these can be investigated in more detail than PDE models. However, if the model is derived as a truncation of a PDE (and the PDE comes from MFE) one must be concerned about the extent to which they depend on the details of the model and whether the results obtained could possibly be an artefact of the truncation.
Figure 2. Three-dimensional phase portraits (projected onto the space spanned by the dipole energy $E_D$, the quadrupole energy $E_Q$, and the averaged square of the velocity perturbations $\langle v^2 \rangle$) for the PDEs showing the difference between (a) Type 1 and (b) Type 2 modulation. In (a) the energy is transferred between the dipole and quadrupole modes whereas in (b) the quadrupole energy is zero and modulation occurs as a result of a transfer of energy between the dipole mode and the velocity perturbations.
Solar observations, however, indicate that the sun does have a global mode (for both the magnetic field and the velocity perturbations) and that this mode oscillates with a basic period of 22 years and is modulated in time. Without making any comments/assumptions about how this large-scale mode is generated from small scales it is still possible to comment on the dynamics of this mode given the symmetries of the system. It is then important to relate the dynamics back to the PDE models and determine which of those models are displaying generic behaviour.

2.1. New model

The model we present here is based purely on symmetry considerations, the (generic) assumption that the dynamo mode is overstable (Knobloch 1994) and the knowledge that the magnetic Lorentz force acts quadratically to drive velocity perturbations. Full details of the derivation of the model can be found in KTW98. We consider evolution equations for the amplitude of the dipolar magnetic field ($z_1$), of the quadrupolar magnetic field ($z_2$), the symmetric part of the velocity perturbation driven by the Lorentz force ($v$) and the antisymmetric part of this velocity perturbation ($w$). The model we derive is of sixth order (two complex equations for $z_1$, $z_2$ and two real equations for $v$, $w$) and is

$$
\dot{z}_1 = (\mu + \sigma + i\omega_1) z_1 + a |z_1|^2 z_1 + b |z_2|^2 z_1 + c z_2^2 \bar{z}_1,
+ (\epsilon v + \delta v^2 + \kappa w^2) z_1 + (\beta + \gamma v) w z_2,
$$

$$
\dot{z}_2 = (\mu + i\omega_2) z_2 + a' |z_2|^2 z_2 + b' |z_1|^2 z_2 + c' z_1^2 \bar{z}_2
+ (\epsilon' \nu + \delta' v^2 + \kappa' w^2) z_2 + (\beta' + \gamma' v) w z_1,
$$

$$
\dot{v} = -\tau_1 v + e_1 (|z_1|^2 + |z_2|^2),
$$

$$
\dot{w} = -\tau_2 w + e_2 (z_1 \bar{z}_2 + z_2 \bar{z}_1).
$$

Here $a$, $b$, $c$, $\beta$, $\gamma$, $\delta$, $\epsilon$, $\kappa$, $a'$, $b'$, $c'$, $\beta'$, $\gamma'$, $\delta'$, $\epsilon'$, $\kappa'$ are all complex coefficients whilst $\tau_1$, $\tau_2$, $e_1$ and $e_2$ are real. The sixth-order system described by equations (1-4) has two important subsystems. If $z_2$ is set to be zero, so that only the dipole mode is considered, then the system reduces to the one considered by Tobias et al. (1995). Moreover if the interaction of the magnetic field with the velocity perturbations is ignored (by setting $e_1 = e_2 = 0$) then the interactions are purely between dipole and quadrupole fields and the system reduces to that proposed by Knobloch & Landsberg (1996). A more detailed discussion of the properties of the system (1-4) can be found in KTW98. An important property to note, however, is that unless both dipole and quadrupole components are non-zero the antisymmetric part of the velocity perturbations ($w$) decays. This is a symmetry property of the PDE system as well, as shown in Figure 3.

3. Results and comparisons with PDEs

Equations (1-4) contain many parameters that control the evolution of the system. An extensive (although by-no-means exhaustive) investigation of the properties of this system is included in KTW98. Here we simply demonstrate that even the most complicated behaviour found so far in the PDEs can be reproduced in this set of symmetry-based equations.
Figure 3. ‘Torsional Oscillations’: Plots of the velocity perturbations for the PDE model. (a) A mixed mode solution of the type shown in figure 2(a) drives a velocity perturbation that is antisymmetric with respect to the equator whereas in (b) a solution of pure symmetry (dipole) must drive velocity perturbations that are symmetric.

Figure 4(a) shows the projection of a trajectory obtained from the PDE model exhibiting ‘flipping’ behaviour of the type shown in Figure 1(d). This behaviour, discussed in detail by Beer et al. (1998), involves a solution that spends much of the time as a chaotically modulated dipole solution, but occasionally emerges from a particularly deep grand minimum as a quadrupole. Beer et al. (1998) noted that the grand minima may act in this way as a switch between solutions of different symmetries and hence that it is possible that the sun has in the past had episodes with a largely quadrupolar magnetic field. The corresponding behaviour for the ODE model is shown in Figure 4(b); the similarity in the form of the two solutions is striking. The time series for the ‘flipping’ behaviour is shown in Figure 5, with $P = \mp 1$ indicating a pure dipole (quadrupole) mode, respectively.

4. Discussion

We have demonstrated how symmetry considerations in stellar dynamo theory can lead to the derivation of a simple model that reproduces both the behaviour observed in stars and that found in “sufficiently realistic” PDE dynamo models. By “sufficiently realistic” we mean that the model is at least two-dimensional (so that radial diffusion is incorporated correctly) and includes a dynamic nonlinearity (i.e., one which is non-local in space and non-instantaneous in time). Examples of these nonlinearities are the Malkus-Proctor effect and $\Lambda$-quenching.

The model we derived (which is discussed in detail in KTW98) describes the interactions between dipole and quadrupole modes and the velocity perturbations driven by the nonlinear Lorentz force. It is capable of describing both Type 1 and Type 2 modulation (as found in the PDEs) and can also reproduce
Figure 4. Comparison of PDE solutions with the symmetry-based model. (a) Projection of the trajectory from the PDEs corresponding to figure 1(d). (b) Projection of a solution of the ODE model onto a similarly defined subspace (see Knobloch et al. 1998 for details). Both solutions show complicated modulation as well as 'flipping'.
Figure 5. Time series for the ‘flipping’ solution of the ODE model. (a) Dipole energy $\mathcal{E}_D$, (b) quadrupole energy $\mathcal{E}_Q$, (c) total energy $\mathcal{E}$, and (d) parity $\mathcal{P}$. Both the dipole and quadrupole energies are chaotically modulated. After particularly deep minima the solution parity may flip in a manner reminiscent of the PDEs.
the more complicated ‘flipping’ behaviour demonstrated by Beer et al. (1998). Moreover, the presence of invariant subspaces and non-normal parameters in the model makes it a good candidate for displaying ‘in-out intermittency’ as in the models of Covas et al. (1997). Whilst this behaviour does not appear to be relevant to solar modulation, it may be an important feature of dynamos in other stars.

Acknowledgments. SMT is grateful to Trinity College, Cambridge, for funding to attend this conference.

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