MODELING CONVECTION IN THE OUTER LAYERS OF THE SUN: A COMPARISON WITH PREDICTIONS OF THE MIXING-LENGTH APPROXIMATION

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ABSTRACT

The mixing-length theory (MLT) approximation (Vitense 1953) is used in most stellar evolution codes to describe the structure of the outer, highly superadiabatic, layers of the Sun. This procedure is known to be incorrect because of the MLT's inadequacies in describing convection and because of the need to include the strong coupling between radiation and convection in modeling this region. However, it is not known to what extent and precisely in what ways the MLT approximation distorts the structure of the highly superadiabatic peak in the outer convection zone. The purpose of this paper is to compare the statistical results of a more realistic three-dimensional numerical simulation of shallow convection to the predictions of the MLT. The simulations differ from the previous simulations of Chan & Sofia (1989) in that they include a treatment of radiative transfer (in the diffusion approximation). The layers are superadiabatic and exhibit a sharp peak in the temperature gradient. The results we derive from this simulation provide much more information than conventional one-dimensional theories of convective energy transport. We attempt to analyze the information from the simulation to be compared with a traditional “theory” in an effort to establish how much a large eddy simulation can teach us about mean convective transport theories. In this paper we chose to use the mixing-length approximation for comparison.

The standard mixing-length approximation predicts a few linear relationships between local thermodynamic and dynamic quantities, the coefficients of which are functions of the mixing length. In these MLT relations, the ratio of mixing length to the local pressure scale is assumed to be constant over the entire convection zone, including the region of high superadiabaticity where convective energy transfer becomes less efficient.

Subject headings: convection — diffusion — Sun: interior — turbulence

1. INTRODUCTION

Our insufficient knowledge of convection is the source of one of the most important uncertainties in stellar astrophysics. In the study of stellar convection zones, we have to be aware that our knowledge of geofluids can provide only limited insight into astrophysical fluid dynamics. The “fluid” within a star has significant differences from the fluids we are familiar with on Earth. In the interior of most stars the gas is highly compressible and stratified. Stars of spectral type A and cooler have convective envelopes that span a number of pressure scale heights. Because of the relatively inviscid nature of the stellar material and the large length scales, the Reynolds number characterizing the turbulent nature of the motions is large. For the Sun, for example, Re \( \sim 10^{16} \). In convective regions, the Rayleigh number is useful in describing the nature of convection. In the visible layers of the solar convection zone, the Prandtl number, which is the ratio of viscosity to thermal diffusivity, is \( \sim 10^{-5} \), and the Rayleigh number is \( 2 \times 10^{14} \) (Bray, Loughhead, & Durrant 1984). This very low Prandtl number and high Rayleigh number indicate that the solar convection is much more turbulent than any laboratory or terrestrial analog.

With modern supercomputers, it is feasible to solve the fully compressible hydrodynamic equations in three dimensions. These large eddy simulations (LESs) are a powerful numerical tool that can be used as an experimental tool in the absence of the actual experiments. Kim et al. (1995, hereafter Paper I) describes one of these calculations. The numerical simulation was performed for a compressible, radiation-coupled, gravitationally stratified medium, using a realistic equation of state and opacities. The statistical information obtained from the simulation has already started to provide a better understanding of stellar photospheric convection (Paper I).

The main emphasis of this paper is the application to stellar structure and to stellar evolution. It is a step toward improving our understanding of stellar convection and also eventually enabling us to calculate stellar radii from first principles, without recourse to the MLT approximation. The main point of the Kim et al. (1995) numerical simulation was to explore the effect of the coupling between radiation and convection on the properties of convection (shallow convection), particularly in the more superadiabatic layers. The earlier work of Chan & Sofia (1989) was valid for deep convection (nearly adiabatic convection) only and assumed decoupling between the radiation and convection.

The purpose of this paper is to analyze the three-dimensional shallow convection results of Paper I in terms
of one-dimensional parameterization, which is needed to apply it to one-dimensional stellar models. Another purpose is to study the difference between the structure of the highly superadiabatic layer of the Sun produced by the mixing-length theory (MLT) approximation and the numerical simulation. It is ironical that the MLT has been primarily relied upon in stellar structure calculations to describe the superadiabatic peak in the outer layers, because it is the region where it is in principle least applicable. This breakdown of the MLT assumptions is well known. In this paper, we attempt to go beyond merely stating the limitations of the MLT. We attempt to explore in what way, where, and by how much the MLT formalism fails.

On the other hand, in the deep regions, the shortcomings of the MLT when it comes to the details of the convective flows and the thermodynamics of convection have been documented in numerical simulations (Chan & Sofia 1989; Cattaneo et al. 1991). The deep layers, however, are nearly in adiabatic equilibrium, and the details of the treatment of convection are irrelevant to the calculation of the mean temperature gradient, which is the main purpose of this paper.

In § 2, we summarize the numerical simulation of Paper I and place them within the historical context. In § 3, certain approximate relationships between thermodynamic parameters obtained from our simulation are compared with those deduced from deep and efficient convection simulations (Chan & Sofia 1989). In § 4, the statistical properties of the convective energy transport are developed and discussed to investigate whether there are relationships between fluctuating quantities and the mean dynamics and thermodynamic structure, such as those the mixing-length approximation prescribes. In § 5, the results of this study are summarized and discussed.

2. STELLAR CONVECTION AND THE ZONE STRATIFICATION

2.1. The Mixing-Length Approximation

The mixing-length approximation represents an extreme simplification of the actual physical process of convection. The mixing-length approximation was originally developed to describe incompressible, terrestrial convection (Taylor 1915; Prandtl 1925). The basic application of this approximation for convective heat transport in stellar convection zones was developed by Vitense (1953; Böhm-Vitense 1958), and despite many variations, this basic formalism has been used to determine the temperature gradient within the convective layers of a stellar model.

The mixing-length approximation replaces, conceptually, the complicated situation in an actual convection zone consisting of convective elements of assorted sizes, shapes, velocities, and lifetimes, with a group of "average" convective elements, all of which have the same physical properties at any radial distance r from the stellar center. Each convective element is assumed to travel, on the average, through a certain vertical distance, the mixing length, before mixing with the surrounding matter and thereby losing its identity. The characteristic dimension of the convective element is assumed to be equal to the mixing length. In addition, it is assumed that the convective elements at a given r have the same speed, which is supposed to approximate the average speed of the actual convective elements at r. Since the horizontal average of vertical velocities is zero in the prescription of the approximation, the kinetic energy flux of convective elements is zero as well.

The major source of uncertainty in applying the MLT to stellar evolution calculations is that the mixing length itself is not prescribed by the theory. In actual model construction, the mixing length is usually chosen to be proportional to the local pressure (or pressure scale height). Then, the mixing-length ratio, \( \alpha \), is chosen in such a way that calculated solar models have the correct solar radius and luminosity at the solar age. There is no a priori reason why \( \alpha \) should remain the same for stars of all masses, ages, and compositions. In fact, we know that is only approximately true (e.g., see discussions in Demarque, Green, & Guenther 1992; Guenther et al. 1992). Since there is no alternative way to estimate the mixing-length ratio for different cases, however, the same value is usually used for all the other stellar model constructions. The failure of the MLT for low-efficiency convection has been demonstrated by recent studies on helioseismology (see, e.g., Baturin & Miranova 1995).

Even though it is a crude prescription, the simplicity of the MLT makes the approximation popular in stellar structure modeling. Furthermore, its success in isochrone calculations for star clusters has convinced most of the astrophysical community that the approximation is, to first order, a good representation of the gross characteristics of stellar convection.

The fundamental limitation of the approximation is that it has not been designed to depict the truly turbulent nature of the stellar convection. Therefore, it does not prescribe any dynamical properties of convection, including "overshoot" above and below convective zones (see the extensive discussion by Canuto 1993). In practice, it has been used only as a prescription of the temperature gradient in the region with higher superadiabaticity. This is the very region where the validity is least justified and where it needs to be tested.

2.2. Approximate Expressions for the Convective Fluxes

There have been many different approaches to a better convection theory. Within the mixing-length approximation framework, a few alternative prescriptions of convection have been proposed (Latour et al. 1976; Gough 1977; Deupree 1979; Stellingwerf 1982; Kuhfuss 1986; Forestini, Lumer, & Arnold 1991). Each of those prescriptions is developed by taking into account an important feature of convection that is ignored by the mixing-length approximation. Canuto & Mazzitelli (1991, 1992) have also developed an improved convection theory that accounts for the full spectrum of eddy sizes in a turbulent flow and also removes the specification of a mixing-length ratio. Since we are making comparisons to MLT we will defer a description and comparison with their theory to a later paper.

Another approach can be found in the numerical simulations of convection. There have been several research groups who study stellar convection using numerical simulations (Chan & Sofia 1986, 1989; Stein & Nordlund 1989; Nordlund & Dravins 1990; Cattaneo et al. 1991; Ludwig 1994). New and important features of convection have been found from those simulations. Of those studies, Chan & Sofia (1989) went a step further. From numerical simulations, they parameterized the characteristics of the convec-

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tive energy transport as an alternative to the MLT. In the course of their study, they confirmed the validity of some features of the mixing-length approximation in the limit of deep and efficient convection (Chan & Sofia 1987, 1989). Also, Chan & Sofia demonstrated the feasibility of directly relating the mean dynamics to the mean thermodynamic structures of the fluid by simple relations similar to those given by the local mixing-length approximation. Their approximate expressions for the energy fluxes have since been successfully employed for stellar structure modeling (Lydon, Fox, & Sofia 1992, 1993).

One limitation, however, was that their calculation was for deep, efficient convection. Convective instability in most stars is accompanied by a zone of partial ionization of the dominant element, hydrogen. A convective element moving upward through such a region of rapidly decreasing ionization liberates ionization energy, which, in turn, is converted to thermal energy. Thus the buoyancy of the convective elements increases, causing them to continue their upward journey, provided the forces opposing their motion are sufficiently small. The net effect is a substantial departure of the temperature gradient from its adiabatic value. Furthermore, the transition between convective and radiative layers exhibits the same feature. When using the MLT, this is the very region where, for all practical purposes, the mixing-length ratio must be adjusted to fit the observational constraints. Therefore, the validity of any approximation should be tested in the limit of shallow and inefficient convection, where ionization and radiative energy transfer play an important role in the dynamics of convection.

2.3. Simulation for Shallow and Inefficient Convection

In this paper we address some of the questions discussed above by using a numerical model for solar photospheric convection in which the initial condition for the simulation was taken from the upper convective region of a detailed solar structure model (where the mixing-length theory is used). This step provides consistency between the shallow regions, where the photospheric convection is simulated, and the structure in the deeper regions of the star. Paper I studied some statistical properties of the convective motions and compared those results with the study of deep and efficient convection by Chan & Sofia (1989). In particular, radiative energy transport and partial ionization were shown to affect the statistical properties of convection.

Some results of Paper I that are relevant for the topic of this paper can be summarized as follows. First, radiative and convective transport can be decoupled in deep convection. This result confirms one of the assumptions Chan & Sofia used. Second, in the deep region, where \( \nabla \approx \nabla_{ad} \), certain linear relationships, which resemble a mixing-length approximation, do exist between thermodynamic variables. Finally, the simulation shows that the weakening of the convective flows occurs as the radiative transport becomes dominant at the top region of the calculation domain.

3. COMPARISON WITH STUDY OF CHAN & SOFIA (1989)

Using their simulations, Chan & Sofia (1989) tested the validity of the mixing-length approximation. They parameterized their calculations to find a better prescription of convection, and they found a few linear relationships that resemble the mixing-length approximation. These relations have been successfully used as the alternative of the mixing-length approximation in stellar models (Lydon et al. 1992, 1993).

Since Paper I showed how the effect of ionization alters the dynamics of flows, it is important to investigate whether relationships similar to Chan & Sofia's results still exist in our calculations. To compare with their result, the relations between the following parameters were tested:

\[
\begin{align*}
V_z^2 & \quad \text{versus} \quad \frac{Q}{\mu} (\nabla - \nabla_{ad}) , \\
(\nabla - \nabla_{ad}) & \quad \text{versus} \quad \left[ \frac{F}{(C_p/\mu T P)} \right], \\
\frac{T''}{T} & \quad \text{versus} \quad (\nabla - \nabla_{ad}) , \\
V_z^2 & \quad \text{versus} \quad \frac{Q T''}{\mu} , \\
\rho & \quad \text{versus} \quad \frac{Q F_z}{C_p \mu V_z^3} ,
\end{align*}
\]

where \( V_z^2 \) is the root mean square fluctuation from the mean vertical velocity, \( T \) is the temperature, \( Q \) is the expansion coefficient of a convective element at constant pressure \([Q = - [(\partial \ln \rho)/(\partial \ln T)]_p] \), \( \mu \) is the mean molecular weight, \( \nabla \) is the temperature gradient \( \equiv (\partial \ln T)/(\partial \ln P) \), \( \nabla_{ad} \) is the adiabatic gradient, \( C_p \) is the specific heat at constant pressure, \( P \) is the pressure, \( F \) is the total heat flux, \( T'' \) is the root mean square fluctuation from the mean temperature, \( \rho \) is the density, and \( F_z \) is the enthalpy flux.

Before starting the investigation of the relations, recall that the value of \( Q \), as well as the mean molecular weight (which is shown in Fig. 5 of Paper I), is variable in our calculation. The distribution of the temporal and horizontal average is shown in Figure 1.

In Figures 2 through 8, the relations of interest are shown. For example, no linear relation is apparent in Figure 2. If we look at only the region where the \( \nabla - \nabla_{ad} \) values are close to those of Chan & Sofia's simulations, we can find a nearly linear relation, as shown in Figure 3. The existence of tight linear relations in the part of the plot where the \( \nabla - \nabla_{ad} \) values are small, i.e., the deeper region, is

![Figure 1](https://example.com/figure1.png)
Fig. 2. $V_s^2/T$ vs. $Q(V - V_{ad})/\mu$

Fig. 3. $V_s^2/T$ vs. $Q(V - V_{ad})/\mu$. The top and bottom pressure scale height are removed to study the linear part of Fig. 2.

Fig. 4. Superadiabaticity vs. energy flux. The superadiabaticity plotted on the y-axis increases as one moves toward the surface. When $V - V_{ad} \geq 0.01$, i.e., closer to the top, the plot shows deviation from the linear relation in the deeper layers.

Fig. 5. $QT^\gamma$ vs. $\mu V_s^2$. The quantity plotted on the x-axis increases as one moves toward the surface.

Fig. 6. $T'/T$ vs. $V - V_{ad}$. The superadiabaticity plotted on the x-axis increases as one moves toward the surface.

Fig. 7. $\rho$ vs. $[Q(F_e + F_p)]/(C_p \mu V_s^3)$.
consistent with the result of Chan & Sofia (1989). The existence of those linear relationships and their resemblance to equations (3)–(7) do inspire some confidence in the MLT approximation. When \( Q/V = V_{\text{ad}} / \mu \geq 0.03 \) (which means \( V - V_{\text{ad}} \geq 0.01 \)), however, the plot shows deviation from the linear relation even in the deeper layers. This implies that our calculation provides information for exploring the region where the mixing-length approximation starts to break down.

Figures 4 through 6 show clear deviations from the linear relations at the shallower region as well. These deviations can be caused by smaller \( V_r^* \) values and/or by larger \( V - V_{\text{ad}} \) values. Considering the fact that the deviation starts more than two pressure scale heights below the top boundary, the smaller \( V_r^* \) is a result of weakened convective motion. The steadily increasing \( V - V_{\text{ad}} \) must contribute to the deviation also. In any case, it is clear that we need higher order polynomials to fit (e.g., Fig. 2). It is, however, difficult to judge whether the nonlinear relations in that region are an artifact of the top boundary condition, the effect of superadiabaticity, or the effect of radiative energy transport. Considering that the relations at the top are different from the relations at the bottom, it is quite likely that the boundary effects are not the only cause of the deviation.

For the five relations of interest and a few other relationship, least-square fits were calculated, and the coefficients can be found in Table 1.

TABLE 1
LEAST-SQUARES FIT (\( \chi^2 \))

<table>
<thead>
<tr>
<th></th>
<th>Chan &amp; Sofia 1989 (( \mu = 1, Q = 1 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( QT^* \sim 1.87(0.04) \mu V_r^* )</td>
<td>( QT^* \sim 0.90(0.10) V_r^* )</td>
</tr>
<tr>
<td>( F_r \sim 1.86(0.06) C_r \langle P \rangle \langle V_r \rangle )</td>
<td>( F_r \sim 1.25(0.08) C_r \langle P \rangle \langle V_r \rangle )</td>
</tr>
<tr>
<td>( F_r \sim 1.37(0.01) C_r \langle P \rangle V_r^3 )</td>
<td>( F_r \sim 0.72(0.06) C_r \langle P \rangle V_r^3 )</td>
</tr>
<tr>
<td>( \langle V_r^* \rangle \sim -0.89 V_r^* (8.8 \times 10^{-6}) )</td>
<td>( \langle V_r^* \rangle \sim -1.1 V_r^* )</td>
</tr>
<tr>
<td>( \langle T/Hp \rangle \sim 1.42 \Delta V + 0.0035 (4.8 \times 10^{-6}) )</td>
<td>( \langle T/Hp \rangle \sim 1.05 \Delta V + 0.0027 (0.0008) )</td>
</tr>
<tr>
<td>( V_r^* \sim 0.73 Q \Delta V + 0.0057 (3.7 \times 10^{-6}) )</td>
<td>( V_r^* \sim 1.17 \Delta V + 0.0032 (0.0011) )</td>
</tr>
<tr>
<td>( \Delta V \sim 0.84 \left[ \frac{F}{\left[0.8 C_r \langle P \rangle (\mu Q T)^{1/3}\right]} \right]^{2/3} - 0.002 (2.5 \times 10^{-7}) )</td>
<td>( \Delta V \sim 0.9 \left[ \frac{F}{\left[0.8 C_r \langle P \rangle (T)^{1/3}\right]} \right]^{2/3} - 0.0002 (0.0007) )</td>
</tr>
</tbody>
</table>

4. RANGE OF USEFULNESS OF THE MIXING-LENGTH APPROXIMATION IN CONSTRUCTING STELLAR MODELS

In the mixing-length approximation, the convective flux is

\[
F_r = \frac{1}{4\sqrt{2}} C_r Q^{1/2} \rho^{5/2} g^2 T P^{-3/2} \lambda^2 (V - V_r)^{3/2}
\]

(see, e.g., Cox and Giuli 1968, eq. [14.31d]), \( g \) is the local gravity, \( \lambda \) is the mixing length, and \( (V - V_r) \) is the excessive temperature gradient of a convective element. In the usual form of the mixing-length approximation, the mixing length is taken to be the mixing-length ratio times the local pressure scale height; \( \lambda = \alpha \mu \rho \), and the temperature gradient of a convective element is assumed to be \( V_r = V_{\text{ad}} \), where \( V_{\text{ad}} \) is the temperature gradient of an element of matter that moves adiabatically \( \Rightarrow \lambda = (\ln T)(\ln P_{\text{ad}}) \). Note that as the radiative energy transport becomes important, \( V_r \) differs from \( V_{\text{ad}} \). Using hydrostatic equilibrium, which is one of the most important assumptions in the mixing-length approximation, the local pressure scale height is \( H_\rho = P/\rho g \). Thus,
the convective flux can be written in the following form:

\[ F_c = \frac{1}{4\sqrt{2}} \alpha^2 C_p Q^{1/2} T^{1/2} \mu^{1/2} R^{-1/2} (\nabla - \nabla_{ad})^{3/2}, \] (1)

where \( R \) is the gas constant.

Another important equation in the mixing-length approximation gives the mean velocity, \( v_c \), of a convective element,

\[ v_c^2 = \frac{1}{2g} Q(A^2/T) \Delta T \]

(see, e.g., Cox & Giuli 1968, eq. [14.27]). Since the temperature change of a convective element when it moves distance \( \Delta \) is

\[ \Delta T(\Delta) = \Delta \Delta T = \Delta T/H_p (\nabla - \nabla_{ad}) \]

the mean velocity becomes

\[ v_c = \frac{\alpha}{2\sqrt{2}} Q^{1/2} \left( \frac{RT}{\mu} \right)^{1/2} (\nabla - \nabla_{ad})^{1/2} \] (2)

(see, e.g., Cox & Giuli 1968, eq. [14.28b]). From equations (1) and (2), the following equations are derived:

\[ \frac{\Delta T}{T} = \alpha (\nabla - \nabla_{ad}) \] (3)

\[ v_c^2 = \frac{\alpha}{8} Q R \frac{\Delta T}{\mu} \] (4)

\[ \frac{v_c^2}{T} = \frac{\alpha^2}{8} R \frac{Q}{\mu} (\nabla - \nabla_{ad}) \] (5)

\[ F_c = \frac{\alpha^2}{4\sqrt{2}} C_p \left( \frac{\mu Q T}{R} \right)^{1/2} P(\nabla - \nabla_{ad})^{3/2} \] (6)

\[ F_c = \frac{4}{\alpha} C_p \frac{1}{\mu R} \rho \mu v_c^3 \] (7)

The mean velocity of a convective element in the MLT approximation is compared to the root mean square fluctuation velocity, \( V_r \) in our notation. The mean vertical velocity is assumed to be zero. As a result, the mean kinetic energy flux is zero. Therefore, according to the mixing-length approximation, the convective flux is the enthalpy flux.

As a way to test the mixing-length approximation, we study equations (3)–(7). It is important, however, to note that not all of those parameters used in the previous section are compatible with those in the equations. First, the \( T'' \) used in previous section is not \( \Delta T \) in equations (3) and (4). In the Eulerian hydrodynamic simulation and the statistics calculation, it is difficult to estimate the temperature change of a convective (Lagrangian) element as it moves. Second, considering Figure 13 of Paper I, which displays the components of the total heat flux, it is clear that the kinetic energy flux is nonnegligible and must be included in evaluation of the convective energy flux, \( F_c \), in equations (6) and (7). Finally, near the top of the convection zone, the approximation regarding the temperature gradient of convective elements becomes invalid, i.e., \( \nabla' \neq \nabla_{ad}' \). Thus, equations (5) and (6) should be used with care in the radiation-convection transition region.

Figure 7 and 8, as well as Figure 2, confirm the linear relationships, which the mixing-length approximation pre-scribes, but only in the deeper region. Within the higher superadiabatic region, the linear relationships fail.

Before one accepts this as an indication of nonconstancy of the mixing length, one must have a clear understanding of the associated region. The top part of the calculation domain has three distinct characteristics. First, this is the region where the superadiabaticity becomes large, as shown in Figure 6 of Paper I. The steep temperature gradient results in an elevated diffusive flux (the dot-dot-dash line). Second, this is the region where the radiative diffusion term becomes nonnegligible. In Figure 13 of Paper I, the dot-dash line represents the radiative flux. Finally, this region is close to the top boundary. The rigid top boundary condition affects some fraction of the top pressure scale height in the simulations of Chan & Sofia (1989). In the present simulations, however, the down-turn of the distribution of the root mean square of \( V_r \) occurs in a deeper region. From the comparison between Figure 7 of Paper I and Chan & Sofia's Figure 1, one can see the difference in the location of the down-turn in the distribution of \( V_r \) as well. Therefore, it seems that the top boundary is not the only cause of the decrease of the root mean square of \( V_r \) in the simulations.

When radiative cooling becomes dominant, the convection becomes weaker. As a result, the root mean square of \( V_r \) is decreased; this is the region where the mixing-length approximation cannot accurately describe the convection.

Horizontally and temporally averaged values were used to calculate the mixing-length ratios, \( \alpha \), in equations (5), (6), and (7), at each vertical grid, shown in Figure 9. At the bottom, the boundary condition forces the \( \nabla - \nabla_{ad} \) to pass through zero while \( p'' \) and \( T'' \) are nonzero. This produces a meaningless result. Therefore, the mixing-length ratios calculated near the bottom were not included in the figure. Near the top, the efficiency of convective energy transport is reduced as the radiative transport becomes important; as a result, a smaller mixing-length ratio is seen at the left-hand side of Figure 9. Note, however, that this can be, in part, due to the fact that \( P' \) is not \( P_{ad}' \). This figure shows that even well inside of the domain, the mixing-length ratio varies at different depths. Furthermore, it seems that different relationships give somewhat different mixing-length ratios in a single convection zone. Remember that the mixing-length ratio can be considered to be an indicator of efficiency of the convective energy transport. This explains

![Figure 9](image_url)

FIG. 9.—The depth dependency of the mixing-length approximation. The open circle, plus, and asterisk are from eqs. (5), (6), and (7), respectively.

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why, in conventional stellar structure and evolution calculations, the value of $z$ has to be adjusted when the details of the physics in the convection zone (e.g., opacities and equation of state) are changed. Considering that the traditional value of $z$ has been between 1 and 2, the figure seems to indicate that the mixing-length approximation underestimates the contribution of radiative energy transport in shallow regions and overestimates it in the deeper region of the convection zone. This implies that a constant mixing-length ratio, especially in the highly superadiabatic region, is not a good approximation and that one needs a new parameterization that avoids the need of the linearity of a mixing-length ratio.

In summary, the linear relations predicted by the mixing-length approximation can be found in the detailed hydrodynamic simulation only at the deeper parts of the layer. Furthermore, the coefficients found in the linear relations do not prescribe a single $z$, as the mixing-length approximation does.

5. DISCUSSION

We have analyzed the three-dimensional numerical hydrodynamic simulation of Paper I and derived parameterization suitable for use in stellar structure and evolution calculations. This was done in order to explore what can be learned about "mean convective transport theories" from large eddy numerical simulations. Another objective was to ascertain the range of applicability of the MLT approximation, which, despite its conceptual shortcomings, has proved very useful in many astrophysical contexts. First, the existence of certain linear relations, found by Chan & Sofia's deep and efficient convection simulations, were investigated using the statistical properties of the simulated flow. The similar linear relations seem to exist only where $V - V_{\text{ad}} \leq 0.01$. Second, the existence of certain linear relations, predicted by the mixing-length approximation, were tested. Our calculation suggests that the "effective" mixing-length ratio $\alpha$ varies within a convection layer. The closer to the surface layer the mixing-length ratio is, the smaller it is (Fig. 9). Because of the uncertainty associated with the top of the domain in our calculation, we were not able to deduce any quantitative description for the mean convective transport at the highly superadiabatic region. Note, however, that the departure from a constant mixing-length ratio starts well inside of our domain. Therefore, these results have to be viewed as another incentive to develop for a realistic "mean convective transport theory" for the highly superadiabatic region.

These results also have to be viewed from two different directions. From the point of view of the thermodynamics of the convection zone, the details of the numerical modeling are essential, and differences with the mixing-length approximation are significant, even in the deep layers where $V - V_{\text{ad}} \leq 0.01$. Contrary to the mixing-length approximation, the kinetic energy flux in the convection zone is not negligible. This result is consistent with the findings of Chan & Sofia (1989) and Cattaneo et al. (1991). Therefore, the approximation underestimates energy transfer by convective elements. As a result, the temperature gradient, $V$, that the approximation prescribes may not be appropriate for a convection zone. The solid line in Figure 10 shows the temperature gradient that the mixing-length approximation prescribes for the region where we simulated the convection. The dashed line in this figure is the gradient of the region found in our simulation. One caution for Figure 10 is that the initial stratification is from a one-dimensional solar model, in which the energy flux at each layer varies depending on the distance from the center. According to the initial model, the flux at the top is $0.996^2$ times the flux at the bottom of the simulation layer. Our simulation, however, is a three-dimensional calculation in Cartesian coordinates where the energy flux is constant throughout the computational domain.

What are the implications of this conclusion for stellar physics? From the point of view of describing the dynamics and thermodynamics of the convection zone deep layers, the linear fits of Table 1 should provide a more realistic description than the mixing-length approximation. In those applications that are sensitive to the velocity field at a particular point in the convection zone—for example, convective overturn timescales and the calculation of Rossby numbers, or estimates of the interaction between convective motions and oscillation modes—a realistic description of convective motions is likely to result in a significant improvement.

From the point of view of the temperature gradient in stars and its structural effects (and radius determination), the deep layers are practically adiabatic. In this respect, there is little difference between the mixing-length approximation and detailed modeling in the deep layers. This research, as well as Chan & Sofia's, confirms that deep and efficient convection can be described with a simple linear relationship as the mixing-length approximation does. The simulations also demonstrate the importance of understanding the structure of the superadiabatic region. Conventional wisdom has been that, in practice, the mixing-length approximation yields an adequate representation of the superadiabatic region. We point out that the success of the approximation does not mean that the approximation is a good prescription for the superadiabatic region. Rather, this success is due to the fact that the region is usually very thin. For practical purposes, therefore, assuming that the effective mixing-length ratio ($\alpha_{\text{eff}}$) is a constant in this layer, has been sufficient. In the other parts of the convection zone, where $V - V_{\text{ad}} \leq 0.01$, the calculated structures are less sensitive to what mixing-length ratio, $\alpha$, is used. Remember that throughout most of a convective zone,
\( \mathbf{v} \approx \mathbf{v}_{\text{ad}}. \) In such regions, the choice of \( \alpha \) does not make any difference in the model structure of the region.

From a practical point of view, then, in stellar structure modeling, the region with higher superadiabaticity is where a prescription of convection is needed most. Getting the structure of the superadiabatic layers and the shallow layers correct is very important for the study of other Sun-like stars and for red giants that have deeper (thicker) superadiabatic layers. Theoretically, helioseismology can provide constraints for distinguishing among convection theories. The sensitivity of the global \( p \)-modes to details of the highly superadiabatic region near the solar surface is an important constraint. Thus, once solar structure models are constructed using improved expressions for the convective fluxes, the improved expression of "mean convective transport theory" can be tested. Helioseismology promises to provide for the first time the means to test solar models in this subphotospheric region in detail (Baturin & Miranova 1995; Paterno et al. 1993; Guenther 1994). For such a comparison, however, observation of very high \( l \) \( p \)-mode frequencies is required.

Before any attempt to obtain such a prescription is made, the present simulations must be improved further. In particular, the region of most interest—the highly superadiabatic layer near the solar surface—is close to the top of the simulation domain and also the point where the diffusion approximation for radiative transport is known to break down. Thus a study of the details of the highly superadiabatic region and the development of an alternative prescription for convective energy transport in these layers must await improvements in the surface boundary conditions and radiative transport effects. Simulations that include both of these improvements are currently underway.

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