MODELING OF A LINEAR FORCE-FREE MAGNETIC FIELD IN A BOUNDED DOMAIN

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Abstract. A method for the reconstruction of the linear force-free magnetic field in a bounded domain (as a rectangular box, $\Omega$) is presented here. The Dirichlet boundary-value problem for the Helmholtz equation is solved for the $B_z$ component specified at the $\Omega$ boundary. Chebyshev's iteration method with the optimal rearrangement of the iteration parameters sequence was used. The solution is obtained as for the positive-definite, and for the non-sign-definite difference analogue of the differential operator $\nabla^2 u + \alpha^2 u$. Specifying two scalar functions, $B_x$ and $B_y$ on the intersection of the lateral part of the $\Omega$ boundary with one selected plane $z = \text{constant}$, and using $B_z$ inside the $\Omega$, we have found $B_x$ and $B_y$ throughout $\Omega$.

The algorithm was tested with the numerical procedure which gives the analytic solution $\mathbf{B}$ of the linear force-free field (LFFF) equations for the dipole in a half-space. The root-mean-square deviation of the analytic solution $\mathbf{B}$ from the calculated $\mathbf{B}'$ does not exceed 1.0%. Boundary conditions for the $\mathbf{B}'$ calculation were taken as given by the analytic LFFF solution $\mathbf{B}$. Comparison of $\mathbf{B}'$ with $\mathbf{B}''$, which was calculated by the potential non-photospheric boundary conditions, show that they differ significantly. Thus, the specification of boundary conditions at non-photospheric boundaries of the volume under consideration is of particular importance when modeling the LFFF in a bounded volume.

The algorithm proposed here allows one to use the information about magnetic fields in the corona for the modeling of LFFF in a limited domain above an active region on the Sun.

1. Introduction

Along with a study of coronal magnetic fields on the basis of the indirect information provided by radio, white-light, and soft X-ray observations (see, e.g., the review by Alissandrakis, 1994), the modeling of coronal magnetic structures by photospheric field measurements remains a strong tool for the investigation of the solar magnetic field. For the modeling, one usually employs the current-free or force-free approximation (see, e.g., reviews by Seeber and Staudt, 1983; Sakurai, 1989). Recently, Aly and Seeber (1993) proposed a new approach to calculate the magnetic field of the whole solar corona under the assumption that the field is force-free only inside closed domains above active regions, being potential everywhere else. In the modeling of a linear force-free field (LFFF) above a limited region on the solar surface (i.e., above an active region) boundary conditions at lateral parts of the volume are a problem. The solution to the LFFF equations is sensitive to the treatment of lateral boundary conditions (Seeber, 1978, 1982). On the other hand, neither the Fourier transform method, nor the Green function method give the possibility to impose directly arbitrary lateral boundary conditions. A numerical routine, which allows one to do this, should be useful in studying how

changes in lateral boundary conditions influence the LFFF solution, and in using
the information about coronal magnetic fields, which can be obtained from radio,
white-light, and soft X-ray observations.

Here we present a numerical algorithm which renders it possible to calculate
the LFFF in a bounded volume (such as a rectangular box, \( \Omega \)) provided \( B_z \) are
specified at all sites of \( \Omega \), and \( B_x, B_y \) are prescribed on the intersection of the
lateral part of the \( \Omega \) boundary with plane \( z = \text{constant} \). The algorithm is based
on the solution of the Dirichlet boundary-value problem (DBVP) to the Helmholtz
equation by Chebyshev’s iteration method.

2. Basic Formulation

The force-free field can be described by equations

\[
\nabla \times \mathbf{B} = \alpha \mathbf{B} ,
\]

\[
\nabla \cdot \mathbf{B} = 0 .
\]

If the scalar quantity, \( \alpha \), is constant in space, we have a linear force-free field with
the field vector satisfying the Helmholtz equation

\[
\nabla^2 \mathbf{B} + \alpha \mathbf{B} = 0 .
\]

Let us calculate a LFFF inside a rectangular box \( \Omega : \{x_0 \leq x \leq x_0 + L_x, y_0 \leq y \leq y_0 + L_y, z_0 \leq z \leq z_0 + L_z \} \). Let the bottom, \( \Sigma_1 \), bound \( \Omega \) by the plane
\( z = z_0 \); the side \( \Sigma_2 \) bound \( \Omega \) by the plane \( y = y_0 \); the side \( \Sigma_3 \) – by the plane
\( x = x_0 + L_x \); the side \( \Sigma_4 \) – by the plane \( y = y_0 + L_y \); the side \( \Sigma_5 \) – by the plane
\( x = x_0 \); the top \( \Sigma_6 \) – by the plane \( z = z_0 + L_z \).

Let the uniform grid, \( \omega \), inside \( \Omega \) be

\[
\omega = \{x_i = x_0 + (i - 1)h_1, i = 1, \ldots, n_1, L_x = (n_1 - 1)h_1; \\
y_j = y_0 + (j - 1)h_2, j = 1, \ldots, n_2, L_y = (n_2 - 1)h_2; \\
z_l = z_0 + (l - 1)h_3, l = 1, \ldots, n_3, L_z = (n_3 - 1)h_3 \}.
\]

We will calculate the LFFF inside \( \Omega \) under the assumption that \( B_z \) are specified
at all sides of \( \Omega \):

\[
B_z \mid_{\Sigma_1} = \psi_1(x, y) ,
\]

\[
B_z \mid_{\Sigma_2} = \psi_2(x, z) ,
\]

\[
B_z \mid_{\Sigma_3} = \psi_3(y, z) ,
\]
\[ B_z|_{\Sigma_4} = \psi_4(x, z) , \]  
(8)  
\[ B_z|_{\Sigma_5} = \psi_5(y, z) , \]  
(9)  
\[ B_z|_{\Sigma_6} = \psi_6(x, y) ; \]  
(10)  
and \( B_x, B_y \) are specified on the intersection, \( G \), of the lateral part of the boundary of \( \Omega \) with one selected plane \( z = z^*(z_0 \leq z^* \leq z_0 + L_z) \):

\[ B_x|_G = g_1(x, y)|_G , \]  
(11)  
\[ B_y|_G = g_2(x, y)|_G . \]  
(12)  
The method proposed here imposes no constraints on the functions \( \psi \) in (5)–(10) and \( g \) in (11)–(12). Boundary conditions (5)–(12) may be chosen either vanishing, or potential, or force-free, or by some another way using \textit{a priori} information about the field in the corona.

At first we calculated \( B_z \) throughout \( \Omega \), and secondly, using \( B_z \) and (11)–(12), the \( B_x \) and \( B_y \) components were computed.

2.1. \textbf{Calculation of} \( B_z \)

Every component of the LFFF satisfies the Helmholtz equation (3). So for the \( B_z \) component we have to solve the DBVP for the Helmholtz equation:

\[ \nabla^2 B_z + \alpha^2 B_z = 0 , \]  
(13)  
where \( \nabla^2 \) denotes the 3D Laplacian in Cartesian coordinates:

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} . \]

Boundary conditions are specified by (5)–(10).

Let us denote \( u \equiv B_z \). The difference analogue of the differentiation operator \( \mathcal{A}u \equiv \nabla^2 u + \alpha^2 u \) on the grid \( \omega \) can be defined as

\[
\mathcal{A}u = (u_{i+1,j,l} - 2u_{i,j,l} + u_{i-1,j,l})/h_1^2 + \\
+ (u_{i,j+1,l} - 2u_{i,j,l} + u_{i,j-1,l})/h_2^2 + \\
+ (u_{i,j,l+1} - 2u_{i,j,l} + u_{i,j,l-1})/h_3^2 + \alpha^2 u_{i,j,l} , \quad (i = 2, 3, \ldots, n_1 - 1, \; j = 2, 3, \ldots, n_2 - 1, \; l = 2, 3, \ldots, n_3 - 1) .
\]  
(14)  
The spectrum of the operator \( \mathcal{A}u \) is well known (Samarsky and Nikolaev, 1978, p. 470):
\[ \mu_k = \lambda_k - \alpha^2 , \]  

(15)

since eigenvalues \( \lambda_k \) of the 3D Laplacian are given by

\[ \lambda_k = \lambda^{(1)}_{k_1} + \lambda^{(2)}_{k_2} , \quad k = 1, 2, \ldots, n_1 \times n_2 \times n_3 , \]  

(16)

\[ \lambda^{(\gamma)}_{k_\gamma} = \frac{4}{h^2_{\gamma}} \sin^2 \frac{k_\gamma \pi}{2n_\gamma} , \quad k_\gamma = 1, 2, \ldots, n_\gamma ; \quad \gamma = 1, 2, 3 . \]  

(17)

Minimum and maximum eigenvalues of the 3D Laplacian are

\[ \min_k \lambda_k \equiv \delta = \sum_{\gamma=1}^{3} \frac{4}{h^2_{\gamma}} \sin^2 \left(\frac{\pi}{2n_\gamma}\right) , \]  

(18)

\[ \max_k \lambda_k \equiv \Delta = \sum_{\gamma=1}^{3} \frac{4}{h^2_{\gamma}} \cos^2 \left(\frac{\pi}{2n_\gamma}\right) . \]  

(19)

If \( \alpha^2 < \delta \), then \( \mu_k > 0 \) for all \( k = 1, 2, \ldots, n_1 \times n_2 \times n_3 \), i.e., the operator \( Au \) is positive-definite. A solution of this DBVP is well known to exist and to be unique. Stable algorithms to solve difference equations with such operators are developed. Later, in Subsection 2.1.1, we apply one of them for our purpose.

If \( \alpha^2 > \delta \), then, at first, a solution of the DBVP (13), (5)–(10) exists and is unique if and only if \( \alpha^2 \neq \lambda_k , \quad k = 1, 2, \ldots, n_1 \times n_2 \times n_3 \). Secondly, some eigenvalues, \( \mu_k \), are negatives, thus the operator \( Au \) is non-sign-definite. To solve the DBVP with such an operator, specific procedures are required. One of them is demonstrated in Subsection 2.1.2.

For our case, when the bounds of the spectrum of the difference operator \( A \) are known, and one knows nothing about a starting approach \( u_0 \) inside the domain of integration \( \Omega \), it is better to use Chebyshev’s two-level iteration model (Samarsky and Nikolaev, 1978).

2.1.1. The Case of the Positive-Definite Difference Operator, \( \alpha^2 < \delta \)

To solve the equation \( Au = 0 \) with the non-degenerate linear operator \( Au \), which is represented on the grid \( \omega \) by the positive-definite difference operator \( Au \), let us create the explicit iteration scheme (Samarsky and Nikolaev, 1978):

\[ u_{m+1} = u_m + \tau_{m+1} \cdot Au_m , \quad m = 0, 1, 2, \ldots, n , \]  

(20)

with an arbitrary starting function \( u_0 \); \( m \) is a running iteration number, \( n \) is a total number of iterations; \( Au \) is defined by (14); \( \{ \tau_{m+1} \} \) is a sequence of iteration parameters. The latter are given by

\[ \tau_m = \frac{\tau_0}{(1 + \rho_0 \nu_m)} , \]  

(21)
where
\[ \tau_0 = 2/(\gamma_1 + \gamma_2), \quad \rho_0 = (1 - \xi)/(1 + \xi), \quad \xi = (\gamma_1/\gamma_2), \]
(22)
\[ \gamma_1, \gamma_2 \] are the lower and upper bounds of the spectrum of the operator \( \mathcal{A}u \). For our case they are
\[ \gamma_1 = \delta - \alpha^2, \quad \gamma_2 = \Delta - \alpha^2 . \]
(23)
The total number of iterations, \( n \), is determined by the \textit{a priori} given value, \( \epsilon \), characterizing the accuracy of the calculations:
\[ n \geq n_0(\epsilon) = \ln(2/\epsilon)/\ln(1/\rho_1), \]
(24)
where
\[ \rho_1 = (1 - \sqrt{\xi})/(1 + \sqrt{\xi}). \]
(25)
In (21) \( \nu_m \) belong to the set \( \mathcal{M}_n \) of roots of the Chebyshev polynomial \( T_n(x) \) of the first kind of degree \( n \):
\[ T_n(x) = \begin{cases} \cos(n \arccos(x)), & |x| \leq 1, \\ \cosh(n \text{arc}(x)), & |x| \geq 1, \end{cases} \]
(26)
\[ \nu_m \in \mathcal{M}_n, \quad \mathcal{M}_n = \left\{ -\cos\frac{2i-1}{2n}\pi, = i = 1, 2, \ldots, n \right\}. \]
(27)
All elements of \( \mathcal{M}_n \) have to be chosen as \( \nu_m \).
The stability of the iteration process (20) significantly depends on the order of the taking of \( \nu_m \) out of \( \mathcal{M}_n \). The matter is that the norm \( \|S\| \) of an operator of the transition from an iteration to the next iteration can become greater than 1.0 for some \( m \), which leads to the growth of computational errors. It is worthwhile to arrange \( \{\tau_m\} \) in such an order that every transition with \( \|S\| > 1 \) is followed by the transition with \( \|S\| < 1 \). Then computational errors do not accumulate, and the iteration process is stable. A technique for optimal arrangement of \( \{\tau_m\} \) is given in the Appendix.
When the optimal sequence \( \{\tau\} \) is formed and a starting approach \( u_0 \) is specified (say, \( u_0 = 0 \) in all internal nodal points of the grid \( \omega \)), one can begin the iteration process according to scheme (20).

2.1.2. \textit{The Case of the Non-Sign-Definite Difference Operator}, \( \alpha^2 > \delta \)
An approach to solve grid equation \( \mathcal{A}u = 0 \) with a non-sign-definite difference operator may be as follows (Samarsky and Nikolaev, 1978). For the given non-sign-definite operator \( \mathcal{A} \), we set up the operator
\[ \hat{\mathcal{A}} = \mathcal{A}^2 - 2\beta \mathcal{A}, \quad (28) \]

where the parameter \( \beta \) has to be fitted so that \( \hat{\mathcal{A}} \) is a positive-definite operator. The iteration scheme for an equation with such an operator is well known (see, for example, the above subsection). By making use of this scheme and taking into account (28) one may form the iteration scheme to solve the equation with a non-sign-definite operator. It comes only to the complication of the sequence of iteration parameters \( \{\tau_m\} \). Below we present the formulae for the calculation of \( \{\tau_m\} \) for this case.

Let us suppose that \( \alpha^2 \neq \lambda_k \), where \( \lambda_k \) are determined by (16), (17). Then Equation (13) has a unique solution in \( \Omega \) satisfying the boundary conditions (5)–(10) at the \( \Omega \) boundary. Let \( \lambda_{\alpha_1} \) and \( \alpha_{\alpha_2} \) be the eigenvalues \( \lambda_k \) nearest to \( \alpha^2 \), so that

\[ \lambda_{\alpha_1} < \alpha^2 < \lambda_{\alpha_2}. \quad (29) \]

Since \( \alpha^2 > \delta \) and the operator \( \mathcal{A}u \) is a non-sign-definite one, its eigenvalues \( \mu_k = \lambda_k - \alpha^2 \) are located within two intervals such as

\[ \hat{\gamma}_1 \leq \hat{\gamma}_2 < 0 < \hat{\gamma}_3 \leq \hat{\gamma}_4. \quad (30) \]

From (15), (18), (19) and (29) we have

\[ \hat{\gamma}_1 = \delta - \alpha^2, \quad \hat{\gamma}_2 = \lambda_{\alpha_1} - \alpha^2, \quad \hat{\gamma}_3 = \lambda_{\alpha_2} - \alpha^2, \quad \hat{\gamma}_4 = \Delta - \alpha^2. \quad (31) \]

It may be shown (Samarsky and Nikolaev, 1978) that the best evaluation for \( \beta \) in (28) is

\[ \beta \equiv \beta_0 = (\hat{\gamma} + \hat{\gamma}_3)/2, \quad (32) \]

and the limits of the spectrum of eigenvalues of the operator \( \hat{\mathcal{A}} \) are

\[ \gamma_1 = (\alpha^2 - \lambda_{\alpha_1})(\alpha_{\alpha_2} - \alpha^2), \quad (33) \]

\[ \gamma_2 = \begin{cases} (\Delta - \alpha^2)(\Delta + \alpha^2 - \lambda_{\alpha_1} - \lambda_{\alpha_2}) & \text{if } \lambda_{\alpha_1} + \lambda_{\alpha_2} \leq (\Delta + \delta), \\ (\alpha^2 - \delta)(\lambda_{\alpha_1} + \lambda_{\alpha_2} - \alpha^2 - \delta) & \text{if } \lambda_{\alpha_1} + \lambda_{\alpha_2} \geq (\Delta + \delta). \end{cases} \quad (34) \]

The sequence of iteration parameters \( \{\tau_m\} \) is given by

\[ \tau_{2m-1} = -\beta_0 \omega_m - \sqrt{\beta_0^2 \omega_m^2 + \omega_m}, \]

\[ \tau_{2m} = -\beta_0 \omega_m + \sqrt{\beta_0^2 \omega_m^2 + \omega_m}, \quad (35) \]
where
\[ \omega_m = \frac{\omega_0}{1 + \rho_0 \nu_m}, \quad \nu_m \in \mathcal{M}_n = \left\{ -\cos \frac{(2i - 1)\pi}{2n}, 1 \leq i \leq n \right\}, \]
\[ 1 \leq m \leq n, \]
\[ \omega_0 = \frac{2}{\gamma_1 + \gamma_2}, \quad \rho_0 = \frac{1 - \xi}{1 + \xi}, \quad \xi = \frac{\gamma_1}{\gamma_2}. \]

\( n \) is estimated by (24) with \( \rho_1 \) defined by (25), where \( \gamma_1 \) and \( \gamma_2 \) are given by (33)–(34). Note that the length of the sequence of iteration parameters (35) is at least twice as many as \( n \).

Further, as well as in the case of the positive-definite operator \( A \), according to the iteration scheme (20), where the operator \( A \) on the grid \( \omega \) is determined by (14), the approximate solution \( B_z \) to the Helmholtz equation (13) may be calculated.

### 2.2. Calculation of \( B_x \) and \( B_y \)

From (1)–(2) we have
\[ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \alpha B_z, \quad (36) \]
\[ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0. \quad (37) \]

If \( \partial^2 B_y / \partial x \partial y \) is a continuous function, then \( \partial^2 B_y / \partial x \partial y = \partial^2 B_y / \partial y \partial x \), and from (36)–(37) we get
\[ \frac{\partial^2 B_x}{\partial x^2} + \frac{\partial^2 B_x}{\partial y^2} = -\frac{\partial^2 B_z}{\partial x \partial z} - \alpha \frac{\partial B_z}{\partial y}, \quad (38) \]
or
\[ \nabla^2 B_x = -\phi(x, y), \quad (39) \]
where
\[ \phi(x, y) \equiv \frac{\partial^2 B_z}{\partial x \partial z} + \alpha \frac{\partial B_z}{\partial y} \]
is the known function of \( (x, y) \) at every level of the grid \( \omega \) along the \( z \)-axis, since \( B_z \) are now known throughout the volume of integration.

Thus, on the selected plane \( z = z^* \) for \( B_x \) the 2D Dirichlet boundary value problem for the Poisson equation (39) is stated. The boundary conditions are specified by (11). To solve this DBVP the routine proposed can be used in 2D modification. The difference analogue of the differentiation operator.
\[ Au = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \phi(x, y) \]

we specified as

\[ Au = \left( u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right)/h_1^2 + \]

\[ + \left( u_{i,j+1} - 2u_{i,j} + u_{i,j-1} \right)/h_2^2 + \phi_{i,j}, \]

\[ i = 2, 3, \ldots, n_1 - 1, \ j = 2, 3, \ldots, n_2 - 1. \] (40)

Lower and upper bounds of the spectrum of the operator (40) are:

\[ \gamma_1 \equiv \delta = \sum_{\gamma=1}^{2} \frac{4}{h_\gamma^2} \sin^2(\pi/2n_\gamma), \] (41)

\[ \gamma_2 \equiv \Delta = \sum_{\gamma=1}^{2} \frac{4}{h_\gamma^2} \cos^2(\pi/2n_\gamma). \] (42)

Operator (40) is positive-definite, so we can use the iteration routine presented in Subsection 2.1.1. Parameters for the iteration scheme can be determined by the same expressions (21), (22), (24), (25), (27) with \( \gamma_1 \) and \( \gamma_2 \) obtained by (41) and (42).

Similarly, the Dirichlet boundary value problem for the \( B_y \) component at the plane \( z = z^* \) with boundary conditions (12) can be stated. The function \( \phi(x, y) \) will be

\[ \phi(x, y) = \frac{\partial^2 B_z}{\partial z \partial y} - \alpha \frac{\partial B_z}{\partial y}. \] (43)

Now we have \( B_x(x, y, z^*) \) and \( B_y(x, y, z^*) \) at the chosen plane \( z = z^* \). To find \( B_x \) and \( B_y \) through the volume \( \Omega \), we used a representaton of \( \mathbf{B} \) (Seehafer, 1978):

\[ \mathbf{B} = \alpha \mathbf{e}_z \times \nabla P + \nabla \times (\mathbf{e}_z \times \nabla P), \] (44)

where \( P \) is a generating function which satisfies the Helmholtz equation, and \( \mathbf{e}_z \) is the unit vector in the \( z \) direction. From (44) we get

\[ B_x = -\alpha \frac{\partial P}{\partial y} - \frac{\partial^2 P}{\partial x \partial z}, \] (45)

\[ B_y = \alpha \frac{\partial P}{\partial x} - \frac{\partial^2 P}{\partial y \partial z}, \] (46)
\[ B_z = -\alpha^2 P - \frac{\partial^2 P}{\partial z^2} \quad (47) \]

The general solution to Equation (47) can be written in the form

\[ P(x, y, z) = P_0(x, y, z) + C_1(x, y) \cos \alpha z + C_2(x, y) \sin \alpha z \quad (48) \]

where \( C_1(x, y) \) and \( C_2(x, y) \) are unknown functions, and \( P_0(x, y, z) \) denotes a particular solution to Equation (47). Let

\[ P_0(x, y, z) = \frac{\cos \alpha z}{\alpha} \int_{z_0}^{z} B_z(x, y, \zeta) \sin \alpha \zeta \, d\zeta - \]

\[ - \frac{\sin \alpha z}{\alpha} \int_{z_0}^{z} B_z(x, y, \zeta) \cos \alpha \zeta \, d\zeta \quad (49) \]

Putting the expression (48) into (45) and (46) we get

\[ \frac{\cos \alpha z}{\alpha} B_x + \frac{\sin \alpha z}{\alpha} B_y = d_1(x, y) + q_1(x, y, z) \quad (50) \]

\[ \frac{\sin \alpha z}{\alpha} B_x + \frac{\cos \alpha z}{\alpha} B_y = d_2(x, y) + q_2(x, y, z) \quad (51) \]

where

\[ q_1(x, y, z) = -\frac{\sin \alpha z}{\alpha} \frac{\partial^2 P_0}{\partial y \partial z} + \sin \alpha z \frac{\partial P_0}{\partial x} + \frac{\cos \alpha z}{\alpha} \frac{\partial^2 P_0}{\partial x \partial z} + \cos \alpha z \frac{\partial P_0}{\partial y} \quad (52) \]

\[ q_2(x, y, z) = -\frac{\cos \alpha z}{\alpha} \frac{\partial^2 P_0}{\partial y \partial z} + \cos \alpha z \frac{\partial P_0}{\partial x} - \frac{\sin \alpha z}{\alpha} \frac{\partial^2 P_0}{\partial x \partial z} - \sin \alpha z \frac{\partial P_0}{\partial y} \quad (53) \]

can be determined from \( B_z \) throughout \( \Omega \); and

\[ d_1(x, y) = \frac{\partial C_1}{\partial y} + \frac{\partial C_2}{\partial x} \quad (54) \]

\[ d_2(x, y) = -\frac{\partial C_2}{\partial y} + \frac{\partial C_1}{\partial x} \quad (55) \]

are unknown functions which do not depend on \( z \). They can be determined by (50) –(51) written for the chosen plane \( z = z^* \), since \( B_x(x, y, z^*) \) and \( B_y(x, y, z^*) \) are known.

Now we can determine \( B_x(x, y, z) \) and \( B_y(x, y, z) \) throughout the volume \( \Omega \) by the same (50)–(51) written as
\[ B_x(x, y, z) = \alpha \sin \alpha z \left[ d_2(x, y) + q_2(x, y, z) - \frac{\cos \alpha z}{\sin \alpha z} \left( d_1(x, y) + q_1(x, y, z) \right) \right] , \]  
\[ B_y(x, y, z) = \frac{\alpha}{\sin \alpha z} \left[ d_1(x, y) + q_1(x, y, z) + \frac{\cos \alpha z}{\alpha} B_x(x, y, z) \right] . \]  

3. Accuracy of the Proposed Method

To estimate the accuracy of the algorithm it is convenient to use a routine proposed by Cuperman, Ofman, and Semel (1989, referred to as Paper I) for a comparison between numerical and analytical solutions to the LFFF equations.

The exact solution of the LFFF equations in a half-space has been obtained by Chiu and Hilton (1977). We shall use it for the simplest case where \( B_z \) are non-zero only in two pixels of the grid. Following Paper I, let us consider a situation where at the bottom (the plane \( z = 0 \)) of the rectangular box, \( \Omega B_z \) are specified, and \( B_z \neq 0 \) only at two points: \((-d, 0, 0)\) and \((d, 0, 0)\). (The Cartesian coordinate origin is located at the center of the bottom of \( \Omega \), see Figure 1 in Paper I.) Let \( B_z(-d, 0, 0) = -B_z(d, 0, 0) \). Then by making use of (13)–(16) from Chiu and Hilton (1977) for a half-space above some plane \( z = z_0 > 0 \), one can calculate the linear force-free field generated by every ‘pole’ at a point \((x, y, z)\) (see also (22)–(24) in Paper I):

\[ B_{x,j} = \frac{1}{r_j^2} \left[ \frac{x_j}{R_j^2} \cos(\alpha R_j) - \frac{x_j z^2}{R_j^2} \sin(\alpha R_j) + \alpha x_j \sin(\alpha z) + \alpha \frac{y z}{R_j} \cos(\alpha R_j) - \alpha y \cos(\alpha z) \right] , \]  
\[ B_{y,j} = \frac{1}{r_j^2} \left[ \frac{y_j}{R_j^2} \cos(\alpha R_j) - \frac{y z^2}{R_j^2} \sin(\alpha R_j) + \alpha y \sin(\alpha z) - \alpha \frac{x z}{R_j} \cos(\alpha R_j) - \alpha x \cos(\alpha z) \right] , \]  
\[ B_{z,j} = \frac{z_j}{R_j^3} \left[ \cos(\alpha R_j) + \alpha R_j \sin(\alpha R_j) \right] , \]  

where

\[ j = 1, 2; \quad R_j^2 = r_j^2 + z^2, \quad R_j^2 = x_j^2 + y^2, \quad x_1 = x - d, \quad x_2 = x + d . \]
The resultant $B_x, B_y, B_z$ components at a point $(x, y, z)$ are

$$B_x = B_{x,1} - B_{x,2}, \quad (61)$$

$$B_y = B_{y,1} - B_{y,2}, \quad (62)$$

$$B_z = B_{z,1} - B_{z,2}. \quad (63)$$

Following Paper I, the distances $x$, $y$, $z$, $r_j$, $R_j$ in Equations (58)–(60) are normalized to $L_0$ – the depth of the ‘poles’ descending the ‘photospheric’ surface $z = z_0$; thus, $z_0 = 1$. We have taken $\alpha = 1$ and $d = \frac{1}{9}$, the rectangular box $\Omega$: $\{-3 \leq x \leq 3; -3 \leq y \leq 3; 1 \leq z \leq 4\}$. A horizontal mesh of $n_1 \times n_2 = 51 \times 51$ grid points was used; thus, $h_1 = h_2 = 0.12$. The vertical grid spacing used was taken as $h_3 = 0.1$; thus, $n_3 = 31$.

The analytical solution, $\mathbf{B}$, of the LFFF equations inside $\Omega$ was calculated by (58)–(63).

The numerical solution, $\mathbf{B}'$, of the LFFF equations (1)–(3) inside $\Omega$ was obtained by Chebyshev’s iteration method. The boundary conditions (5)–(12) were calculated by the analytical formulae (58)–(63). $z^*$ was taken to be 4.0. On the given mesh, the minimum eigenvalue of the 3D Laplacian $\delta = 1.64$, thus, $\alpha^2 < \delta$, and the 3D difference operator $A u$ defined in (14) is a positive-definite one. With $\epsilon = 10^{-4}$ a total number of iterations $n$ according to (24) has to be not less than $n_0 = 190$.

The deviation of the analytical solution $\mathbf{B}$ from the numerical $\mathbf{B}'$ was estimated by formula (28) of Paper I, which gives the average of the relative error over the entire horizontal plane of the integration domain $\langle \Delta B_\gamma \rangle$ ($\gamma = x, y, z$) as a function of height $z$:

$$\langle \Delta B_\gamma \rangle = \frac{1}{\langle B_\gamma \rangle} \left[ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (B_{\gamma;i,j} - B'_{\gamma;i,j})^2 / (n_1 \times n_2) \right]^{1/2}. \quad (64)$$

The functions calculated in this way are presented in Figure 1. One can see that the maximum values of $\langle B_\gamma \rangle$ are about 0.9% at $z = 1.1$ for $\langle \Delta B_x \rangle$, about 0.6% at $z = 1.1$ for $\langle \Delta B_y \rangle$, and about 0.4% at $z = 3.2$ for $\langle \Delta B_z \rangle$.

Thus, the proposed numerical method ensures a good enough accuracy of the linear force-free field reconstruction.

4. Potential Non-Photospheric Boundary Conditions

Expressions (58)–(63) give an analytical representation of a field that is linear force-free in a half-space $z > z_0$. The solution $\mathbf{B}'$ obtained in the previous section is a fragment of such a field inside the volume $\Omega$ (with an accuracy better than 1%), since the boundary conditions (5)–(12) were defined by (58)–(63).

Let us specify other boundary conditions (5)–(12), say, potential field values. Then
\textbf{Figure 1.} Average over the entire horizontal plane of the integration domain of the relative error $\langle \Delta B_\gamma \rangle$ ($\gamma = x, y, z$) of the numerical solution $\mathbf{B}'$ relative to the analytic solution $\mathbf{B}$ as a function of $z$.

\begin{align*}
\psi_i &= B_z(\text{pot}) \quad \text{for } i = 2, \ldots, 6 \quad \text{in (6) -- (10)} , \\
g_1(x,y)|_G &= B_z(\text{pot}) \quad \text{in (11)} , \\
g_2(x,y)|_G &= B_y(\text{pot}) \quad \text{in (12)} .
\end{align*}

Let $\psi_1(x,y)$ in (5) be the same as in the $\mathbf{B}'$ calculations. Let $z^*$ be also equal to 4.0. Taking such boundary conditions, we calculated $\mathbf{B}''$ and compared it with $\mathbf{B}'$. The potential field was calculated by $B_z$ (at $z_0 = 1$) as a boundary condition, using our modification of the Schmidt integration method (Abramenko, 1986).

Starting from the same $B_z$ magnetogram at $z = z_0 = 1.0$, the fields $\mathbf{B}'$ and $\mathbf{B}''$ exhibit quite different behaviors with height. The $B_z$ components and the orientation of the transverse field vector $\mathbf{B}_\perp$ of $\mathbf{B}'$ and $\mathbf{B}''$ in the $(x,y)$ planes at three heights are shown in Figure 2. For $B'_z$ (Figure 2(a)), areas of the main fluxes decrease and areas of opposite polarity increase with height. Whereas for $B''_z$ the sign preserves up to $z = 3.9$. Vortices of $\mathbf{B}'_\perp$ and $\mathbf{B}''_\perp$ (Figure 2(b)) are also quite different. Such an obvious difference between $\mathbf{B}'$ and $\mathbf{B}''$ in vertical and transverse field components implies that the magnetic flux tube structures of $\mathbf{B}'$ will deviate from those of $\mathbf{B}''$. This is confirmed by Figure 3, where the lines of force for $\mathbf{B}'$ and $\mathbf{B}''$ are shown. Field lines of $\mathbf{B}'$ and $\mathbf{B}''$ were integrated from the start points with the same $(x,y)$ coordinates, but the main flux tube of $\mathbf{B}''$ is more extensive, higher, and has another twist.
Figure 2a–b. Contours of $B_z$ magnetic field component (a) and azimuth of total transverse field vector (b) in the ($x, y$) planes situated at the heights indicated on the frames for $B'$ – linear force-free field with analytic force-free boundary conditions and for $B''$ – linear force-free field with potential non-photospheric boundary conditions. Solid (dashed) lines indicate positive (negative) fields. Contour levels are drawn at $\pm 10, \pm 30, \pm 10, \pm 50, \pm 100, \pm 200, \pm 500, \pm 1000, \pm 1500, \pm 2000$ units; the contour $\pm 1$ is added at $z = 3.9$. 
Thus, magnetic field configurations $\mathbf{B}'$ and $\mathbf{B}''$ calculated by the same numerical routine at the same volume by the same photospheric $B_z$, but having different boundary conditions at non-photospheric sides of the volume, differ significantly.
5. Conclusions

We have proposed here a technique which enables us to calculate a linear force-free magnetic field inside a bounded domain (such as a rectangular box, $\Omega$) provided the $B_z$ component is prescribed at all sides of $\Omega$ and $B_x$, $B_y$ are specified on the intersection of the lateral part of the $\Omega$ boundary, with one selected plane $z = $ constant. Such a routine may be useful when modeling a LFFF above a limited region of the solar photosphere, and there is some a priori information about the field at non-photospheric boundaries of the volume studied. The routine also may be used under the assumption that $B_z$ vanishes at the vertical and top sides of the volume $\Omega$.

The method given here renders it possible to broaden the integration domain without decreasing the region of accessible $\alpha$ except for one requirement: $\alpha$ must not coincide precisely with any eigenvalue of the 3D Laplacian. The algorithm does not imply any periodicity of the observational data and of the solution beyond the magnetogram as the Fourier transform method does.

The simplest numerical example (Section 4) shows that taking into account a priori information about the field values in the corona may change considerably the results of LFFF modeling in an active region.
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Appendix. Arrangement of the Set $M_n$

Let $n$ be the number of iterations estimated by (24). Let us denote $\Theta^m$ as a set of $m$ integer numbers: $\Theta^m = \{\Theta_1^{(m)}, \Theta_2^{(m)}, \ldots, \Theta_m^{(m)}\}$.

Starting from $\Theta^1 = \{1\}$, let us determine a procedure forming a set $\Theta^n$ by the following way. Individual steps of this procedure are transitions of two kinds: a transition from $\Theta^m$ to $\Theta^{2m}$ and a transition from $\Theta^{2m}$ to $\Theta^{2m+1}$.

The transition from $\Theta^{2m}$ to $\Theta^{2m+1}$ is to add the odd number $2m + 1$ behind the last element of $\Theta^{2m}$.

The transition from $\Theta^m$ to $\Theta^{2m}$ is a more complex one. If it is followed by $\Theta \to \Theta^{4m}$, or if it is the last step in the $\Theta^n$ formation, then

$$\Theta^{(2m)}_{2i-1} = \Theta^{(m)}_i, \quad \Theta^{(2m)}_{2i-1} + \Theta^{(2m)}_{2i} = 4m, \quad i = 1, 2, \ldots, m;$$

if it is followed by $\Theta^{2m} \to \Theta^{2m+1}$, then

$$\Theta^{(2m)}_{2i-1} = \Theta^{(m)}_i, \quad \Theta^{(2m)}_{2i-1} + \Theta^{(2m)}_{2i} = 4m + 2, \quad i = 1, 2, \ldots, m.$$

To make things clear let us consider a simple example for $n = 15$. A sequence of transitions from $\Theta^1$ to $\Theta^{15}$ will be so far: $\Theta^1 \to \Theta^2 \to \Theta^3 \to \Theta^6 \to \Theta^7 \to \Theta^{14} \to \Theta^{15}$. This sequence is obvious if one looks at it in the inverse order. According to the above rule, the sets $\Theta^m (m = 1, 2, 3, 6, 7, 14, 15)$ are:

$$\Theta^1 = \{1\};$$

$$\Theta^2 = \{1, 5\};$$

$$\Theta^3 = \{1, 5, 3\};$$

$$\Theta^6 = \{1, 13, 5, 9, 3, 11\};$$

$$\Theta^7 = \{1, 13, 5, 9, 3, 11, 7\};$$

$$\Theta^{14} = \{1, 29, 13, 17, 5, 25, 9, 21, 3, 27, 11, 19, 7, 23\};$$

$$\Theta^{15} = \{1, 29, 13, 17, 5, 25, 9, 21, 3, 27, 11, 19, 7, 23, 15\}.$$

Using the formed set $\Theta^n$, let us arrange the set
\[ \mathcal{M}_n = \left\{ -\cos \frac{2i - 1}{2n} \pi, \ i = 1, 2, \ldots, n \right\} \]

in the following way:

\[ \mathcal{M}_n^* = \left\{ -\cos(\beta_i), \ \beta_i = \frac{\pi}{2n} \Theta_i^{(n)}, \ i = 1, \ldots, n \right\}. \]

Now, when forming the sequence of iteration parameters by (21), one can take consecutively elements of the ordered set \( \mathcal{M}_n^* \) as \( \nu_m \), then the iteration process is a stable one, and minimum calculation errors are ensured.

References