NET CIRCULAR POLARIZATION IN MAGNETIC SPECTRAL LINES PRODUCED BY VELOCITY GRADIENTS: SOME ANALYTICAL RESULTS

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Abstract. The net circular polarization in a spectral line due to the combined effect of magnetic fields and velocity gradients is analyzed for a few schematic situations. In some particular cases, its dependence on the magnetic field, velocity field and line parameters can be expressed analytically.

Key words: Magnetic fields – Polarization

1. Introduction

Let us consider an atomic spectral line formed in a static stellar atmosphere permeated by a magnetic field. Provided the line is unbinned and atomic orientation can be neglected, the circular polarization Stokes $V$ parameter is antisymmetrical about the central wavelength $\lambda_0$ of the line (see e.g. Landi Degl’Innocenti and Landi Degl’Innocenti, 1981). It follows that the net (i.e. wavelength-integrated) circular polarization (NCP) is zero. However, the broadband observations by Illing, Landman, and Mickey (1974), as well as the spectropolarimetric observations by Stenflo et al. (1984) clearly show that this is not the case for sunspots and solar active regions.

In this paper the NCP produced by velocity gradients is investigated. The NCP has a very complicated dependence both on the parameters specifying the model atmosphere (including the velocity gradient and the magnetic field) and on the spectral line itself. Here we will not attempt a systematic investigation of this dependence, but will rather restrict attention to a few schematic situations where some analytical results can be derived.

The interpretation of observational data generally requires more sophisticated models than those considered in this paper. However, we believe that an approach based on simple models is the most appropriate to give a deeper insight into the involved topic of radiative transfer in the presence of magnetic and velocity fields. This is why in the following we consider,

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for instance, a model where the NCP is generated by a variable velocity field and a constant magnetic field, although such a model is believed to be inadequate to reproduce the observations (see e.g. Skumanich and Lites, 1987).

2. The Transfer Equation for Polarized Radiation

The transfer equation for polarized radiation in a magnetic spectral line is well established (see e.g. Landi Degl’Innocenti and Landi Degl’Innocenti, 1972, 1975). We rewrite it here in order to fix the notations that will be needed later.

Consider an isolated spectral line formed in an atmosphere where a magnetic field and a macroscopic velocity field are present. The line – characterized by the rest wavelength $\lambda_0$ – originates from the (electric-dipole) transition between two energy levels having angular momentum quantum numbers and Landé factors $(J_\ell, g_\ell)$ and $(J_u, g_u)$, respectively. At each optical depth, the two levels are split according to the local value of the magnetic field. We assume that the Zeeman effect regime holds and that no atomic polarization is present: i.e. the magnetic sublevels of each $J$-level are evenly populated and the off-diagonal elements of the atomic density matrix are zero. Under these assumptions, the transfer equation for the Stokes vector $\mathbf{I} = (I, Q, U, V)\dagger$ (defined according to Shurcliff, 1962) characterizing a radiation beam travelling along the direction $\Omega$ is

$$\frac{d\mathbf{I}}{d\tau} = \mathbf{C I} - \mathbf{j}, \quad (1)$$

where $\tau$ is the continuum optical depth at the line wavelength measured along $\Omega$, and where the propagation matrix $\mathbf{C}$ and the emission vector $\mathbf{j}$ are given by

$$\mathbf{C} = \begin{pmatrix} 1 + k_I & k_Q & k_U & k_V \\ k_Q & 1 + k_I & f_V & -f_U \\ k_U & -f_V & 1 + k_I & f_Q \\ k_V & f_U & -f_Q & 1 + k_I \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} S_c + k_I S_L \\ k_Q S_L \\ k_U S_L \\ k_V S_L \end{pmatrix}.$$

Here $S_c$ and $S_L$ are the continuum and line source functions, while the $k$ and $f$ coefficients, related to absorption and anomalous dispersion, respectively, can be written in the form

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\[ k_I = \kappa_L \frac{1}{2} \left[ \eta_p \sin^2 \vartheta + \frac{1}{2} (\eta_b + \eta_r) (1 + \cos^2 \vartheta) \right] \]

\[ k_Q = \kappa_L \frac{1}{2} \left[ \eta_p - \frac{1}{2} (\eta_b + \eta_r) \right] \sin^2 \vartheta \cos 2\varphi \]

\[ k_U = \kappa_L \frac{1}{2} \left[ \eta_p - \frac{1}{2} (\eta_b + \eta_r) \right] \sin^2 \vartheta \sin 2\varphi \]

\[ k_V = \kappa_L \frac{1}{2} [\eta_r - \eta_b] \cos \vartheta \]

\[ f_Q = \kappa_L \frac{1}{2} \left[ \rho_p - \frac{1}{2} (\rho_b + \rho_r) \right] \sin^2 \vartheta \cos 2\varphi \]

\[ f_U = \kappa_L \frac{1}{2} \left[ \rho_p - \frac{1}{2} (\rho_b + \rho_r) \right] \sin^2 \vartheta \sin 2\varphi \]

\[ f_V = \kappa_L \frac{1}{2} [\rho_r - \rho_b] \cos \vartheta \]

with

\[ \eta_b = \eta_{-1}, \quad \eta_p = \eta_0, \quad \eta_r = \eta_{+1}, \quad \rho_b = \rho_{-1}, \quad \rho_p = \rho_0, \quad \rho_r = \rho_{+1} \]

and (using 3-j symbols)

\[ \eta_q = \sum_{M_u M_u} 3 \left( \begin{array}{ccc} J_u & J_u & 1 \\ -M_u & M_u & -q \end{array} \right)^2 \frac{1}{\sqrt{\pi}} H(v - v_A + v_B (g_u M_u - g_\ell M_\ell), a), \]

\[ \rho_q = \sum_{M_u M_u} 3 \left( \begin{array}{ccc} J_u & J_u & 1 \\ -M_u & M_u & -q \end{array} \right)^2 \frac{1}{\sqrt{\pi}} L(v - v_A + v_B (g_u M_u - g_\ell M_\ell), a), \]

where \( q = M_\ell - M_u = 0, \pm 1 \). In these expressions, \( \kappa_L \) is the ratio between line and continuum absorption coefficient, \( \vartheta \) and \( \varphi \) the inclination and azimuth angles specifying the magnetic field direction (cf. Landi Degl’Innocenti and Landi Degl’Innocenti, 1972, Figure 1), \( H \) the Voigt function and \( L \) the associated dispersion profile

\[ H(v, a) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(v-y)^2 + a^2} \, dy, \quad L(v, a) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} (v-y)}{(v-y)^2 + a^2} \, dy, \]

\( a \) being the damping constant. Note that \( L(v, a) = 2F(v, a) \), where \( F \) is generally referred to as Faraday-Voigt function. Finally, \( v, v_A \) and \( v_B \) are the wavelength distance from line center, the Doppler shift due to the line-of-sight velocity \( w_A \) of the ambient medium, and the Zeeman splitting, all expressed in Doppler width units

\[ v = \frac{\lambda - \lambda_0}{\Delta \lambda_D}, \quad v_A = \frac{\Delta \lambda A}{\Delta \lambda_D} = \frac{\lambda_0}{\Delta \lambda_D} \frac{w_A}{c}, \]

\[ v_B = \frac{\Delta \lambda B}{\Delta \lambda_D} = \frac{\lambda_0^2 e_0}{4 \pi mc^2 \Delta \lambda_D} |\mathbf{B}|, \]

with \( e_0 \) the absolute value of the electron charge, \( m \) the electron mass, and \( c \) the velocity of light. In accordance with the astrophysical sign convention, a positive \( w_A \) value means a red-shifted line.
3. The Net Circular Polarization in two simple cases

In order to characterize the NCP in a spectral line, we define the dimension-less parameter $\nu$ to be the ratio between the wavelength-integrated emerging $V$ Stokes parameter and the fraction of continuum subtracted by the line,

$$\nu = \frac{\int V(\tau = 0, \lambda) \, d\lambda}{\int [I_c(\tau = 0) - I(\tau = 0, \lambda)] \, d\lambda},$$

where $I_c$ is the continuum intensity at the line wavelength and the integrals extend over the line profile. As apparent from the definition, the $\nu$ parameter is unaffected by a global shift of the $V$ and $I$ profiles. This means that, as far as NCP is concerned, what is important is not the line-of-sight velocity itself, but the line-of-sight velocity gradient.

3.1. The Milne-Eddington Atmosphere with a Small Velocity Gradient

Under the Milne-Eddington approximation (plane-parallel atmosphere with $C = \text{const.}$ and $S_c = S_L = B_P = B_0(1 + \beta t)$), Equation (1) reduces to

$$\mu \frac{dI}{dt} = C (I - B_P U),$$

(5)

where $\mu$ is the cosine of the heliocentric angle, $t$ the continuum optical depth measured along the vertical, and $U = (1, 0, 0, 0)^t$. All the parameters on which the matrix $C$ depends are independent of optical depth; in particular, the reduced Doppler shift $\nu_A$ defined in Equations (4) is constant (or zero if $\nu_A = 0$).

Let us now suppose that a small velocity gradient $\delta \nu_A(t)$ is present, where small means

$$\frac{\lambda_0}{\Delta \lambda_D} \frac{\delta \nu_A(t)}{c} \ll 1$$

for any $t$. We can expand the propagation matrix $C$ to first order in the gradient

$$C \rightarrow C + \left( \frac{\partial C}{\partial \nu_A} \right) \delta \nu_A(t) = C - \frac{\lambda_0}{c} \left( \frac{\partial C}{\partial \lambda} \right) \delta \nu_A(t),$$

and look for a perturbative solution of the form $I + \delta I$. Substituting the above expression for $C$ into Equation (5) and equating the zero and first order terms, we obtain a zero order equation (which is identical to Equation (5)) corresponding to the static (or constant-velocity) atmosphere, and a first order equation

$$\mu \frac{d}{dt} \delta I = C \delta I - \frac{\lambda_0}{c} \frac{\partial C}{\partial \lambda} (I - B_P U) \delta \nu_A(t),$$

(6)
which has the same form as the usual transfer equation, with an emission vector depending on the solution of the zero order equation. The latter is given by the well-known Unno-Rachkovsky formula

\[ I(t, \mu) = B_0 \left[ (1 + \beta t) I + \beta \mu C^{-1} \right] U , \tag{7} \]

where \( I \) is the unit matrix and \( C^{-1} \) the inverse of the matrix \( C \). The solution to Equation (6) can be written, using the evolution operator defined in Landi Degl’Innocenti and Landi Degl’Innocenti (1985), in the form

\[ \delta I(t, \mu) = \int_t^\infty e^{-\frac{1}{\mu} (t'-t)} C \frac{\lambda_0}{c} \frac{\partial C}{\partial \lambda} (I - B_{\text{P}} U) \delta w_A(t') \frac{dt'}{\mu} , \tag{8} \]

the exponential factor being defined by its Taylor expansion.

If we now make the further assumption that the velocity gradient is linear in the optical depth (\( \delta w_A(t) = w_A^{(1)} t \)), the integral in Equation (8) can be performed analytically. Since for any constant matrix \( M \), we can write

\[ \int_a^b e^{-xM} x \, dx = \left\{ [a1 + M^{-1}] e^{-aM} - [b1 + M^{-1}] e^{-bM} \right\} M^{-1} , \]

we obtain for the first order correction to the emerging Stokes parameters

\[ \delta I(0, \mu) = -B_0 \beta \mu^2 \lambda_0 \frac{w_A^{(1)}}{c} C^{-1} \frac{\partial C^{-1}}{\partial \lambda} U . \tag{9} \]

It should be noticed that the integrand in Equation (8) is, apart from the factor \( \delta w_A(t') \), the response function for the Stokes vector \( I \) to a velocity perturbation, evaluated for a Milne-Eddington atmosphere. The concept of response function for the Stokes parameters has been introduced by Landi Degl’Innocenti and Landi Degl’Innocenti (1977). Expressions for \( \delta I(0, \mu) \), basically equivalent to Equation (9), have been obtained by Landolfi (1987) and by Sánchez Almeida (1992). The derivation presented here provides however a much simpler expression for this quantity.

Direct calculation of the matrix product in the right-hand side of Equation (9) leads to the following expression for the NCP parameter

\[ v = \mu \frac{\lambda_0}{\Delta \lambda_D} \frac{w_A^{(1)}}{c} \frac{\int \Delta^{-2} A \, dv}{\int \left\{ 1 - \Delta^{-1}(1 + k_I)((1 + k_I)^2 + f_Q^2 + f_P^2) \right\} dv} , \tag{10} \]

where \( v \) is the reduced wavelength defined in Equation (4) and
\[\Delta = (1 + k_I)^4 + (1 + k_I^2)(f_Q^2 + f_V^2 - k_Q^2 - k_V^2) - (k_Q f_Q + k_V f_V)^2\]

\[\mathcal{A} = \left\{(1 + k_I)^3(k_Q k_V + f_Q f_V) + (1 + k_I)[-k_Q k_V(f_Q^2 - f_V^2) + f_Q f_V(k_Q^2 - k_V^2)]\right\}\frac{\partial k_Q}{\partial v}

\[= -(1 + k_I)^3(k_Q^2 + f_Q^2)\frac{\partial k_V}{\partial v} + \left\{(1 + k_I)^3(k_V f_Q - k_Q f_V) + (1 + k_I)[k_Q f_V(k_Q^2 + f_Q^2) + k_V f_Q(k_Q^2 + f_Q^2)]\right\}\frac{\partial f_Q}{\partial v}

\[-(1 + k_I)(k_Q f_Q + k_V f_V)(k_Q^2 + f_Q^2)\frac{\partial f_V}{\partial v},\]

with

\[k_Q = k_Q(\varphi = 0), \quad f_Q = f_Q(\varphi = 0).\]

It can be seen that the \(v\) parameter is independent of the azimuth angle \(\varphi\).

A symmetry relation concerning the dependence on the angle \(\vartheta\) follows from Equations (10) and (2),

\[v(\pi - \vartheta) = -v(\vartheta).\]

Hence \(v(\pi/2) = 0\). Moreover, \(v(0) = v(\pi) = 0\). Thus the NCP parameter is zero if the magnetic field is perpendicular or parallel to the propagation direction. The former result is obvious, since Equations (2) give \(k_V = f_V = 0\) for \(\vartheta = \pi/2\), so that the quantity \(\mathcal{A}\) in Equation (10) vanishes. In other words, for \(\vartheta = \pi/2\) the profile \(V(\lambda)\) is identically zero. The latter result has been obtained by various authors (see e.g. Skumanich and Lites, 1987).

Under the assumption of small velocity gradient, Sánchez Almeida, Collados, and del Toro Iniesta (1989) have proved that it always holds in LTE.

For weak magnetic field \((v_B \ll 1)\), the \(\eta_q, \rho_q\) profiles defined in Equations (3) can be expanded in power series of \(v_B\) (see Landi Degl’Innocenti and Landi Degl’Innocenti, 1973). To the lowest order we obtain from Equation (10)

\[v = -\frac{1}{16} \mu \frac{\lambda_0}{\Delta \lambda_D} \frac{u_A^{(1)}}{c} v_B^5 \bar{g} \bar{G}^2 \sin^4 \vartheta \cos \vartheta \frac{\mathcal{I}_5(\kappa_L, a)}{\mathcal{I}_1(\kappa_L, a)},\]  

where \(\bar{g}\) is the effective Landé factor of the line and

\[\bar{G} = \bar{g}^2 - \frac{1}{80} (g_e - g_u)^2 \left\{16 [J_\ell(J_\ell + 1) + J_u(J_u + 1)] - 7 [J_\ell(J_\ell + 1) - J_u(J_u + 1)]^2 - 4\right\},\]

\[\mathcal{I}_1(\kappa_L, a) = \int_1^{\kappa_L \eta / (1 + \kappa_L \eta)} dv,\]

\[\mathcal{I}_5(\kappa_L, a) = \int \frac{\bar{g}^2 (\eta'' + \rho'' \eta)^m}{(1 + \kappa_L \eta)^5} dv,\]
Fig. 1. The absolute value of the ratio $I_5(\kappa_L, a)/I_1(\kappa_L, a)$ is plotted against $\kappa_L$ on a Log-Log scale. The three curves are labeled by the value of $a$. The ratio $I_5(\kappa_L, a)/I_1(\kappa_L, a)$ is positive on the right of the figure (full lines) and negative on the left (dashed lines).

with

$$\eta = H(v, a)/\sqrt{\pi}, \quad \rho = L(v, a)/\sqrt{\pi},$$

where the primes in the integrand of $I_5(\kappa_L, a)$ denote the derivatives with respect to the reduced wavelength $v$.

According to Equation (11), the NCP is a strongly increasing function of $\vartheta$ which attains the maximum value for $\cos \vartheta = 1/\sqrt{5}$ (i.e. $\vartheta \simeq 63^\circ$). As to the dependence on the Zeeman pattern, it can be shown that for assigned $\bar{g}$, the $\bar{G}$ factor – and, therefore, the NCP – is maximum for Zeeman triplets. Finally, the dependence of the NCP on $\kappa_L$ and $a$ is contained in the ratio $I_5/I_1$. The integral $I_1$ is a positive quantity (proportional to the equivalent width of the line), while the sign of $I_5$ cannot be established without a numerical evaluation of the integral. This shows that $I_5$ is positive (for all $a$ values) provided $\kappa_L$ is larger than about 1.5. It follows that, apart from very weak lines, the NCP originating from an atmosphere with $w_A^{(1)} > 0$ obeys, in the weak field regime, the sign rule

$$\text{sign}(v) = \text{sign}(-\bar{g} \cos \vartheta).$$

It is important to notice that just the opposite sign rule would be obtained if the $\rho$ terms in the expression for $I_5(\kappa_L, a)$ were neglected. This is a striking example of the significance of magneto-optical effects.

The order of magnitude of the NCP predicted by Equation (11) can be estimated with the help of Figure 1. It can be seen that $v$ scales approximately as $\kappa_L^3$.

Another result, concerning the case of very strong magnetic field, can be derived from Equation (10): provided the $H$ and $L$ profiles corresponding
to the transition between each couple of magnetic sublevels \((J_JM_J, J_MM_M)\) do not overlap, \(v\) is zero. However, this is true only for really strong fields, because \(L(v, a)\) is a weakly decreasing function of \(v\) \((L(v, a) \sim v^{-1})\), while \(H(v, a) \sim v^{-2}\).

3.2. A DISCONTINUOUS MODEL WITH VELOCITY AND MAGNETIC FIELD GRADIENTS

The next case we consider is a discontinuous model formed by a thin slab laying on top of a semi-infinite atmosphere. We consider vertical propagation \((\mu = 1)\) and assume that a (constant) line-of-sight velocity \(w_{sys}\) is present in the slab, while the underlying atmosphere is static; the slab and the atmosphere are permeated by different magnetic fields, \((\mathbf{B})_{sys} \equiv (B_s, \theta_s, \varphi_s)\) and \((\mathbf{B}) \equiv (B, \theta, \varphi)\) respectively, also assumed as constant with optical depth. This is essentially the model considered by Illing, Landman, and Mickey (1975).

The transfer of radiation across the slab is described by the equation

\[
\frac{dI}{dt} = \mathbf{C}^{(s)} \mathbf{I} - \mathbf{j}^{(s)} ,
\]

whose formal solution is

\[
I(0) = \int_{0}^{t_s} O(0, t) \mathbf{j}^{(s)} \, dt + O(0, t_s) \mathbf{I}^{(b)} ,
\]

(12)

where \(t_s\) is the optical thickness of the slab, \(O\) the evolution operator, and \(\mathbf{I}^{(b)}\) the boundary Stokes vector, characterizing the radiation emerging from the underlying atmosphere. Since \(t_s \ll 1\), we can expand the evolution operator in power series up to first order,

\[
O(0, t) = e^{-t} \mathbf{C}^{(s)} \simeq 1 - t \mathbf{C}^{(s)} \quad (0 \leq t \leq t_s) .
\]

To the lowest order in \(t_s\) we get for the NCP parameter, the expression

\[
v = -t_s \frac{\int[k_{\mathbf{V}}^{(s)} I^{(b)} + f_U^{(s)} Q^{(b)} - f_Q^{(s)} U^{(b)} + k_I^{(s)} V^{(b)}]d\lambda}{\int[I_c^{(b)} - I^{(b)}]d\lambda} ,
\]

(13)

where \(I^{(b)}\), \(Q^{(b)}\), \(U^{(b)}\), \(V^{(b)}\) are centered at the rest wavelength \(\lambda_0\) while \(k_I^{(s)}\), \(k_{\mathbf{V}}^{(s)}\), \(f_Q^{(s)}\), \(f_U^{(s)}\) are centered at \(\lambda_0(1 + w_{sys}^{(s)}/c) = \lambda_0 + \Delta\lambda_{A}^{(s)}\).

In order to further simplify the problem, we assume that the boundary Stokes parameters are given by the Seares formulae.
\[ I^{(b)} = B_0[(1 + \beta) - \beta k_I] \quad U^{(b)} = -B_0 \beta k_U \]
\[ Q^{(b)} = -B_0 \beta k_Q \quad V^{(b)} = -B_0 \beta k_V. \]

These expressions – which are indeed rough approximations – can be derived from Equation (7) under the limit of weak spectral line \((\kappa_L \ll 1)\). From Equations (13), (14) and (2) we have

\[
v = \frac{\kappa_\perp(s) t_s}{4\Delta\lambda_D} \left\{ \frac{1}{2} \left[ \cos\vartheta_s(1 + \cos^2\vartheta) + \cos\vartheta(1 + \cos^2\vartheta_s) \right] \times \int (\eta_r^{(s)} - \eta_b^{(s)}) d\lambda \right. \\
+ \cos\vartheta_s \sin^2\vartheta \int (\eta_r^{(s)} - \eta_b^{(s)}) \eta_p d\lambda \\
+ \cos\vartheta \sin^2\vartheta_s \int (\eta_r - \eta_b) \eta_p^{(s)} d\lambda \\
+ \frac{1}{2} \left[ \cos\vartheta_s(1 + \cos^2\vartheta) - \cos\vartheta(1 + \cos^2\vartheta_s) \right] \times \int (\eta_r^{(s)} - \eta_b^{(s)}) \eta_r d\lambda \\
- \sin^2\vartheta_s \sin^2\vartheta \sin 2(\varphi - \varphi_s) \times \int \left[ \rho_p^{(s)} - \frac{1}{2} \left( \rho_b^{(s)} + \rho_r^{(s)} \right) \right] \left[ \eta_p - \frac{1}{2} (\eta_b + \eta_r) \right] d\lambda \right\},
\]

where all the quantities with the index \((s)\) refer to the slab and the other quantities to the underlying atmosphere. We will now consider the three special cases which are obtained from this equation by setting 2 of the 3 magnetic field parameters to the same value. The case \((\vartheta_s = \vartheta, \varphi_s = \varphi)\) will be denoted as \(\Delta B\)-effect (the fourth and fifth term in Equation (15) vanish); similarly, we will consider the \(\Delta \vartheta\)-effect \((B_s = B, \varphi_s = \varphi; \text{terms 1 and 5 are zero})\) and the \(\Delta \varphi\)-effect \((B_s = B, \vartheta_s = \vartheta; \text{the first four terms are zero})\). For simplicity, we restrict attention to Zeeman triplets and assume the Doppler width and the damping constant in the slab, and in the atmosphere to be the same.

Equation (15) contains convolutions of \(H\) functions and of \(H\) and \(L\) functions. These can be evaluated analytically, yielding

\[
\int_{-\infty}^{\infty} H(v - v_0, a) H(v - v'_0, a) dv = \sqrt{\frac{\pi}{2}} H \left( \frac{v_0 - v'_0}{\sqrt{2}a}, \sqrt{2a} \right) \\
\int_{-\infty}^{\infty} H(v - v_0, a) L(v - v'_0, a) dv = \sqrt{\frac{\pi}{2}} L \left( \frac{v_0 - v'_0}{\sqrt{2}a}, \sqrt{2a} \right).
\]

The following expressions for the three effects mentioned above are obtained:

\(\Delta B\)-effect
\( v \frac{\kappa_{1}^{(s)} t_{s}}{4\sqrt{2\pi}} \cos \vartheta \left\{ (1 + \cos^{2} \vartheta) \left[ H(v_{rr}, \hat{a}) - H(v_{bb}, \hat{a}) \right] \\
+ \sin^{2} \vartheta \left[ H(v_{rp}, \hat{a}) + H(v_{pr}, \hat{a}) - H(v_{bp}, \hat{a}) - H(v_{pb}, \hat{a}) \right] \right\}, \) \hspace{1cm} \text{(17)}

\( \Delta \vartheta - \text{effect} \)

\[ v \frac{\kappa_{1}^{(s)} t_{s}}{4\sqrt{2\pi}} \left\{ \cos \vartheta_{s} \sin^{2} \vartheta - \cos \vartheta \sin^{2} \vartheta_{s} \right\} \left[ H(v_{rp}, \hat{a}) - H(v_{bp}, \hat{a}) \right] \\
+ \frac{1}{2} \left[ \cos \vartheta_{s} (1 + \cos^{2} \vartheta) - \cos \vartheta (1 + \cos^{2} \vartheta_{s}) \right] \left[ H(v_{rb}, \hat{a}) - H(v_{br}, \hat{a}) \right] \}, \] \hspace{1cm} \text{(18)}

\( \Delta \varphi - \text{effect} \)

\[ v \frac{\kappa_{1}^{(s)} t_{s}}{4\sqrt{2\pi}} \sin^{4} \vartheta \sin 2(\varphi - \varphi_{s}) \times \left[ \frac{3}{2} L(v_{pp}, \hat{a}) - L(v_{rp}, \hat{a}) - L(v_{bp}, \hat{a}) + \frac{1}{4} L(v_{rb}, \hat{a}) + \frac{1}{4} L(v_{br}, \hat{a}) \right], \] \hspace{1cm} \text{(19)}

where

\[ v_{rr} = \frac{1}{\sqrt{2 \Delta \lambda_{D}}} \left[ \Delta \lambda^{(s)}_{A} + \bar{g} (\Delta \lambda^{(s)}_{B} - \Delta \lambda_{B}) \right] \]

\[ v_{bb} = \frac{1}{\sqrt{2 \Delta \lambda_{D}}} \left[ \Delta \lambda^{(s)}_{A} - \bar{g} (\Delta \lambda^{(s)}_{B} - \Delta \lambda_{B}) \right] \]

\[ v_{pp} = \frac{1}{\sqrt{2 \Delta \lambda_{D}}} \Delta \lambda^{(s)}_{A} \]

\[ v_{rp} = \frac{1}{\sqrt{2 \Delta \lambda_{D}}} \left[ \Delta \lambda^{(s)}_{A} + \bar{g} \Delta \lambda^{(s)}_{B} \right] \]

\[ v_{bp} = \frac{1}{\sqrt{2 \Delta \lambda_{D}}} \left[ \Delta \lambda^{(s)}_{A} - \bar{g} \Delta \lambda^{(s)}_{B} \right] \]

\[ v_{pr} = \frac{1}{\sqrt{2 \Delta \lambda_{D}}} \left[ \Delta \lambda^{(s)}_{A} - \bar{g} \Delta \lambda_{B} \right] \]

\[ v_{pb} = \frac{1}{\sqrt{2 \Delta \lambda_{D}}} \left[ \Delta \lambda^{(s)}_{A} + \bar{g} \Delta \lambda_{B} \right] \]

\[ v_{rb} = \frac{1}{\sqrt{2 \Delta \lambda_{D}}} \left[ \Delta \lambda^{(s)}_{A} + \bar{g} (\Delta \lambda^{(s)}_{B} + \Delta \lambda_{B}) \right] \]

\[ v_{br} = \frac{1}{\sqrt{2 \Delta \lambda_{D}}} \left[ \Delta \lambda^{(s)}_{A} - \bar{g} (\Delta \lambda^{(s)}_{B} + \Delta \lambda_{B}) \right] \]

\( \hat{a} = \sqrt{2} a \).

As far as the \( \Delta B - \text{effect} \) is concerned, the sign of the first line of Equation (17) is easily evaluated. Since \( H(v, a) \) decreases monotonically with increasing \(|v|\), we have \( H(v_{bb}, \hat{a}) > H(v_{rr}, \hat{a}) \) when \( \Delta \lambda^{(s)}_{A} \) and \( \bar{g} (\Delta \lambda^{(s)}_{B} - \Delta \lambda_{B}) \) have the same sign. For slightly inclined magnetic fields (i.e., \( \vartheta \simeq 0 \) or \( \vartheta \simeq \pi \)) we thus obtain the sign rule

\[ \text{sign}(v) = \text{sign} \left[ \bar{g} \Delta \lambda^{(s)}_{A} \cos \vartheta (\Delta \lambda_{B} - \Delta \lambda^{(s)}_{B}) \right] \hspace{1cm} (\Delta B - \text{effect}). \] \hspace{1cm} \text{(20)}

By similar reasoning, and observing that the two factors in Equation (18) depending on the inclination angles have the same sign as \( (\vartheta - \vartheta_{s}) \), we get
\[
\text{sign}(\nu) = \text{sign} \left[ -\tilde{g} \Delta \lambda_A^{(s)} (\vartheta - \vartheta_s) \right] \quad (\Delta \vartheta - \text{effect}). \tag{21}
\]

Comparison of Equations (20) and (21) shows that an increase in the optical depth of the transverse component of the magnetic field affects the NCP parameter in the same way as a decrease of the magnetic field strength, and vice versa. A relation similar to Equation (20) has been derived by Pahlke and Solanki (1986) and Solanki and Pahlke (1988). Sánchez Almeida, Collados, and del Toro Iniesta (1989) have demonstrated that it always holds in LTE, provided the magnetic field gradient and the velocity gradient are small. A relation similar to Equation (21) is given in Solanki and Montavon (1993).

As for the \( \Delta \varphi \)-effect, Equation (19) shows that \( \nu \) has a complicated dependence on the Doppler shift and Zeeman splitting. Numerical evaluation of the expression in square brackets leads to the following result: provided the Doppler shift is sufficiently small (\( \Delta \lambda_A^{(s)} < 1.5 \Delta \lambda_D \)), we can write

\[
\text{sign}(\nu) = \text{sign} \left[ \Delta \lambda_A^{(s)} (\varphi - \varphi_s) \right] \quad (\Delta \varphi - \text{effect}). \tag{22}
\]

It should be pointed out that Equation (22), unlike Equations (20) and (21), is independent of the Landé factor (the transformation \( \tilde{g} \rightarrow -\tilde{g} \) merely changes \( v_{rp} \) into \( v_{bp} \) and \( v_{rb} \) into \( v_{br} \)). Therefore, the use of two spectral lines with a positive and a negative Landé factor respectively, allows in principle to distinguish the \( \Delta \varphi \)-effect from the \( \Delta B \) and \( \Delta \vartheta \)-effects.

Finally, Equations (20), (21), and (22), as well as Equation (10), predict that a sign switch of the velocity gradient produces a sign switch of the NCP. Sánchez Almeida and Lites (1992) have proved that this is always the case under LTE conditions.

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References