THE FIRST AND SECOND ORDER MOMENTS OF THE
POLARIZATION PROFILES OF HYDROGEN LINES

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Abstract. The main properties of the first- and second-order moments of polarized hydrogen lines, forming in the presence of stationary electric and magnetic fields, are reviewed. The analytical results presented here apply directly to the case of optically-thin emission lines in the LTE regime. Some applications of such results to electric- and magnetic-field diagnostics in (solar) plasmas are then briefly considered.

Key words: Spectropolarimetry – Zeeman Effect – Stark Effect – Hydrogen Lines

1. Introduction

The detection and measurement of (stationary) electric and magnetic fields is a main issue in the understanding of the equilibrium condition and dynamical evolution of solar plasmas.

Historically, however, the magnetic field always played a dominant role in solar physics investigation. This is essentially due to the fact that, on the sun, relatively high-intensity magnetic fields exist (e.g., in sunspots), which may determine easily detectable circular-polarization signature in magnetic-sensitive lines, because of the linear Zeeman effect.

On the contrary, though the existence of macroscopic electric fields in the solar atmosphere was long ago suggested (Wien, 1916), and is presently provided by most of MHD modelling theories of solar structures (like flares), very few attempts of measuring them—through the linear polarization induced by the Stark effect on electric-sensitive lines—have been made so far (e.g., Dravins, 1973; Foukal et al., 1988; Foukal and Behr, 1995). Such more involute stage of the electric-field diagnostics may be ascribed to different reasons.

First of all, the attempts of measuring electric fields in many cases could only determine upper limits to the electric-field intensities possibly occurring in different structures of the solar atmosphere. Those limits show that the electric fields on the sun might be relatively weak (e.g., Foukal and Hinata,

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1991; Foukal and Behr, 1995). Low electric-field intensities ask for a refinement of the electrogaph techniques, and require the choice of particularly electric-sensitive lines.

Beside this, the Stark effect does not contribute to first order of perturbation (i.e., there is no a linear Stark effect) for most of the observable lines in the solar atmosphere, with the only (important) exception of lines which are originated in transitions of hydrogen or hydrogen-like atoms. On the other hand, the quadratic Stark effect usually becomes important (i.e., observable) only at much higher electric-field intensities than the ones which are of concern in solar plasma investigation.

Thus, the possibility of an electric-field diagnostics based on the Stark effect ultimately depends on the choice of highly electric-sensitive hydrogen (or hydrogen-like) lines, and on their actual observability in the solar atmosphere (Casini and Foukal, 1995).

Anyway, joint electric- and magnetic-field diagnostics is needed to test the field topologies provided by MHD models of solar plasma structures (Foukal and Hinata, 1991; Foukal and Behr, 1995). The results presented here for the hydrogen lines can be used to conceive a possible observing procedure finalized to that purpose, at least for the most simple field topologies (Casini, 1995b).

2. First and Second Order Moments of Hydrogen lines

The theory of radiative transfer for polarized radiation was formalized within the framework of (non relativistic) quantum electrodynamics by Landi Degl’Innocenti (1983). Its application to the problem of formation of hydrogen lines in the presence of simultaneous, stationary electric and magnetic fields was considered only a decade later by the authors (Casini and Landi Degl’Innocenti, 1993). In this last work, one of the main concerns was that of writing a numerical code for the complete calculation of the Stokes profiles of hydrogen lines.

Until now such code has only been applied to the case of optically-thin emission lines, forming in a LTE atmosphere. Under such simplifying assumptions the atomic density matrix of any level \( \ell \), \( \rho^{(\ell)}_{\lambda\lambda'} \), is simply proportional to the unit matrix of rank \( 2\ell^2 \) (taking into account the electronic spin), and does not need to be determined by solving the statistical equilibrium equations for the atomic system (see the contribution of Landi Degl’Innocenti in these proceedings). Moreover, since in the optically-thin case the observed intensity of a line is simply proportional to the optical depth of the layer in which the line is formed (plus a possible intensity contribution from background sources), nor is there a need for solving the
radiative-transfer problem for polarized radiation. Then, the numerical code provides directly the (normalized) Stokes profiles of such a line.

In spite of these rude simplifications, the solution can still be a rather time consuming one, if very high transitions of the hydrogen atom are involved. For instance, the complete calculation of the 15–9 transition would require the calculation of the four Stokes parameters for the 72 900 \( (= 2^2 \times 15^2 \times 9^2) \) fine-structure components of the line.

For purposes which do not explicitly require the knowledge of the frequency dependence of the Stokes profiles, the information provided by integral properties of those profiles, like the first- and second-order frequency moments, may be sufficient. In fact, those moments already determine the center of gravity of the Stokes profiles and their characteristic width—estimated by the standard deviation. Both these quantities are affected by the external fields, so that their estimate can in principle be applied to the problem of the diagnostics of electric and magnetic fields in astrophysical plasmas. In addition, by determining the analytical form of the moments, the contribution of the external fields to the polarization of hydrogen lines can be quantified—at least to lowest orders—in a simple and straightforward way.

In the absence of the broadening mechanisms typically present in a thermal plasma (i.e., for infinitely narrow line components), the \( q^{\text{th}} \)-order frequency moments—with respect to a reference frequency \( \bar{\omega} \)—of the Stokes profiles of an optically-thin emission line in the LTE regime can be written in the form

\[
\langle \omega^q (i) \rangle = \sum_\alpha \tilde{s}_\alpha (i) (\omega_\alpha - \bar{\omega})^q, \quad i = 0, 1, 2, 3, \tag{1}
\]

where the index \( i \) enumerates the four Stokes parameters \( I, Q, U, V \). In the above equation, the \( \tilde{s}_\alpha (i) \) are the relative strengths—for the four Stokes parameters—of the line component at the frequency \( \omega_\alpha \), the summation being extended to all the line components.

If we choose as the reference frequency \( \bar{\omega} \) the center of gravity of the intensity profile \( (i = 0) \) in the absence of external fields, one finds (Casini and Landi Degl’Innocenti, 1994a,b)

\[
\langle \omega (i) \rangle = -\delta_{i3} \frac{\mu_0}{\hbar} B \cos \vartheta_B, \quad i = 0, 1, 2, 3, \tag{2}
\]

and

\[
\langle \omega^2 (0) \rangle = \langle \omega^2 (0) \rangle_{\text{ff}} + \frac{e^2 \alpha_0^2 E^2}{\hbar^2} \left[ A_0 (n, m) + \frac{1}{2} (1 - 3 \cos^2 \vartheta_E) A_2 (n, m) \right] + \frac{1}{2} \frac{\mu_0^2 B^2}{\hbar^2} (1 + \cos^2 \vartheta_B), \tag{3}
\]
\begin{align}
\langle \omega^2(1) \rangle &= \frac{3}{2} \frac{e_0^2 a_0^2 E^2}{\hbar^2} \sin^2 \vartheta_E \cos 2(\varphi_E - \gamma) A_2(n, m) \\
&\quad - \frac{1}{2} \frac{\mu_0^2 B^2}{\hbar^2} \sin^2 \vartheta_B \cos 2(\varphi_B - \gamma), \\
\langle \omega^2(2) \rangle &= \langle \omega^2(1) \rangle \left\{ \begin{array}{l}
\cos 2(\varphi_E - \gamma) \rightarrow \sin 2(\varphi_E - \gamma) \\
\cos 2(\varphi_B - \gamma) \rightarrow \sin 2(\varphi_B - \gamma)
\end{array} \right\}, \\
\langle \omega^2(3) \rangle &= 0.
\end{align}

(Actually a contribution to \( \langle \omega(i) \rangle \), for \( i = 0, 1, 2 \), is present if one accounts for the quadratic Zeeman effect. Such contribution, negligible for magnetic-field intensities typical of the solar atmosphere, is evaluated in Casini and Landi Degl'Innocenti, 1994a.)

In the above equations, the field polar angles, \( \vartheta \), are measured from the line-of-sight. The field azimuthal angles, \( \varphi \), and the position angle of the reference direction for Q-polarization, \( \gamma \), are measured from an arbitrarily chosen direction on the plane of the sky (normal to the line-of-sight). The dimensionless coefficients \( A_0(n, m) \) and \( A_2(n, m) \), and the second-order moment of the intensity profile in the field-free case, \( \langle \omega^2(0) \rangle_{\text{ff}} \), are tabulated for all the hydrogen transitions up to the level \( \ell = 50 \) (Casini, 1995a).

### 3. Discussion and Conclusions

From inspection of Equations (2)–(6), one can draw some immediate conclusions.

To first order, only the circular-polarization profile is affected by a (longitudinal) magnetic field. The fact that the electric field does not bring any contribution to first order tells us that the magnetic-field measurements by longitudinal magnetographs are still reliable in the presence of electric fields.

To second order, both the electric and the magnetic field contribute to the intensity and linear-polarization profiles. In particular, we see that only the transverse components of the fields contribute to linear polarization. Instead, the circular-polarization profile is unaffected. Another important property is that the dependence of the second-order moments on the fields is purely quadratic, i.e., there are no mixed terms of the form \( EB \).

However, the most important fact is that, for the hydrogen lines, only the electric-field contribution to line polarization is dependent on the line considered—through the coefficients \( A_0(n, m) \) and \( A_2(n, m) \). Instead, the magnetic field always brings the same contribution to the Stokes (frequency) profiles of hydrogen lines. In particular, the introduction of the effective Landé factor, \( g_{\text{eff}} \), does not make sense for hydrogen lines, since it has been
shown (Casini and Landi Degl’Innocenti, 1994a) that $g_{\text{eff}} = 1$ for any hydrogen line. This difference in the behavior of the electric field and of the magnetic field in determining the polarization signature of hydrogen lines is particularly important since on it relies the possibility of a joint electric-and magnetic-field diagnostics (Casini, 1995b).

From another standpoint, the fact that the electric-field contribution to line polarization is line-dependent gives a meaning to the concept of electric-sensitivity (or Stark-sensitivity) for the hydrogen lines. In particular, one can try to define a Stark-sensitivity parameter for hydrogen lines which could help in the choice of transitions particularly suited to electric-field investigation. A first rigorous attempt in that sense, based on the results for the second-order moments, has been made by Casini (1995b).

A main advantage of the integral approach to the problem of electric-field diagnostics, as compared to the complete calculation of the Stokes profiles, is the very fast computation of the moments from Equations (2)–(6) (though some relevant coefficients, namely $A_2(n,m)$ and $\langle \omega^2(0) \rangle_{\text{eff}}$, must still be calculated numerically; see Casini, 1995a). Indeed, this fact enables one to consider very high transitions of the hydrogen atom, which are particularly suited to electric-field investigation due to their high Stark-sensitivity (Casini, 1995b).

A final remark should be made on the fact that Equations (2)–(6) hold only in the absence of broadening mechanisms. The generalization of such equations to the case of arbitrarily broadened profiles has however been dealt with in Casini and Landi Degl’Innocenti (1994b).

In a recent work (Casini and Landi Degl’Innocenti, 1995), the given expressions for the first- and second-order moments have also been applied to the generalization of the weak-field solution of the LTE radiative-transfer problem for polarized hydrogen lines in the case in which both the electric and the magnetic field are present. It is shown that the lowest non-vanishing orders (in a Taylor expansion of the line profile with respect to frequency) of the circular- and linear-polarization profiles are proportional respectively to the first and second derivatives (with respect to frequency) of the lowest-order intensity profile. More remarkably, such proportionality relations take the same form as those already known in the purely magnetic case, which have been widely used in the calibration of solar magnetographs so far. Eventually, the only difference would lie in the structure of the proportionality factors, which must now account for the general presence of electric as well as magnetic fields. Indeed, these factors turn out to be precisely the first- and second-order moments given by Eqs. (2)–(6).

These results are particularly important in conceiving reliable calibration procedures for vector magnetographs. In particular, it follows from the above—as it has already been stated in the opening of this section—that the circular-polarization profile is not affected (up to second order) by the
presence of electric fields, thus longitudinal magnetographs can be safely calibrated through the well-known relation between $V$ and the first derivative of $I$ even in the presence of electric fields. On the contrary, one should consider the contribution of the electric field in the relation between $Q$ ($U$) and the second derivative of $I$, since this would affect the reliability of the calibration procedure for transverse magnetographs.

References