NONLOCAL ESCAPE-INTEGRAL APPROXIMATIONS FOR THE LINE FORCE IN STRUCTURED LINE-DRIVEN STELLAR WINDS

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ABSTRACT

We develop a nonlocal, integral escape-probability formalism for approximating both the direct and diffuse line force in a structured, radiatively driven stellar wind. Our approach represents a direct generalization of the local Sobolev escape-probability methods commonly applied in smooth, steady wind models. It naturally incorporates previous nonlocal force methods based on pure-absorption or smooth source function (SSF) approximations for the line transport. However, it also leads to the development of a new “escape-integral source function” (EISF) method, which, for the first time, takes into account the dynamical effects of gradients in the perturbed source function. Perturbation analyses, formulated here in terms of the perturbed escape probability, demonstrate how key aspects of the linear wind instability, including line-drag and phase-propagation reversals, are incorporated in the various nonlocal force approximations. The methods here thus provide the basis for further, more complete simulations of the nonlinear wind structure resulting from this strong line-driven flow instability.

Subject headings: circumstellar matter — line: formation — radiative transfer — stars: atmospheres — stars: early-type — stars: mass loss

1. INTRODUCTION

The winds of hot, luminous OB stars are understood to be driven by line scattering of the star’s continuum radiation flux. A central component in dynamical models of these winds is the computation of the line-driving force. Following the original suggestion of this line-driving mechanism by Lucy & Solomon (1970), Castor, Abbott, & Klein (1975; hereafter CAK) developed a simple but powerful formalism, still the basis of models today, for treating the cumulative effect of a large ensemble of driving lines. An important simplification in this formalism was the application of the Sobolev (1957, 1960) approximation to provide a strictly local solution of the line transport. For the smooth, monotonically accelerating flow assumed in typical steady state wind models, the accuracy of the Sobolev approach has since been quantitatively confirmed by direct comparison with full comoving frame transfer solutions (Paudlach, Puls, & Kudritzki 1986).

An essential general requirement for the applicability of a Sobolev treatment is that the length scale $H$ for any flow variations must be larger than the Sobolev length $L \equiv n_{th}(dv/dr)$, over which the flow speed $v$ increases by an ion thermal speed $n_{th}$. For a smooth outflow, a characteristic flow variation scale is just $H \approx v/(dv/dr)$, implying the Sobolev approach applies throughout the region $v > n_{th}$. For the relatively heavy metal ions (mainly iron group elements) important for line driving, the thermal speed is significantly smaller than the isothermal sound speed $a$, with typically $n_{th}/a \approx 0.1$–0.2. Thus the Sobolev line force accurately describes the driving near the sonic point, where the mass-loss rate is fixed, and does not seriously fail until well into the subsonic region, where gas pressure provides the primary support against gravity. (See, e.g., Sellmaier et al. 1993.)

However, such a Sobolev approach is not well suited for modeling the dynamics of the extensive small-scale structure that several kinds of evidence suggest may be intrinsic to such line-driven winds. The extended black absorption troughs of saturated UV lines indicate the wind velocity fields are highly nonmonotonic, and this implies an extensive nonlocal coupling of the multiple resonances points in the line transfer (Lucy 1982a; Puls et al. 1993). Soft X-ray emission (Harnden et al. 1979; Chlebuski 1989) further suggests the presence of numerous shocks distributed throughout these winds (Lucy 1982b; Hillier et al. 1993; Cohen et al. 1996). Such extensive wind structure could arise naturally from the line-driving process itself, which linear stability analyses have shown to be highly unstable (Lucy & Solomon 1970; MacGregor, Hartmann, & Raymond 1979; Carlberg 1980; Lucy 1984; Owocki & Rybicki 1984, 1985, hereafter OR I, OR II; Rybicki 1987; Rybicki, Owocki, & Castor 1990; Castor 1991). This instability is strongest for small-scale perturbations with length scales $\lambda$ near or below the Sobolev length $L$ and vanishes in the Sobolev limit $\lambda/L \rightarrow \infty$ (OR I; Abbott 1980). As such, simulations of the nonlinear evolution of this instability must necessarily be based on line-force computations that do not assume the Sobolev approximation.

This inapplicability of Sobolev approximation represents a serious complication for instability simulations, since the line force can no longer be specified from strictly local flow quantities but rather must in general involve integrals that account for potential nonlocal couplings, such as the multiple line resonances that occur in a nonmonotonic velocity field. However, a fully self-consistent transfer solution for the large number of driving lines involved is not practical, as it would be far too costly to compute at each time step of any numerical simulation of the wind structure. It is thus essential to develop line-force approximations that retain essential elements of the structured-flow line transfer but can be evaluated fast enough for practical application within a time-dependent hydrodynamics simulation. A principal aim of this paper is to provide a general
formalism for such approximations, based on an integral escape-probability (see, e.g., Kalkofen 1984) treatment of the line transfer, which represents a straightforward generalization of the local Sobolev (1957, 1960) escape-probability method.

Previously, there have been two levels of approximation used in computing the nonlocal line force in instability simulations. The initial models of Owocki, Castor, & Rybicki (1988; hereafter OCR) were based on a pure-absorption treatment, which computes the force from numerical integration of the frequency-dependent attenuation of the direct radiation from the stellar core but ignores the dynamical role of the diffuse, scattered radiation field. In a smooth, supersonic wind amenable to a Sobolev line-transfer treatment, the diffuse line radiation attains a near fore-aft symmetry and so does not contribute much to the line force (Castor 1970, 1974; Lucy 1971). However, Lucy (1984) showed that small-scale velocity perturbations in an otherwise smooth wind do experience a strong net “line drag” from this mean diffuse radiation. This can significantly reduce the strong instability associated with the perturbed direct radiation force, particularly near the wind base.

To take into account such line-drag effects in instability simulations, Owocki (1991) introduced the “smooth source function” (SSF) method, which allowed for approximation of the net diffuse force in terms of the fore-aft asymmetries in the line escape near wind structures. A great advantage of this SSF method is that it utilizes only integral quantities already available from the computation of the direct force term and so requires only a modest (∼10%–20%) increase in computation time relative to pure-absorption models. Such SSF models show a markedly reduced level of variability, particularly near the wind base, while still exhibiting extensive structure in the outer wind (Owocki 1991, 1993, 1994; Feldmeier 1994, 1995). As in the OCR pure-absorption models, this structure is again dominated by reverse shocks that separate high-speed, low-density rarefactions from slower, high-density compressions. This is commensurate with the dominant instability being in the wave mode that propagates inward relative to mean flow, with velocity and density perturbations anticorrelated (OR 1).

A major shortcoming of this SSF treatment is its neglect of the diffuse force component associated with gradients in the perturbed source function. Although the linear perturbation analysis of OR II showed that such terms have little direct effect on the growth rate of the instability, they can reverse the direction of phase propagation for small-scale perturbations, implying that some unstable perturbations should now be outward propagating, with a positive correlation between density and velocity (OR II; Puls 1994). In principal, this could potentially have a dramatic effect on the fundamental nature of the resulting wind structure, perhaps even leading to extensive forward shocks in which the highest speed material is quite dense, in marked contrast to the reverse shocks and negative correlation between density and velocity seen in all previous instability simulations (OCR; Castor 1991; Owocki 1994; Feldmeier 1994, 1995).

To provide the basis for incorporating such gradient effects in instability simulations, we describe here an extension of the SSF treatment of the diffuse line force, which we term the “escape-integral source function” (EISF) method. While the earlier nonlocal force treatments arbitrarily assumed that the line-scattering source function was either zero (pure absorption) or had a specified, locally constant form (SSF), this EISF method estimates the source function directly in terms of the integral escape probability. Although still not a fully self-consistent transfer solution, this does allow for the source function to reflect flow variations in a way that incorporates the essential aspects of both the line-drag and the phase-reversal effects.

The development here provides a general framework (§§ 2–3) for progressing through these various force approximations, beginning with the local Sobolev approach (§ 3.1), then to the pure-absorption and SSF methods (§ 3.2), and finally to the new EISF treatment (§ 3.3). Through a perturbation analysis of the line force (§ 4), expressed now in terms of the appropriate perturbed escape probability, we demonstrate how the successive nonlocal force approximations (pure absorption, SSF, EISF) incorporate in turn each of the essential perturbation effects (direct instability, line drag, and propagation reversal; see §§ 4.1–4.3). In addition to the force from single lines, we also derive ( §§ 5.1–5.4) expressions for the cumulative force from a CAK-type line ensemble with a power-law number distribution in line strength, and then describe (§ 5.5) methods for efficient numerical evaluation of these force expressions. A perturbation analysis (§ 6.1) shows that these ensemble force components have characteristics quite similar to the analogous single-line forms. In particular, a phase analysis (§ 6.2) demonstrates explicitly that the ensemble EISF force quite naturally incorporates the phase reversal effect and so provides a good basis for future dynamical simulations of how this might alter the correlation properties of wind structure.

2. BASIC FORMULAE

In a one-dimensional, spherically symmetric outflow, we write the line force per unit mass as

$$g(r) = \frac{4\pi k e_{\text{th}} v_0}{c^3} \bar{H}(r),$$

(1)

where $v_0$ is the line frequency, $e_{\text{th}}$ is the ion thermal speed, and $c$ is the speed of light. Here we characterize the line strength in terms of the mass absorption coefficient $\kappa \equiv \chi_\ell / \rho$, where $\rho$ is the mass density and $\chi_\ell$ is the profile-integrated line-opacity divided by the thermal Doppler width (as defined, e.g., by eq. [2] of Puls et al. 1993). The frequency-integrated flux at radius $r$ is

$$\bar{H}(r) = \frac{1}{2} \int_{-1}^{+1} d\mu \int_{-\infty}^{\infty} dx \phi(x - \mu u(r)) I(x, \mu, r),$$

(2)

where $u(r) = v(r) / e_{\text{th}}$ is the wind velocity in thermal-speed units, and $\phi$ is the line profile function, taken here to be a Gaussian. $I(x, \mu, r)$ is the specific intensity at observer’s frame frequency $x$, measured from line center in thermal Doppler-width units, along a ray with direction cosine $\mu$ (i.e., relative to a local, outward radius vector).

Taking advantage of the assumed spherical symmetry, the required angle integrations can be formulated in terms of a series of rays, each defined by its minimum distance to the origin, i.e., “impact parameter” $p$ (see, e.g., Mihalas 1978, pp. 252–253), with the position along the ray given by the coordinate $z \equiv \mu r$, where $r = (p^2 + z^2)^{1/2}$. We thus define the frequency-
dependent line-optical depth between two points \( z_1 \) and \( z_2 \) along a ray \( p \),

\[
t(x, p, z_1, z_2) = \int_{\min \{z_1, z_2\}}^{\max \{z_1, z_2\}} \kappa \rho(u) [x - \mu u(r')] dz',
\]

(3)

where now \( r' \equiv (p^2 + z'^2)^{1/2} \) and \( \mu' \equiv z'/r' \).

In general, the intensity consists of both a direct component from the star, and a diffuse component from radiation scattered (or created) within the wind. The direct component can be written

\[
I_{\text{dir}}(x, p, z) = I_\odot(p) e^{-\tau(x, p, z_A, z)} ; \quad p < R_\odot
\]

\[
= 0 ; \quad p > R_\odot,
\]

(4)

where \( z_A \equiv (R_\odot^2 - p^2) \) is the ray coordinate at the stellar surface \( r = R_\odot \), and the core intensity \( I_\odot \) can vary with \( p \) if there are center-to-limb variations. The diffuse component is given by the formal solution

\[
I_{\text{diff}}(x, p, z) = \int_0^{(x, p, z, z_0)} S(r') e^{-\tau(x, p, z, z')} dr(x, p, z, z'),
\]

(5)

where \( S \) is the line source function. Since \( \mu = z/r \), the sign of \( z \) determines whether this intensity is defined along an outward (+) or inward (-) ray, with the boundary value of the ray coordinate given by

\[
z_b = z_a ; \quad z > 0, p < R_\odot
\]

\[
= -\infty ; \quad \text{otherwise}.
\]

(6)

Note that all such quantities depending on local ray parameters \( (p, z) \) can equivalently be written as functions of \( (\mu, r) \), e.g.,

\[
I_{\text{diff}}(x, p, z) \rightarrow I_{\text{diff}}(x, \mu, r).
\]

The frequency-integrated intensity is

\[
\bar{I}(\mu, r) \equiv \int_{-\infty}^{\infty} dx \phi(x - \mu u(r))[I_{\text{dir}}(x, \mu, r) + I_{\text{diff}}(x, \mu, r)],
\]

(7)

If we assume pure, isotropic scattering with complete frequency redistribution (CRD), then the source function \( S(r) \) is given by the frequency-integrated mean intensity,

\[
S(r) = \bar{I}(r) \equiv \langle \bar{I}(\mu, r) \rangle,
\]

(8)

where the angle brackets signify angle integration, e.g.,

\[
\langle \bar{I}(\mu, r) \rangle = \frac{1}{2} \int_{-1}^{+1} d\mu \bar{I}(\mu, r).
\]

(9)

For simplicity, we shall henceforth ignore center-limb variations, and so assume the star has a constant intensity \( I_\odot \) over a disk defined by

\[
D(\mu) = 1 ; \quad \mu_\odot < \mu < 1,
\]

\[
= 0 ; \quad \mu < \mu_\odot,
\]

(10)

where \( \mu_\odot \equiv (1 - R_\odot^2/r^2)^{1/2} \). A useful special case is defined by the limit \( t(x, p, z, z') \rightarrow 0 \), for which the line force in equation (1) reduces to the optically thin result,

\[
\dot{g}_{\text{thin}}(r) = \frac{4\pi \kappa v_0 l_\odot}{c^2} \langle \mu D(\mu) \rangle
\]

\[
= \frac{4\pi \kappa v_0 l_\odot}{c^2} \frac{1 - \mu_\odot^2}{4}
\]

\[
= \frac{\kappa v_0 l_\odot}{4\pi r_0^2 c^2} L_\odot(v_0)
\]

(11)

with \( L_\odot(v_0) \) the stellar continuum luminosity near the line frequency \( v_0 \).

3. ESCAPE PROBABILITY FORMS FOR THE SOURCE FUNCTION

In general, a scattering source function like that defined in equation (8) is solved using differential or integral methods (e.g., complete linearization, or accelerated lambda iteration; see, e.g., Mihalas 1978; Hubeny 1993) that account for potential nonlocal couplings associated with scattering. However, in a stellar wind, the strong velocity gradient can lead to a localization of the line transport, so that the line radiation field can be approximated in terms of an escape probability (cf. Kalkofen 1984),
which we define here as

\[ b(\mu, r) \equiv \int_{-\infty}^{\infty} dx \phi(x - \mu u(r)) e^{-x} \, d\tau. \] (12)

As written, the expression in equation (12) still retains a certain nonlocalness, through the integral in equation (3) defining the optical depth \( \tau(x, p, z, z_0) \). Before describing methods that use this integral form for \( b(\mu, r) \), let us review the Sobolev (1957, 1960) theory that, in certain circumstances, will allow this escape probability to be approximated in terms of strictly local quantities.

3.1. The Sobolev Source Function and Line Force

For a monotonically accelerating, supersonic wind, the variation of the line–optical-depth integrand in equation (3) is dominated by the velocity dependence of the profile function. The basis of the Sobolev approximation is thus to convert this spatial integral into an integral over comoving frame frequency \( x - \mu u(r) \),

\[ t(x, p, z, z_0) \approx \tau(\mu, r) \Phi(x - \mu u(r)) , \] (13)

where

\[ \Phi(x) \equiv \int_{-\infty}^{\infty} dx' \phi(x') \] (14)

is the profile integral function \( [\text{=erfc}(x)/2 \text{ for a Gaussian profile}] \). Here \( \tau(\mu, r) \) is the Sobolev optical depth along direction cosine \( \mu \), given by

\[ \tau(\mu, r) \equiv \kappa \rho L_{\text{u}} \equiv \frac{\kappa \rho}{d(\mu u)/dz} = \frac{\kappa \rho r}{u(1 + \sigma u^2)} , \] (15)

where \( L_{\text{u}} \equiv 1/[d(\mu u)/dz] \) is the Sobolev length along \( \mu \). The last equality applies for a spherically symmetric outflow, with \( \sigma \equiv (r/u)(du/dr) - 1 \) (Castor 1970); it shows that the Sobolev optical depth is an even function of \( \mu \). Applying equation (13) in equation (12), we then find the frequency integral can be readily done, yielding for the Sobolev escape probability

\[ b_{\text{Sob}}(\mu, r) = \frac{1 - e^{-\tau(\mu, r)}}{\tau(\mu, r)} . \] (16)

The direct component of frequency-integrated intensity (eqs. [4] and [7]) can thus now be written

\[ I_{\text{dir}}(\mu, r) = I_{\text{u}}(\mu)b_{\text{Sob}}(\mu, r) = I_{\text{u}}(\mu) \frac{1 - e^{-\tau(\mu, r)}}{\tau(\mu, r)}. \] (17)

In a similar way, we can also convert the spatial integration in the formal solution of equation (5) to an integral in comoving frame frequency, yielding for the diffuse intensity

\[ I_{\text{d}}(x, \mu, r) = S(r)[1 - e^{-\tau(\mu, r)\Phi(x - \mu u)}] . \] (18)

When integrated over frequency, this becomes

\[ I_{\text{d}}(\mu, r) = S(r)[1 - b_{\text{Sob}}(\mu, r)] . \] (19)

By further integrating equation (19) over the angle and applying equations (7), (8), and (17), we can solve for the Sobolev source function,

\[ S_{\text{Sob}}(r) = \frac{\langle b_{\text{Sob}}(\mu, r)I_{\text{u}}(\mu) \rangle}{\langle b_{\text{Sob}}(\mu, r) \rangle} . \] (20)

This shows that the source function (and mean line intensity) is fixed by the equilibrium ratio between the stellar core radiation’s penetration probability over the frequency- and angle-averaged escape probability. For optically thin lines, we then have

\[ S_{\text{Sob}}(r) = S_{\text{thin}}(r) \equiv I_{\text{i}} \frac{1 - \mu^2}{2} = \frac{I_{\text{i}}}{2(1 + \mu^2)} \frac{R_{\text{i}}^2}{r^2} ; \quad \tau \ll 1 . \] (21)

For optically thick lines, the thin source function is modified by a factor that accounts for possible differences in the angle dependence for core penetration versus escape,

\[ S_{\text{Sob}}(r) = S_{\text{thin}} \left( \frac{1 + \sigma E_\alpha}{1 + \sigma/3} \right) ; \quad \tau \gg 1 , \] (22)

where we have used the notation (cf. OR II)

\[ E_n \equiv \frac{n}{n - 1} - \mu_n^{-1} = \frac{1}{n} \sum_{j=1}^{n} \mu_j^{-1} . \] (23)

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Note that \( \tau(\mu, r) \), and hence \( b_{\text{Sob}}(\mu, r) \) and \( I_{\text{diff}}(\mu, r) \), are all even functions of \( \mu \). This means the diffuse flux \textit{vanishes} in the Sobolev approximation (Sobolev 1957, 1960; Castor 1970, 1974; Lucy 1971). Hence the line flux (2), and the line force (1), arise solely from the \textit{direct} component of intensity (17). The Sobolev line force is thus given by

\[
g_{\text{Sob}}(r) = \frac{4\pi c^2 v_0}{c^2} \left( \frac{\mu b(\mu)}{1 - e^{-\tau(\mu, r)}} \right) \tau(\mu, r) \quad (24)
\]

In the limit \( \tau(\mu, r) \to 0 \), this reduces to the optically thin force expression (11).

3.2. \textit{The Smooth Source Function (SSF) Method}

Note that while the optical depth defined in equation (3) depends on frequency and angle, as well as radius, the scattering source function in equation (8) is a function of radius only. By the assumptions of complete redistribution in both frequency and angle, the source function is \textit{averaged} over these quantities. Compared to optical depth terms, it should thus be relatively insensitive to the details of dynamical variations in, e.g., the velocity, implying that its radial variation should likewise remain relatively \textit{smooth}.

To take advantage of this idea, let us now apply the \textquotedblleft smooth source function\textquotedblright{} (SSF; Owocki 1991) approximation, whereby in the formal solution for diffuse intensity (eq. [5]), we make the substitution

\[
S(r') \to S(r) \quad (25)
\]

with the function \( S(r) \) left free to be specified, e.g., using Sobolev theory. This makes the integral in equation (5) analytic, yielding

\[
I_{\text{diff}}(x, p, z) = \left( S(r)[1 - e^{-\tau(x, p, z, r)}] \right) \quad (26)
\]

Note that this has a very similar form to the analogous Sobolev expression (18), except that now the optical depth is given by the more general, nonlocal equation (3), instead of by the local Sobolev result (13). Similarly, averaging over the profile function \( \phi(x) \) and using equations (7) and (12), we find that the frequency-integrated diffuse intensity is given by

\[
I_{\text{diff}}(\mu, r) = \left( S(r)[1 - b(\mu, r)] \right) \quad (27)
\]

which appears identical to the Sobolev expression (19), except that the escape probability \( b(\mu, r) \) is given now by the general, nonlocal integral (12), instead of the local Sobolev form (16).

In addition to retaining a nonlocal quality in the optical depth and escape probability, a key distinction here from the Sobolev approach is that the optical depth need not necessarily be fore-aft symmetric. Accordingly, the escape probability (12) and diffuse intensity (27) are likewise not necessarily even in \( \mu \), implying that the frequency-integrated diffuse flux \( H_{\text{diff}} \) can now be nonzero. Applying equations (1), (2), and (27), the associated diffuse line force is given by

\[
g_{\text{diff}}(r) = -\frac{4\pi c^2 v_0}{c^2} \left( \mu b(\mu, r) \right) \quad (28)
\]

For comparison, the direct component of the line force has the general form

\[
g_{\text{dir}}(r) = \frac{4\pi c^2 v_0}{c^2} \left( b(\mu, r) \right) I_\mu \quad (29)
\]

As discussed further below (§ 4), a perturbation analysis of these SSF force expressions shows that the direct term properly incorporates the line-driven instability (Lucy & Solomon 1970; MacGregor et al. 1979; Carlberg 1980; OR 1), while the diffuse term contains the \textquotedblleft line-drag\textquotedblright{} effect (Lucy 1984; OR 2). When summed over a line distribution (§ 5), this SSF approach thus provides a useful framework for carrying out simulations of the nonlinear evolution of this line-driven flow instability (Owocki 1991, 1993, 1994; Feldmeier 1994, 1995). However, because it effectively ignores any variations in the source function, this method cannot account for source function perturbation terms, which have been shown (OR II; Puls 1994) to have an important influence on the propagation and correlation characteristics of flow perturbations.

We next describe a method for incorporating source function gradient effects ignored in this SSF approach.

3.3. \textit{The Escape-Integral Source Function (EISF) Method}

As a further generalization, let us next retain the source function \( S(r') \) within the formal solution integral (5) for the diffuse intensity but now approximate its spatial variation in terms of the escape probability \( b(\mu, r) \),

\[
S(r') \approx \frac{\langle b(\mu, r) \rangle I_\mu}{\langle b(\mu, r) \rangle} \quad (30)
\]

This form is chosen in analogy with the Sobolev result (20), but note that \( b(\mu, r) \) is now to be computed from the more general, nonlocal integral (12) rather than from the local Sobolev expression (16). We thus refer to this as the \textquotedblleft escape-integral source function\textquotedblright{} (EISF).

The diffuse intensity is now obtained by applying equation (30) in the formal solution spatial integral of equation (5),

\[
I_{\text{diff}}(x, p, z) = \int_0^{I(t(x, p, z, r))} \frac{\langle b(\mu, r) \rangle I_\mu}{\langle b(\mu, r) \rangle} e^{-\tau(x, p, z, r)} dt(x, p, z, z') \quad (31)
\]
The corresponding diffuse force is then derived by computing the frequency-integrated flux from equation (2) and applying this in equation (1),

$$g_{\text{diff}}(r) = \frac{4\pi \nu v_0}{c^2} \left( \mu \int_{-\infty}^{\infty} dx \phi(x - \mu a) \int_{0}^{\infty} y(x, p, z) \frac{\langle b(\mu, r) \rangle}{b(\mu, r)} e^{-\frac{r(x, p, z)}{\tau(\mu, r)}} dt(x, p, z) \right).$$  \hfill (32)

A great advantage of this approach is that it utilizes the same angle-averaged escape probabilities needed for computing the direct and diffuse forces in the SSF method. As in the SSF case, this requires nested integrations over frequency, angle, and (for the optical depth) space. Note, however, that in this EISF method there is now a second nested level of spatial integration required for computing the diffuse intensity.\(^1\) As compared to the SSF approach, for which this diffuse intensity integral is done analytically (see eq. [31]), this EISF method is thus significantly (about an order of magnitude) more costly in computer time. (See § 5.4.)

4. PERTURBATION ANALYSIS

Let us now analyze how these line forces are affected when a small-amplitude perturbation, e.g., in velocity, is introduced into an otherwise smooth, supersonic flow that is well described by the Sobolev theory summarized in § 3.1.

4.1. Instability of the Direct Component of the Line Force

The response of the direct force to perturbations was analyzed extensively by OR I, who derived a general "bridging law" that unified the unstable response of small-scale, optically thin perturbations (MacGregor et al. 1979; Carlberg 1980) with the stable response of perturbations on a large enough scale to be treated within the Sobolev approximation (Abbott 1980). For a line that is optically thick in the mean flow (i.e., with $\tau(\mu, r) \gg 1$), we can apply this OR I analysis to write an analogous bridging law for the dependence of perturbed escape probability $b(\mu, r)$ on the perturbed velocity $\delta u$. Taking a radial velocity perturbation of the form $\delta u = k \Phi(x)$, where $k$ is the radial wavenumber, the perturbation $\delta b(\mu, r)$ can be calculated in complete analogy to the OR I derivation of the bridging law, however, with the following replacements:

$$\delta b(\mu, r) = b_{\text{thin}} (\mu, r) \quad (\text{OR I, eq. [15]}),$$

$$\frac{\delta v}{v_{\text{th}}} = \mu \delta u \quad \text{accounting for the projected velocity component},$$

$$k = \mu k,$$

$$X_b = \frac{2x_b}{L} \gg \frac{2x_b}{L}. \quad (33)$$

Here the mean, blue-edge absorption frequency is defined by

$$x_b \equiv \frac{1}{b_{\text{Sob}}(\mu, r)} \int_{- \infty}^{\infty} dx \phi(x)e^{-\frac{r(x)}{\tau(\mu, r)}},$$

which, for the assumed Gaussian profile, can be solved approximately through the transcendental relation (cf. OR I, eq. [38]),

$$x_b \approx \sqrt{\ln \left[ \frac{\tau(\mu, r)}{2\pi^{1/2} x_b} \right]}. \quad (34)$$

Accounting finally for $\omega_b = 2x_b \nu_{\text{th}} / v_{\text{th}}$, we may then write

$$\frac{\delta b(\mu, r)}{b_{\text{Sob}}(\mu, r)} = 2x_b \frac{\mu k}{\nu_{\text{th}}} \frac{1}{\mu} \frac{i\mu k}{i\mu k + 2x_b/L} \delta u; \quad \tau(\mu, r) \gg 1. \quad (35)$$

Here $L$ is the Sobolev length along $\mu$ (defined through eq. [15]), and the Sobolev escape probability for the assumed case of an optically thick line reduces to

$$b_{\text{Sob}}(\mu, r) \approx \frac{1}{\tau(\mu, r)}; \quad \tau(\mu, r) \gg 1. \quad (36)$$

Assuming a fixed (unperturbed) mass-absorption opacity $\kappa$, we can then write the perturbed direct force in the form (cf. eq. [29]),

$$\delta g_{\text{diff}}(r) = \frac{4\pi \nu v_0}{c^2} \left( \nu I_*\nu(\mu) \delta b(\mu, r) \right)$$

$$= \frac{4\pi \nu v_0}{c^2} \left( \nu I_*\nu(\mu) \frac{2x_b}{\tau(\mu, r)} \left( \frac{\mu k}{i\mu k + 2x_b/L} \right) \right) \delta u. \quad (37)$$

\(^1\) Actually, in the case of single line it is possible to carry out the two spatial integrals in parallel, avoiding this extra nesting. Unfortunately, this trick cannot be extended to line-ensemble forms used in actual numerical models. See § 5.4.

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Far from the star, the radiation becomes radially streaming \((\mu_* \to 1)\), and we can trivially integrate equation (37) to recover
the OR I result,
\[
\frac{\delta g_{\text{air}}}{\delta u} = 2x_* g_{\text{Sob}} \frac{ik}{ik + 2x_*/L}; \quad \mu_* \to 1 ,
\]
(38)
where \(L \equiv L_1 \equiv 1/(\text{d}u/\text{d}r)\) is the radial Sobolev length, and \(x_*\) generally denotes an average of \(x\) over the stellar disk, with just \(x_* = x_1\) in this case. As discussed by OR I, the Sobolev length \(L\) represents roughly a “bridging length” between unstable short-wavelength perturbations \((kL \gg 1)\); for which \(\delta b \sim \delta u\) and asymptotically stable, long-wavelength perturbations \([kL \ll 1]\); for which \(\delta b \sim ik \delta u \sim d(\delta u)/dr\). In the unstable limit \(kL \gg 1\), we can find the asymptotic value of \(\delta g_{\text{air}}\) in the somewhat more general case that the stellar core has a uniform brightness over a finite disk \(\mu_* < \mu < 1\),
\[
\frac{\delta g_{\text{air}}}{\delta u} \approx 2x_* g_{\text{Sob}} \frac{E_3 + \sigma E_5}{E_2 + \sigma E_4}; \quad kL \gg 1 ,
\]
(39)
where the values of \(E_a\) are defined in equation (23).

Since the perturbed force varies directly with the perturbed velocity (i.e., \(\delta g_{\text{air}} \sim \delta u\)), such short-scale perturbations are highly unstable, with a typical amplification length of about a Sobolev length \(L\). In highly supersonic winds with a terminal speed \(v_\infty > 100v_\text{th}\), initially small-amplitude perturbations could thus potentially be amplified by some \(\sim v_\infty/v_\text{th} > 100\) e-folds within the wind! This implies such perturbations should quickly reach nonlinear amplitude. (See, e.g., OCR.)

4.2. Line-Drag Effect of the Mean Diffuse Radiation Field

We next review how this extremely strong instability can be reduced somewhat near the wind base by the Lucy (1984) “line-drag” effect, which is associated with the interaction of the perturbation with the mean, diffuse radiation field. Because source function gradient terms are not involved, this effect is contained already in the SSF form of the diffuse force, equation (28) (Owocki 1993). Ignoring perturbations in \(S\) or \(\kappa\), the part of the perturbed diffuse force leading to line drag is
\[
\delta g_{\text{drL}}(r) = - \frac{4\pi v_\text{th} v_0 \kappa}{c^2} S(r) \langle \mu \delta b(\mu, r) \rangle .
\]
(40)
Apart from the minus sign, and the dependence on \(S(r)\) instead of \(I_\lambda(\mu)\), this has a very similar form to the perturbed direct force (37). The minus sign implies a tendency to reduce or negate the effect of the direct term, including the instability. For the case of an optically thick line with source function (22), the ratio of the diffuse to direct perturbed force in the short-wavelength limit is
\[
\frac{\delta g_{\text{drL}}}{\delta g_{\text{air}}} = \frac{4 + \sigma E_3}{1 + \sigma E_3}, \quad kL \gg 1.
\]
(41)
For \(r \to R_\ast\), we have \(\mu_* \to 0\) and \(E_n \to 1/\sigma\), and so
\[
\delta g_{\text{drL}} \to -\delta g_{\text{air}}; \quad r \to R_\ast ,
\]
(42)
implying that this diffuse line drag can exactly cancel the direct instability near the stellar surface. On the other hand, for \(r/R_\ast \to \infty\), we have \(\mu_* \to 1\), \(E_n \to 1\), and \(\sigma \to -1\), and so
\[
\delta g_{\text{drL}} \to -0.2\delta g_{\text{air}}; \quad r/R_\ast \to \infty ,
\]
(43)
implying that far from the star the drag term only reduces the direct instability growth rate by \(\sim 20\%\). As summarized by OR II (see their Fig. 1), the residual net growth rate is thus still quite large throughout most of the acceleration region of the wind.

4.3. Source Function Perturbations and the Reversal of Phase Propagation

Let us next consider the effects of perturbations in the scattered radiation field itself. As we now need to account for source function gradient terms, we base our analysis on the EISF expressions derived in § 3.3. From the formal solution integral (eqs. [5] and [31]), the component of the diffuse intensity associated with a perturbed source function \(\delta S\) is
\[
\delta I_\delta(x, \mu, r) = \int_{0}^{\infty} \frac{\delta S e^{i\lambda(r-r')} e^{-i\lambda(x, p, z, z')} dt(x, p, z, z')} ,
\]
(44)
where the optical depth variation in the unperturbed flow can be approximated by the local, Sobolev form
\[
t(x, p, z, z') \approx \tau(\mu, r)\Phi(x - \mu u(r)) - \Phi(x - \mu u(r)) ,
\]
(45)
with the boundary optical depth \(t(x, p, z, z_0\)) likewise approximated by its Sobolev expression (13). This suggests we rewrite the optical depth integral in equation (44) in terms of the comoving-frame frequencies \(\bar{x} \equiv x - \mu u(r)\) and \(\bar{x}' \equiv x - \mu' u(r')\),
\[
\delta I_\delta(x, \mu, r) = \delta S\tau(\mu, r) \int_{\bar{x}} d\bar{x}' \hat{\phi}(\bar{x}) e^{i\epsilon\bar{x} . (\bar{x}' - \bar{x}) + i\mu(r)(\lambda(x') - \lambda(x))} ,
\]
(46)
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where the spatial wavenumber of the perturbation has been converted into frequency units through \( \kappa = - k/(d \bar{\chi}/d \bar{z}) = k \nu \). After integrating over frequency and angle, the associated component of the perturbed force is given by

\[
\delta g_0 = \frac{4 \pi \kappa \nu_0 v_0}{c^2} \delta S \langle \mu A(\mu) \rangle ,
\]

where \( A(\mu) \) is a complex function first derived by OR II (cf. their eq. [35]),

\[
A(\mu) = \tau(\mu, r) \int_{-\infty}^{\infty} d\bar{\chi} \phi(\bar{z}) \int_{-\infty}^{\infty} d\bar{\chi} \phi(\bar{z}) e^{i \mu \kappa (\bar{z} - \bar{z}) - i \tau(\mu, r) \Phi(\bar{z}) - \Phi(\bar{z})}.
\]

Since \( \tau(\mu, r) \) and \( \kappa \) are even functions of \( \mu \), the real (imaginary) part of \( A(\mu) \) is an even (odd) function of \( \mu \). Thus, \( \langle \mu A \rangle \) is purely imaginary, while \( \langle A \rangle \) is purely real.

In their fully self-consistent solution of the perturbation equation for scattering-line transfer, OR II showed that \( \delta S \) itself depends on \( \langle A \rangle \). (See their eq. [38].) Using the present escape-probability notation, this can be written as

\[
\delta S = \frac{\langle [I^e(\mu) - S_{\text{sob}}] \delta b(\mu) \rangle}{1 - \langle A(\mu) \rangle}.
\]

Alternatively, if, in place of this self-consistent perturbed transfer solution, we apply the EISF method, then we find from equation (30) that the perturbed source function can be approximated by

\[
\delta S \approx \frac{\langle [I^e(\mu) - S_{\text{sob}}] \delta b(\mu) \rangle}{\langle b_{\text{sob}}(\mu) \rangle}.
\]

Comparison shows that in the EISF method the complicated perturbation integral \( 1 - \langle A \rangle \) is simply approximated by the angle-averaged escape probability \( \langle b_{\text{sob}} \rangle \), as given by equation (16). For an optically thick line, it is shown in the Appendix that \( 1 - \langle A \rangle \) scales roughly as

\[
1 - \langle A \rangle \approx \langle b_{\text{sob}} \rangle + \min \left(k/\kappa \rho, 1\right).
\]

Since \( \langle b_{\text{sob}} \rangle \approx 1/\kappa \rho L \), we see that the EISF approach, which essentially drops the second term in (51), is strictly justified only for long-wavelength perturbations with \( kL \ll 1 \), since then \( \min \left(k/\kappa \rho, 1\right) = \min \left(kL/\tau, 1\right) = kL/\tau \\approx \langle b_{\text{sob}} \rangle \). For short-wavelength perturbations \( k \gg 1 \), it leads to an overestimate of \( \delta S \) and so tends to exaggerate the associated perturbed force. Nonetheless, the EISF approximation does generally preserve the correct sense of effects associated with such source function perturbations.

As discussed by OR II (see their Figs. 3 and 4; see also chap. 2 of Puls 1994, and § 6.2 below), one principal effect is a reversal in the direction of phase propagation for short-wavelength perturbations \( kL > 1 \). Since \( \langle A \rangle \) is real, the phase of \( \delta S \) is set by \( \delta b \), which, for \( kL \gg 1 \), is nearly in phase with \( \delta u \), as follows from equation (35). However, since \( \langle A \mu \rangle \) is purely imaginary (and negative), we then see from equation (47) that \( \delta g_0/\delta u \) must likewise be negative imaginary. This counteracts the positive imaginary component of the direct force \( \delta d \), (which, in this limit \( kL \gg 1 \), becomes vanishingly small anyway; cf. equation [37]), causing the perturbations now to become outward- rather than inward-propagating.

A potentially important consequence of this lies in the associated shift in the relative phase of the perturbed velocity and density. In the pure-absorption or SSF cases, the inward-propagating nature of the unstable waves means that the velocity and density are anticorrelated. As such, the nonlinear wind structure that forms tends to be dominated by high-speed, low-density rarefactions, and low-speed, high-density clumps (e.g., see OCR; Owocki 1994; Feldmeier 1994, 1995). However, when one takes account of the phase reversal arising from \( \delta S \) terms, these unstable, short-wavelength perturbations now have a perturbed density \( \delta n \) that varies in phase with the perturbed velocity \( \delta u \). (See § 6.2.) In principal, this could have important implications for the basic nature of the resulting nonlinear wind structure. A follow-up paper (Owocki & Puls 1996) will examine this question and the influence of the EISF overestimate of the perturbed source function.

5. SUMMATION OVER A LINE ENSEMBLE

Let us next consider the cumulative force from a large ensemble of lines. Following CAK (see also Abbott 1982), we assume here these lines have a flux-weighted number distribution given by

\[
\frac{dN(\kappa)}{d\kappa} = \frac{1}{\kappa_0} \left( \frac{\kappa}{\kappa_0} \right)^{1/2} e^{-\kappa/\kappa_{\text{max}}},
\]

Here \( \kappa_0 \) is related to the usual CAK constant \( k \) through \( k = \Gamma(\alpha)(\nu_0 c/\kappa_0)^{1/\alpha} \), with \( \alpha \) the CAK exponent and \( \kappa_0 \) the electron scattering coefficient. The cutoff at \( \kappa_{\text{max}} \) is introduced to allow us to limit the maximum line strength (OCR). In accordance with CAK, we further assume the line distribution in frequency is independent of the distribution in line strength, with

\[
\frac{dN(\nu)}{d\nu} = d\nu/\nu.
\]

If we ignore line-overlap effects, the cumulative force is then just given by the integral sum

\[
g(\kappa) = \int_{\nu_0}^{\infty} d\nu \int_{\nu_0}^{\infty} d\kappa \frac{dN(\kappa)}{d\kappa} g_0(\kappa, r),
\]

\[
\int_{\nu_0}^{\infty} d\nu \int_{\nu_0}^{\infty} d\kappa \frac{dN(\kappa)}{d\kappa} g_0(\kappa, r).
\]
where \( g_{\kappa,r} \) represents the opacity and frequency dependence of the corresponding single-line force. Since the frequency dependence is only via the term \( (v_o I_\kappa) \), we can perform the frequency integration separately to obtain
\[
g_{\Delta} = \int_0^\infty \frac{dN}{dk} \frac{dN}{dk} g_{\Delta}(r),
\]
where \( g_{\Delta} \) is now the single line force with \( (v_o I_\kappa) \) replaced by the frequency-integrated stellar intensity, \( I_{\kappa}^{\text{int}} \).

\[
I_{\kappa}^{\text{int}} = \int_0^\infty dv I_\kappa.
\]

### 5.1. The Ensemble Direct Force and Escape Probability

For the direct component of the line force, the opacity dependence occurs through the factor \( \kappa b_{\Delta}(\mu, r) \), where the \( \kappa \) subscript emphasizes that the single-line escape probability \( b(\mu, r) \) depends on opacity through the optical depth (cf. eqs. [12] and [3]). We thus define an ensemble-summed escape probability, \( b_{\Delta}(\mu, r) \)
\[
b_{\Delta}(\mu, r) = \int_0^\infty dk \frac{dN}{dk} \frac{\kappa}{\kappa_0} b_{\Delta}(\mu, r)
\]
\[
= \Gamma(\alpha) \int_0^\infty dx \frac{\phi[x - \mu u(r)]}{[\tau_0(\kappa, \rho) + \kappa_0 / \kappa_{\text{max}}]}.
\]
where \( \tau_0 \) is the optical depth of a line with \( \kappa = \kappa_0 \), as given by equation (3), with the \( \kappa \) of individual lines assumed to be spatially constant (and ignoring any differences in the thermal speed for the contributing ions.) In terms of the continuum-integrated intensity \( I_{\kappa}^{\text{int}} \), the direct component of the line force then becomes (cf. eq. [29])
\[
\begin{align*}
  g_{\text{air.}} &= \frac{4\pi k_0 v_{\text{th}} I_{\kappa}^{\text{int}}}{c^2} \langle \mu D(\mu)b_{\Delta}(\mu, r) \rangle \\
  &= g_{\text{thin.}} \langle \mu D(\mu)b_{\Delta}(\mu, r) \rangle \langle \mu D(\mu) \rangle,
\end{align*}
\]
where in general the ensemble escape probability (eq. [57]) is computed using the integral optical depth (eq. [3]).

### 5.2. The Ensemble Sobolev Force and Escape Probability

For the Sobolev case of § 3.1, the ensemble escape reduces to the purely local form
\[
b_{\text{Sob.}}(\mu, r) = \frac{\Gamma(\alpha)}{(1 - \alpha) \tau_0} \left[ \frac{1 + 1}{\tau_{\text{max}}} \right]^{1 - \alpha} \left[ \frac{1}{\tau_{\text{max}}} \right]^{1 - \alpha},
\]
where \( \tau_0 \) and \( \tau_{\text{max}} \) are the local Sobolev optical depths (eq. [15]) for lines with \( \kappa = \kappa_0 \) and \( \kappa = \kappa_{\text{max}} \), respectively. For the usual case of an optically thick opacity cutoff \( (\tau_{\text{max}} \gg 1) \), application of equation (59) in (58) yields the CAK point-star \( (\mu_{\kappa} \to 1) \) force,
\[
g_{\text{CAK}} = \frac{L_{\kappa}^{\text{int}} v_{\text{th}}}{4\pi r^2 c^2} \Gamma(\alpha) \frac{1}{1 - \alpha} \left( \frac{1}{\kappa_0 \rho} \right)^{1/\alpha} \left( \frac{R_\kappa}{r} \right)^{\alpha}.
\]

which applies far from the stellar surface. (Here \( L_{\kappa}^{\text{int}} \) is the frequency-integrated stellar luminosity.) Nearer the star (i.e., where \( \mu_{\kappa} < 1 \)), this expression is modified by a finite-disk correction factor (Friend & Abbott 1986; Pauldrach, Puls, & Küdritzki 1986),
\[
F(\alpha) = \frac{1 - (1 - \beta)^{1 + \alpha}}{\beta(1 + \alpha)} ; \quad \beta = \left( 1 - \frac{u}{r} \right) \left( \frac{R_\kappa}{r} \right)^2.
\]

### 5.3. The Ensemble Diffuse Force for a Smooth Source Function

For the diffuse component of the line force, the summation over the line ensemble is complicated by the fact that, in general, the source function itself depends on the opacity. However, in many cases this dependence is relatively weak and so is of limited importance. For example, the Sobolev source function (20) becomes opacity independent in both the optically thin and thick limits (cf. eqs. [21] and [22]), with the limiting values being identical in the case of isotropic expansion \( \sigma = 0 \) and differing only by a factor of order unity otherwise. In generalizing the SSF approach of § 3.2 to an ensemble of lines, let us thus assume an opacity-independent form for \( S(r) \), e.g., the optically thin expression, equation (22). As \( S(r) \) is proportional to \( I_{\kappa} \), we
can integrate over the line distribution in frequency in the same way as was done for equation (54). Defining \( S^{\text{out}}(r) = \int S(r)dv = P^\text{in}_{\text{w}}S(r)/I_\lambda \), the SSF form of the ensemble diffuse force is thus (cf. eq. [28])

\[
g_{\text{diff},a}(r) = -\frac{4\pi\kappa_0 v_{\text{th}} S^{\text{out}}(r)}{c^2} \langle \mu b_a(\mu, r) \rangle .
\]  

(62)

5.4. The EISF Force for an Ensemble Source Function

Let us now define an ensemble form of the EISF method that allows us to retain the spatially varying character of the scattering source function. As in the SSF case above, we again require that the source function itself be independent of \( \kappa \), thus allowing the force to be computed in terms of an ensemble-summed escape probability \( b_\lambda \). In analogy with the single-line expression (30), we define the ensemble source function as

\[
S_\lambda(r) = \frac{\langle b_\lambda(\mu, r) f^{\text{out}}(\mu) \rangle}{\langle b_\lambda(\mu, r) \rangle}.
\]  

(63)

Since this is independent of opacity, the associated diffuse force is computed from a line-ensemble form of the formal solution for the diffuse intensity,

\[
g_{\text{diff},a}(r) = \frac{4\pi\kappa_0 v_{\text{th}} 2\Gamma(\alpha)}{c^2} \left\langle \mu \int_{-\infty}^{\infty} dx\phi[x - \mu u(r)] \int_0^{\gamma(x, p, \mu, z)} \frac{S_\lambda(r)dx}{[\Gamma_0(x, p, \mu, z) + \kappa_0/\kappa_{\text{max}}]^{1+z}} \right\rangle.
\]  

(64)

Without an opacity cutoff (i.e., for \( \kappa_0/\kappa_{\text{max}} \rightarrow 0 \)), the formal solution integral here formally diverges as \( z' \rightarrow z \); however, the flux-moment integration over angle ensures that the associated ensemble force always remains finite. As noted in § 3.3, evaluation now requires a total of four nested integrations, one each over frequency and angle, plus two over space to compute the optical depth and formal solution. As compared to the SSF approach, for which the analogous diffuse intensity integral is done analytically, this EISF method is thus more an order of magnitude more costly in computer time. Note that if we ignore the spatial variations of the ensemble source function and so let \( S_\lambda(r) \Rightarrow S^{\text{out}}(r) \), then the EISF force (64) reduces to the SSF form (62).

5.5. Outward/Inward Ray Formulation of the Flux-Moment Angle Quadrature

In practice, the flux-moment-angle integrations required in the general ensemble line-force expressions can be most readily carried out in terms of a ray parameter \( y \equiv (p/R_\star)^2 \). The direct force expression (58) is thereby written as

\[
g_{\text{diff},a}(r) = g_{\text{thin},a} \int_0^1 dy \mu_b(\mu, r)\]

\[
= g_{\text{thin},a} \Gamma(\alpha) \int_0^1 dy \int_{-\infty}^{\infty} dx\phi[x - \mu u(r)] I_a^{\text{out}}(x, y, r),
\]  

(65)

where \( \mu_b(r) = [1 - y(R_\star/r)^2]^{1/2} \) is the local direction cosine of the ray at radius \( r \). The outward-streaming (+) optical depth from the star is (cf. eq. [53])

\[
t_+^a(x, y, r) \equiv \frac{\kappa_0}{\kappa_{\text{max}}} + \frac{\kappa_0 \phi(x)}{\kappa_{\text{el}}} + \int_r^{R_\star} \frac{dr'}{\mu_b(r')} \kappa_0 \rho(r') \phi[x - \mu_b(r')u(r')] ,
\]  

(66)

where the boundary condition at \( r \rightarrow R_\star \) incorporates both the opacity cutoff term, plus a Schuster-Schwarzchild electron-scattering reversing layer (OCR; Jeffries 1968). Numerical tests show that just a single ray quadrature with \( y = 0.5 \) yields a direct force that is typically within a few percent of the full, finite-disk integration (Poe, Owociki, & Castor 1990).

For the SSF diffuse force of equation (62), this \( y \)-integration should, in principle, extend beyond the stellar disk to a maximum value \( y_{\text{max}} = (r/R_\star)^2 \). However, in practice, a convenient approximation (cf. Owociki 1991, 1993) is to also limit this diffuse force integration to rays with \( y \leq 1 \), but now taking separate account of both outward (+) and inward (−) streams,

\[
g_{\text{diff},a}(r) \approx g_{\text{thin},a} \Gamma(\alpha) \int_0^1 dy \int_{-\infty}^{\infty} dx\phi[x - \mu u(r)][t_-^a(-x, y, r) - t_+^a(x, y, r)].
\]  

(67)

Here the inward-streaming optical depth is given by

\[
t_-(x, y, r) \equiv \frac{\kappa_0}{\kappa_{\text{max}}} + \int_r^{R_\star} \frac{dr'}{\mu_b(r')} \kappa_0 \rho(r') \phi[x - \mu_b(r')u(r')] ,
\]  

(68)

wherein the definition in terms of \(-x\) has enabled us to use the profile symmetry \([\phi(x) = \phi(-x)]\) to write the frequency dependence of the inward stream in terms of the same comoving frequency \((x - \mu u)\) as for the outward stream. Thus, in this way, once the outward-stream optical depth has been computed over the full model range \( R_\star < r < R_{\text{max}} \), the inward-stream optical depth can be readily obtained, without any further integration, from the relation

\[
t_-(x, y, r) - t_-(-x, y, R_{\text{max}}) = t_+(x, y, R_{\text{max}}) - t_+(x, y, r).
\]  

(69)
Since in practice the model is always limited to some finite maximum radius \( R_{\text{max}} \), the effect of the medium over \( R_{\text{max}} < r < \infty \) must be incorporated through the outer boundary condition \( t_-(-x, y, R_{\text{max}}) \). Following suggestions by J. I. Castor (1991, private communication), we typically set this inward optical depth at \( R_{\text{max}} \) by reflecting the outward optical depth about the local comoving frequency \( x - \mu_{\text{u}}(R_{\text{max}}) \), i.e.,
\[
t_-(-x, y, R_{\text{max}}) = t_+ [2\mu_{\text{u}}(R_{\text{max}}) - x, y, R_{\text{max}}].
\] (70)

This tends to nullify the diffuse force near the outer boundary, since the frequency-integrated contributions of the outward and inward terms in equation (67) become nearly equal there.

The EISF line-ensemble force defined in equation (64) can likewise be formulated in terms of \(+/-\) integrations over a ray quadrature in the coordinate \( y \),
\[
g_{\text{diss},a}(r) = g_{\text{thin},a} \frac{\Gamma(y)}{2(1 + \mu)\tau_{\text{thin}}} \int_0^1 dy \int_{-\infty}^\infty dx \phi(x - \mu_{\text{u}}(r)) [a_-(x, y, r) - a_+(x, y, r)],
\] (71)

where
\[
a_{\pm}(x, y, r) \equiv \alpha \left[ t_{\pm}(x, y, r) \right] \left[ t_{\pm}(x, y, r) + \kappa_0/\kappa_{\text{max}} \right]^{-z}.
\] (72)

Note that if we take \( S_{\pm}(r) \rightarrow S^\text{int}(r) \), then we obtain \( a_{\pm} = S^\text{int}(r)[t_{\pm}^z + (\kappa_0/\kappa_{\text{max}})^{-z}] \). Applying this in equation (71), we again see that the EISF force reduces in this case to the SSO form (67).

6. PERTURBATION ANALYSIS OF ENSEMBLE FORMS

6.1. Perturbed Ensemble Force

The perturbation analysis for the ensemble line force is quite analogous to that done in § 4 for a single, optically thick line. Following the OR 1 results for the perturbed component of the ensemble line-absorption force (see their eq. [51]), we can write a "bridging law" for the perturbed ensemble-escape probability (cf. eq. [35]),
\[
\delta b_{\Sigma}(\mu, r) \approx 2x_+ \frac{\mu_{\text{u}}}{\mu_{\text{u}} + 2x_+/L_{\mu}} \mu \delta u.
\] (73)

Here \( x_+ \) is an escape-probability weighted average of the blue-edge absorption frequencies in the line ensemble (cf. Gayley & Owocki 1994),
\[
x_+ = \int_0^\infty d\kappa \frac{dN}{d\kappa} \frac{\kappa_{\text{u}}(\mu, r)}{\kappa_0} \mu_{\text{u}}
\approx (1 - z) \int_{-\infty}^\infty d\phi(x) x \Phi(x)^{-z} ; \quad \tau_{\text{max}} \gg 1.
\] (74)

The perturbed direct force for the ensemble is then obtained by (cf. eqs. [37] and [58]),
\[
\delta g_{\text{diss},a}(r) = \frac{4\pi\kappa_0\nu_{\text{th}}}{c^2} \left\langle \mu T^\text{int}(a_{\Sigma}(\mu, r)) \right\rangle

= \frac{4\pi\kappa_0\nu_{\text{th}}}{(1 - z)c^2} \left\langle \mu T^\text{int}(\mu) \right\rangle \frac{2x_+}{\tau_{\text{u}}(\mu, r)} \frac{\mu_{\text{u}}}{\mu_{\text{u}} + 2x_+/L_{\mu}} \delta u.
\] (75)

This reduces to limiting forms identical to equations (38) and (39), except that now \( g_{\text{Sob}} \Rightarrow g_{\text{cak}} \), and \( x_{\Sigma} \Rightarrow x_+ \).

The perturbed diffuse force from the ensemble is likewise similar to the analogous single-line case, with a drag component derived from the SSO form (cf. eq. [40]),
\[
\delta g_{\text{diss},a}(r) = \frac{4\pi\nu_{\text{th}}\kappa_0}{c^2} S^\text{int}(\mu) \left\langle \mu \delta b_{\Sigma}(\mu, r) \right\rangle,
\] (76)

where \( S^\text{int}(r) \) is the assumed opacity-independent "smooth source function" in the unperturbed flow.

This drag effect is also obtained in the EISF approach, with the modification that \( S^\text{int}(r) \Rightarrow S_\Sigma\text{Sob}(r) \) in equation (76). In addition, however, perturbations in the ensemble escape probability now also lead to perturbations in the source function itself (cf. eq. [50]),
\[
\delta S_\Sigma = \frac{\langle I^{\text{int}(\mu)} - S_{\Sigma\text{Sob}} \rangle \delta b_{\Sigma}(\mu, r)}{\langle b_{\Sigma\text{Sob}} \rangle}.
\] (77)

The associated component of the perturbed force can then be written in a form similar to the analogous single-line equation (47),
\[
\delta g_{\text{diss},a} = \frac{4\pi\kappa_0\nu_{\text{th}}}{c^2} \delta S_\Sigma \langle \mu A_{\Sigma}(\mu) \rangle.
\] (78)
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where now

\[ A_s(\mu) \equiv \frac{\Gamma(1 + \alpha)}{\tau_0(r, \mu, r)^2} \int_{-\infty}^{\infty} d\xi \phi(\xi) \int_{-\infty}^{\infty} d\xi' \phi(\xi') \frac{e^{i\mu \kappa_s(\xi - \xi')}}{[\Phi(\xi') - \Phi(\xi') + 1/\tau_{\max}]^{1+\alpha}.} \tag{79} \]

Like its single-line analog (cf. eq. [48]), \( \langle \mu A_s \rangle \) is negative imaginary. As detailed in the next subsection (see also § 4.3), this leads to a reversal in the direction of phase propagation for short-wavelength perturbations with \( kL \gg 1 \). Defining the negative real quantity \( \eta_s \equiv \langle \mu \Im (A_s) \rangle \langle b_{s, \text{Sob}} \rangle \), we find by collecting equations (75)–(79) that the total perturbed force in the ensemble EISF model is

\[ \delta g_s = \frac{4\pi \kappa_0 \nu_h}{c^2} \langle [f^\text{im}(\mu) - S_{s, \text{Sob}}](\mu + \eta_s) \delta b_s \rangle. \tag{80} \]

6.2. Phase Propagation Reversal by Perturbed Scattering Gradients

Let us now examine explicitly how the phase between velocity and density variations is affected by the source function gradient terms in the diffuse line force. We begin with the linearized equations for conservation of mass and momentum for the case of a sinusoidal perturbation ( \( \sim e^{i(kr - \omega t)} \)) with wavenumber \( k \) and frequency \( \omega \),

\[ -i\omega \delta \rho + ik \rho \delta v = 0, \tag{81} \]

and

\[ -i\omega \delta \omega = \delta g. \tag{82} \]

Here \( \rho, v, \) and \( g \) represent the density, radial velocity, and radiative driving force, with \( \delta \) forms denoting the first-order perturbation values of these. For simplicity, we have ignored pressure terms in the momentum balance, since these play no essential role in the phase shift effects of interest here and are in any case generally small compared the radiative force term in these highly supersonic, radiatively driven outflows. Equation (82) thus implies \( \omega = i \delta g/\delta v \), showing immediately that the growth rate of the instability, represented by the imaginary part of \( \omega \), is given simply by the real part of \( \delta g/\delta v \).

The correlation between velocity and density variations is given by their relative phase angle \( \phi_{rp} \),

\[ C_{rp} = \cos \phi_{rp} = \frac{\Re (\delta \nu/\delta \rho)}{|\delta \nu/\delta \rho|} = \frac{\Re (\omega/k)}{|\omega/k|}, \tag{83} \]

where the last equality follows directly from the perturbed mass conservation equation (81). This shows that the sign of the phase propagation speed \( \Re (\omega/k) \) quite generally sets the sign of the velocity-density correlation \( C_{rp} \). For the usual convention that \( k \) is real, equations (82) and (83) then imply

\[ C_{rp} = -\frac{\Im (\delta g/k \delta v)}{|\delta g/k \delta v|}. \tag{84} \]

For the simple OR I bridging law (cf. eq. [38]), we then find

\[ C_{rp} = \frac{-1}{\sqrt{1 + (kL/2x_s)^2}}. \tag{85} \]

The associated radiative wave mode is thus always inward propagating, with velocity and density anticorrelated.

Including the force associated with the gradient of the perturbed source function can reverse this phase anticorrelation. This is easiest to illustrate analytically if we approximate the required angle integrations using a simple two-stream radiative transfer model, with \( \mu = \pm 1 \). The total perturbed line-ensemble force (cf. eq. [80]) can then be written

\[ \delta g_{s, \text{eak}} = \frac{\delta b_{s, \text{Sob}}}{b_{s, \text{Sob}}} = \frac{\delta b_{s, \text{Sob}}^+ - \delta b_{s, \text{Sob}}^-}{2b_{s, \text{Sob}}} [1 + i \eta_s], \tag{86} \]

where \( s \equiv 1/(1 + \mu_s) \), \( \delta b_{s, \text{Sob}}^+ \equiv \delta b_{s, \text{Sob}}(\mu = \pm 1, r) \) (cf. eq. [73]), \( b_{s, \text{Sob}} = b_{s, \text{Sob}}(\mu = 1, r) \), and \( \eta_s = \Im [A_s(\mu = 1)]/b_{s, \text{Sob}} \).

Numerical evaluation of the frequency integrals in equation (79) shows that, for the usual case \( \tau_{\max} \gg 1 \), the wavenumber dependence of \( \eta_s \) can be fitted roughly by

\[ \eta_s \approx -\frac{kL}{(1 + kL)^{1+\alpha}}. \tag{87} \]

Application in equation (86) then leads to a scattering-modified bridging law,

\[ \frac{\delta \rho_s}{\delta u} \approx g_{s, \text{eak}} \frac{kL}{1 + (kL/2x_s)^2} \left\{ (1 - s) \frac{kL}{2x_s} \right\} + \left[ 1 - s \frac{\alpha(kL)^2}{2x_s(1 + kL)^{1+\alpha}} \right]. \tag{88} \]

Note that setting \( s = 0 \) recovers the standard, pure-absorption bridging law (cf. eq. [38]). The first and second terms with the factor \( s \) represent respectively the scattering line drag and the gradients in the perturbed scattered radiation. For pertur-
bations with wavenumbers $k$ above a critical value,

$$k_c \approx \frac{1}{L} \left[ \frac{2x_c(1 + \mu_k)}{\alpha} \right]^{1/(1 + \alpha)}, \quad (89)$$

the gradient term causes $\text{Im} (\delta g/k \delta v)$ to reverse sign, implying (from eqs. [83] and [84]) that such perturbations should now have an \textit{outward} phase propagation, with a \textit{positive} correlation between velocity and density. This thus demonstrates explicitly that the phase reversal effects alluded to in § 4.3 (also in OR II and Puls 1994) are indeed also incorporated in this ensemble EISF form for the diffuse force. The ensemble EISF force method thus forms a good basis for dynamical simulations aimed at determining whether such phase reversal effects might lead to a wind structure that is qualitatively different from earlier pure-absorption or SSF models.

7. CONCLUDING SUMMARY

We have developed an integral escape-probability formalism for approximating the line-driving force in a structured, radiatively driven stellar wind. This represents a significant generalization of local CAK/Sobolev methods for computing the line force. It provides a common framework for nonlocal, integral methods, including the previous pure-absorption and SSF approximations, as well as a new EISF approach that takes into account the dynamical effect of gradients in the diffuse radiation field. The EISF expression for the diffuse force thus incorporates both essential aspects of the response to small perturbations, namely line drag and phase reversal. Together with the corresponding integral expression for the direct force, this forms the most complete basis yet for simulating the nonlinear structure arising from the strong line-driven instability of such winds. Results will be given in our follow-up paper (Owocki & Puls 1996).

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APPENDIX A

APPROXIMATE VALUES FOR $\langle A(\mu) \rangle$ AND $\langle \mu A(\mu) \rangle$

To estimate the accuracy of EISF method in the single-line case, let us analyze the behavior of $1 - \langle A(\mu) \rangle$, where $A(\mu)$ is the function defined in equation (48). In this Appendix, we derive some useful relations for the optically thick case $\tau(\mu, r) \gg 1$. We consider the (purely) imaginary part of $\langle \mu A(\mu) \rangle$, since this in connection with $1 - \langle A(\mu) \rangle$ controls the behavior of the perturbed force (cf. eq. [47]). With respect to the resulting reversal of the phase propagation, the quantity first defined by OR II,

$$\eta = \frac{-i\langle \mu A(\mu) \rangle}{1 - \langle A(\mu) \rangle}, \quad (90)$$

plays the crucial role (see especially OR II, eq. [43]).

First, let us rewrite $A(\mu)$ as

$$A(\mu) = \tau(\mu, r) \int_{-\infty}^{\infty} d\tilde{x} \phi(\tilde{x}) \int_{-\infty}^{\infty} d\tilde{x} \phi(\tilde{x}) e^{-\tau(\mu, r)[\Phi(\tilde{x}) - \Phi(\tilde{x})]}$$

$$- \tau(\mu, r) \int_{-\infty}^{\infty} d\tilde{x} \phi(\tilde{x}) \int_{-\infty}^{\infty} d\tilde{x} \phi(\tilde{x}) e^{-\tau(\mu, r)[\Phi(\tilde{x}) - \Phi(\tilde{x})]} [1 - e^{-\tau(\mu, r)[\beta(\tilde{x} - \tilde{x})]}], \quad (91)$$

where we have defined

$$i\mu K_\mu = i\mu \tau(\mu, r)K = \tau(\mu, r)\beta,$$

$$\beta = i\mu K,$$

$$K = \frac{k}{\kappa \rho}. \quad (92)$$

The integrals in the first term of (90) are elementary, while those in the second just represent the well-discussed interaction function $U(\tau, \beta)$ introduced by Hummer & Rybicki (1985; see also Puls & Hummer 1987)

$$A(\mu) = 1 - \frac{1 - e^{-\tau(\mu, r)}}{\tau(\mu, r)} - U[\tau(\mu, r), \beta]. \quad (93)$$

In the case of optically thick lines, the $U$-function becomes independent of $\tau$ (see, e.g., Puls & Hummer 1987, eq. [33a])

$$U(\tau \gg 1, \beta) \approx \beta \int_{-\infty}^{\infty} d\tilde{x} \frac{\phi(\tilde{x})}{\beta + \phi(\tilde{x})}. \quad (94)$$
and so, after resubstituting for $\beta$ and performing the angle averages, we obtain

$$1 - \text{Re} \langle A(\mu) \rangle = \langle b_{sob} \rangle + \int_{-\infty}^{\infty} d\tilde{x} \phi(\tilde{x}) \left[ 1 - \frac{\phi(\tilde{x})}{K} \arctan \left( \frac{K}{\phi(\tilde{x})} \right) \right],$$

$$\text{Im} \langle \mu A(\mu) \rangle = -\frac{1}{K} \int_{-\infty}^{\infty} d\tilde{x} \phi(\tilde{x}) \left[ 1 - \frac{\phi(\tilde{x})}{K} \arctan \left( \frac{K}{\phi(\tilde{x})} \right) \right]; \quad \mu(\mu) \gg 1 . \quad (94)$$

So far, equation (93) is (almost) exact in the optically thick case. In order to obtain analytic expressions for the limiting cases, we split the integrals in dependence of the magnitude of $K/\phi$ into parts about a border $x_0$, where for a Doppler profile

$$K > \pi^{-1/2} : \quad x_0 = 0 ,$$

$$K < \pi^{-1/2} : \quad x_0 = \left[ -\ln \left( \pi^{1/2} K \right) \right]^{1/2} . \quad (95)$$

Expanding $\arctan (K/\phi)$ up to 3rd order for small arguments, the integrals can then be approximated by

$$1 - \text{Re} \langle A(\mu) \rangle \approx \langle b_{sob} \rangle + \text{erfc} \left( x_0 \right) + \frac{2}{3} K^2 \pi^{1/2} \int_{0}^{x_0} d\tilde{x} \phi(\tilde{x}) ,$$

$$\text{Im} \langle \mu A(\mu) \rangle \approx -\frac{1}{(2\pi K^{1/2})} \text{erfc} \left( 2^{1/2} x_0 \right) - \frac{2}{3} K x_0 . \quad (96)$$

Hence, for large $K \pi^{1/2} > 1$, i.e., $k/(\kappa \rho) \gtrsim 1$, we have $x_0 = 0$ and consequently

$$1 - \text{Re} \langle A(\mu) \rangle \approx \langle b_{sob} \rangle + 1 ,$$

$$\text{Im} \langle \mu A(\mu) \rangle \approx -\frac{1}{(2\pi K^{1/2})} , \quad (97)$$

while for smaller $K \pi^{1/2} < 1$ and $x_0 \gtrsim 2$ we can expand the complementary error functions and the integral, \( \int dx \exp (x^2) \approx \exp (x_0^2/(2x_0)) \), and find

$$1 - \text{Re} \langle A(\mu) \rangle \approx \langle b_{sob} \rangle + K \frac{4}{3x_0} ,$$

$$\text{Im} \langle \mu A(\mu) \rangle \approx -K \left( \frac{2x_0}{3} + \frac{1}{2x_0} \right) . \quad (98)$$

Combining both results, we finally obtain the relation (51)

$$1 - \langle A \rangle \approx \langle b_{sob} \rangle + \min \left( k/(\kappa \rho), 1 \right) ,$$

whereby the behavior of $-\eta$ as plotted by OR II (their Fig. 2) is readily understood, since for small (large) $K$, $-\eta$ increases (decreases) with $K$ (note that the denominator remains bounded) and shows a maximum somewhere at $K \approx 1$.

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