INVERS10: A New Code for Magnetic Doppler Imaging

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Abstract:
We present a new Magnetic Doppler Imaging computer code designed to simultaneously reconstruct both the 3D structure of magnetic fields and the temperature structure of active regions on the surfaces of late-type stars. We show results from numerical experiments and demonstrate that the full set Stokes parameters is needed for proper reconstruction of the magnetic field vector.

1. Introduction

For the last three years we have been developing a new code designed to reconstruct the distribution of magnetic field vectors and temperature on the surfaces of active late-type stars. The code will also map magnetic field and chemical spots on the surfaces of chemically peculiar stars. The reconstruction process extracts information contained in the rotational modulation of the four Stokes parameters or polarization profiles. The goal is to create a tool that produces reliable Magnetic Doppler Images (MDI) by solving the inverse problem in a consistent way. Surface inhomogeneities in magnetic field and temperature are both taken into account when computing the Stokes profiles at each rotational phase.

The core of any Doppler Imaging code is the local line profile calculation. This part of the code should be able to solve the equation of radiative transfer (RT) for specific intensities in a fast and accurate manner. To appreciate the stringent requirements imposed on the RT solver, recall that in the case of an active late-type star, high excitation atomic lines and molecular bands may be present in the same spectrum, and the relative strength of blends may change dramatically with depth and across the surface. Therefore the RT solver should include sophisticated physics in other to produce realistic profiles. The situation is even more complex in the presence of magnetic fields, where orientation changes relative to the line of sight as the star rotates. When considering magnetic fields, the number of free parameters forces us to solve the RT “on the fly” rather than pre-calculating tables of local line profiles (Piskunov & Rice 1993), which means that the magnetic RT solver must be extremely fast.

In the next section we describe and evaluate different methods for solving magnetic RT, comparing their speed, accuracy and stability. After that we briefly outline the rest of our MDI code and demonstrate the first results of numerical experiments with the INVERS10 code.
2. Magnetic RT Solver

Three algorithms for RT solver in the presence of magnetic field have been considered: Runge-Kutta, Feautrier and Diagonal Element Lambda Operator (DELO). All three methods have been implemented by different people and detailed descriptions can be found in Landi Degl’Innocenti (1976, Runge-Kutta), Auer et al. (1977, Feautrier), and Rees et al. (1989, DELO). We strongly recommend those papers for anybody who wants to find out specific details about each algorithm. Here we just give a short outline of the magnetic RT problem and the approach taken by different algorithms.

The magnetic RT problem is a first order system of ordinary differential equations:

\[
\frac{d\mathbf{I}}{dz} = -K \mathbf{I} + \mathbf{j},
\]

(1)

where \( \mathbf{I} = (I, Q, U, V) \) is the vector of Stokes parameters, \( \mu \) is the limb angle, \( K \) is the total absorption matrix and \( \mathbf{j} \) is the total emission vector:

\[
K = \begin{pmatrix}
    k_c + k_I \cdot \phi_I & k_I \cdot \phi_Q & k_I \cdot \psi_U & k_I \cdot \psi_V \\
    k_I \cdot \phi_Q & k_c + k_I \cdot \phi_I & k_I \cdot \psi_V & -k_I \cdot \psi_U \\
    k_I \cdot \psi_U & -k_I \cdot \psi_V & k_c + k_I \cdot \psi_I & k_I \cdot \psi_Q \\
    k_I \cdot \psi_V & k_I \cdot \psi_Q & k_c + k_I \cdot \psi_I & k_I \cdot \psi_Q
\end{pmatrix}
\]

(2)

\[
\mathbf{j} = \begin{pmatrix}
    k_c \cdot S_c + k_I \cdot S_I \phi_I \\
    k_I \cdot S_I \phi_Q \\
    k_I \cdot S_I \psi_U \\
    k_I \cdot S_I \psi_V
\end{pmatrix},
\]

(3)

where \( k_c \) and \( k_I \) are the continuum and line opacity and \( S_c \) and \( S_I \) are the continuum and line source functions. In cases where the continuum is formed in LTE \( S_c = B_\nu \) (we also assume no polarization in the continuum). We note for later use that the diagonal elements of the absorption matrix are dominant which provides the basis for DELO algorithm.

The Zeeman splitting depends on the strength of the magnetic field and the Landé factors of \( \pi \)- and \( \sigma \)-components. The amplitude of the Stokes parameters depends on the orientation angles of magnetic vector (the angle \( \gamma \) between magnetic vector and the line of sight, and the position angle \( \chi \)) via the absorption coefficients \( \phi \)'s and anomalous dispersion coefficients \( \psi \)'s. \( \psi \)'s are responsible for magneto-optical effects. The relation of \( \phi \)'s and \( \psi \)'s to the line profiles of the Zeeman components is given by:

\[
\phi_I = \frac{1}{2} \phi_p \sin^2 \gamma + \frac{1}{4} (\phi_r + \phi_b)(1 + \cos^2 \gamma)
\]

\[
\phi_Q = \frac{1}{2} [\phi_p - \frac{1}{2}(\phi_r + \phi_b)] \sin^2 \gamma \cos 2\chi
\]

\[
\phi_U = \frac{1}{2} [\phi_p - \frac{1}{2}(\phi_r + \phi_b)] \sin^2 \gamma \sin 2\chi
\]

\[
\phi_V = \frac{1}{2} (\phi_r - \phi_b) \cos \gamma
\]

(4)
\[
\psi_Q = \frac{1}{2} \left[ \psi_p - \frac{1}{2} (\psi_r + \psi_b) \right] \sin^2 \gamma \cos 2\chi \\
\psi_U = \frac{1}{2} \left[ \psi_p - \frac{1}{2} (\psi_r + \psi_b) \right] \sin^2 \gamma \sin 2\chi \\
\psi_V = \frac{1}{2} (\psi_r - \psi_b) \cos \gamma 
\]

where indices \( p, b, r \) stand for \( \pi \)-components and \( blue \) and \( red \) \( \sigma \)-components.

The wavelength dependence of \( \phi_p, \phi_b, \) and \( \phi_r \) are given by the Voigt function \( V(a, v) \) while \( \psi_p, \psi_b, \) and \( \psi_r \) are proportional to the Faraday-Voigt function \( F(a, v) \). Humlíček (1982) gives very fast and accurate complex approximation for \( V(a, v) \) and \( F(a, v) \).

### 2.1. Runge-Kutta Magnetic RT Integrator

Runge-Kutta techniques for solving the radiative transfer equations (1) integrate the Stokes parameters from the bottom of the atmosphere where an initial condition is set. A detailed description of the algorithm and its computer implementation (the MALIP code) has been given by Landi Degl’Innocenti (1976). He also analyses the main problems of the techniques. The advantage of Runge-Kutta is that the accuracy of the integration is checked on every step, so one can set the required accuracy \textit{a priori}. We would also like to point out that the RT equation is one of a few cases where 6th order Runge-Kutta has a substantial advantage over the conventional 4th order scheme because the accuracy can be checked without refining the step size. The main disadvantage is that different parts of the right hand side have a different depth dependence, and in order to achieve high accuracy the algorithm is forced to use very small steps in \( z \). To summarize: \textit{Runge-Kutta techniques are accurate but slow. The results are especially useful as a reference for other methods.} Now we shall turn to finite differences integration techniques which are more promising in terms of speed.

### 2.2. Feautrier Magnetic RT Integrator

The Feautrier method for solving RT equation operates by splitting the intensity into two oppositely directed beams. The resulting equation is a second order ODE with two boundary conditions (one at the bottom and one at the surface of the atmosphere). Since the finite difference approximation involves 3 adjacent points for each step the method has good stability and convergence properties. Application of the Feautrier method to non-magnetic RT requires the solution of a system of linear equations that form a tri-diagonal matrix. Although the accuracy cannot be checked at each step and the properties of the residual errors are much more complex than in the case of Runge-Kutta, refining the depth grid generally leads to a fast convergence and an accurate result. The Feautrier method has been extended to handle magnetic RT by Auer et al. (1977). In that case the tri-diagonal matrix is replaced by a block tri-diagonal, where each block is a \( 4 \times 4 \) matrix. The equations can be solved by analogy with the non-magnetic case, but back-substitution requires lots of \( 4 \times 4 \) matrix inversions and multiplications. The net result is a significant accumulation of numerical errors. For the centers of Zeeman components where the Voigt function is maximal and the Faraday-Voigt function is close to zero the difference between diagonal
and non-diagonal elements in the absorption matrix reaches several orders of magnitude and with all the inversions, multiplications, and subtractions this scheme is bound to be numerically unstable. The alternative is to treat the block tri-diagonal matrix as a band diagonal matrix. The band should include 15 diagonals in order to cover all the blocks. The resulting scheme is robust against numerical errors for the price of only 20% degradation in speed. Comparison with Runge-Kutta shows that for the same conditions, the Feautrier RT solver is about 30 times faster if the required accuracy is $10^{-3}$. That is not quite fast enough for MDI, as the typical disk integration procedure requires approximately $10^3$ surface elements and the magnetic RT equation must be solved for each of them.

2.3. DELO Magnetic RT Solver

Twelve years after the formulation of magnetic Feautrier algorithm, Rees et al. (1989) proposed a lambda operator method serving as a one-way magnetic RT integrator. It is based on the fact that the absorption matrix is dominated by its diagonal elements. The principle can be easily illustrated in the non-magnetic case, but the DELO method is most impressive when integrating Stokes parameters.

In the non-magnetic case we can write the formal solution of the RT equation connecting the intensities at optical depths $\tau_k$ and $\tau_{k+1}$:

$$I(\tau_k) = I(\tau_{k+1}) \cdot e(\tau_k - \tau_{k+1}) + \int_{\tau_k}^{\tau_{k+1}} e^{-\tau + \tau_k} S(\tau) d\tau$$  \hspace{1cm} (5)

where $S(\tau)$ is the source function. If we assume that the source function in our depth interval is linear in $\tau$ and can be expressed as $S(\tau) = [(\tau_{k+1} - \tau) S_k + (\tau - \tau_k) S_{k+1}]/(\tau_{k+1} - \tau_k)$, then the integration in equation (5) can be performed analytically and we obtain a recurrence relation of the type:

$$I(\tau_k) = P_k + Q_k \cdot I(\tau_{k+1})$$  \hspace{1cm} (6)

with a boundary condition at the bottom of the atmosphere.

Generalization to the magnetic case is straightforward. After we implemented this method, we found it to be free of numerical instabilities and about 6 times faster than the Feautrier method (both are a direct result of much fewer matrix inversions). On the down side, we found that the convergence properties of the DELO method are not as good as for Feautrier (not surprising as the latter is a second order finite difference method), and it takes a much finer grid (4 – 8 times smaller stepsize) to reach an accuracy of about $10^{-3}$, thereby compromising the integration speed. After extensive experiments, we noticed that an adaptive depth grid can remedy the problem. It is much more efficient to refine the grid where the total opacity coefficient $k_e + k_l$ (diagonal elements of the opacity matrix) shows the largest variation. Once implemented, this techniques proved to a be a winner. It usually takes about 10% of additional grid points to reach the accuracy of $10^{-3}$. **DELO with adaptive refinement of the depth grid is the fastest technique with good stability and convergence properties.**
3. The Structure of INVERS10

With the new powerful magnetic RT solver based on the DELO method, we are able to compute local Stokes profiles “on the fly” rather than pre-calculating the interpolation tables. For each rotational phase, our new MDI code computes the specific intensity (Stokes) profiles for the local magnetic field and local model atmosphere, and derivatives with respect to field components (radial, and the two tangential) and the local temperature: $\partial I/\partial B_r$, $\partial I/\partial B_m$, $\partial I/\partial B_p$, and $\partial I/\partial T_{\text{eff}}$. The disk integration of the flux profiles takes into account the rotational Doppler shifts and the radial-tangential macroturbulence. After disk integration the discrepancy and the regularization functions are computed together with the gradient vector. We use a modified conjugate gradient procedure to improve the solution. The modification makes use of the gradient vector during 1D optimization, since the gradient vector can be computed with very little effort whenever the discrepancy function is evaluated. The overall procedure is efficient enough to reach a convergence for a typical size MDI problem (10 spectral lines, 100 wavelength points, 20 rotational phases) in 20-30 CPU hours on a fast workstation (an HP 9000 C-180 in our case) with about 15 minutes per function evaluation.

4. Numerical Experiments

The best way to assess the reliability of an inverse code is with numerical experiments in which we setup an artificial star with known surface structure, compute a set of “observed” profiles, and then use them as input data for the inversion. Below we show the results of 3 such experiments with INVERS10. In all cases we have used a rather strong field (8000 G) and a hot star (8000 K), but quite different filling factors. For this temperature we used the Fe\,II 6141 Å line, which has a rather typical Zeeman pattern with 6 $\pi$ and 10 $\sigma$ components. The effective Landé factor is 1.5. The “observed” profiles were computed for 10 equispaced rotational phases on a very fine surface grid using the Feautrier algorithm. The simulated profiles have been broadened by the instrumental profile corresponding to a resolving power of 60,000. The $v \sin i$ of the star was set to 30 km s$^{-1}$ with an inclination $i$ of 70°.

In the first experiment we attempted to reconstruct the central dipolar field. The magnetic axis was tilted 90° from the rotational axis and the polar field was 8000 Gauss. All four Stokes parameters were used in the inversion. The initial guess had the correct temperature but zero field. Figure 1 shows the results of successful reconstruction. The cross-talk between magnetic field and temperature is less than 150 K in the temperature map and less then 200 G in the magnetic map.

In the next experiment we used the same test star, but only two Stokes parameters ($I$ and $V$) were used in the inversion. The result is shown in Figure 2. The reconstructed magnetic field differs significantly from dipolar (most of the field vectors are directed along lines of constant latitudes in stellar coordinates, lower panel on Fig. 2a) while the cross-talk reached the level of 1000 K in the temperature map.
Figure 1a. The reconstruction of the global magnetic field.

For a model star with simple dipolar magnetic field (upper panel) we have computed the profiles of all four Stokes parameters at 10 rotational phases. The polar field was 8000 G and the surface temperature 8000 K. The reconstruction (lower panel) shows very accurate reproduction of the magnetic vectors practically all around stellar surface. The residual variations of the temperature are less than 150 K. Yellow and black arrows show the negative and positive field vectors.

Figure 1b. The comparison of the Stokes profiles for maps in Figure 1a. All four computed Stokes profiles (blue line) show very good fit to the simulated "observations" (black line).
**Figure 2a.** The importance of complete polarization information.
For a model star with simple dipolar magnetic field (upper panel) we have computed the intensity and circular polarization profiles at 10 rotational phases. The polar field was 8000 G and the surface temperature 8000 K. The reconstruction (lower panel) produces the I and V profiles identical to the dipolar case. The temperature variations in the reconstruction (nearly 1000 K) successfully compensate for the discrepancies between the true and the derived magnetic field. Yellow and black arrows show the negative and positive field vectors.

**Figure 2b.** The comparison of the Stokes profiles for maps in Figure 2a. The computed Stokes I and V profiles (blue line) match very closely the simulated "observations" (black line) while the linear polarization differs substantially.
Figure 3a. The small scale field reconstruction. The two temperature spots with opposite polarity magnetic fields (upper panel) on the surface of the test star produced very characteristic Stokes profiles which were successfully used to reconstruct the original temperature and magnetic vector distribution (lower panel). Yellow and black arrows show the negative and positive field vectors.

Figure 3b. The comparison of the Stokes profiles for maps in Figure 3a. Black line are the “observed” profiles and blue is the result of the inverse problem solution. Note that polarization signal is only present when we “see” the magnetic spots.
In the last experiment two small cool spots (7000 K, 1000 K cooler then the rest of the surface) were located at zero longitude with symmetrical placement relative to the equator. Both spots had a radial magnetic field of 4000 Gauss, but opposite polarity. The results, shown in Figure 3, demonstrate that 4 Stokes parameters even with very modest phase coverage can be used to recover realistic field and temperature structures.

5. Conclusions

Although many more experiments will be required to investigate all the properties of the new code, even now it is clear that we can reliably reconstruct the vector magnetic field and that observations of all four Stokes parameters are required. It is also clear that the MDI problem must be solved in a consistent way rather then by separately imaging magnetic field and temperature, since (at least with incomplete observations) one of the variables can successfully mimic the other.

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References