Derivation of Accelerated Electron Spectra by Inversion of Bremsstrahlung Spectra from a Thick Target of Nonuniform Ionisation

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Abstract. The problem of inferring flare electron spectra at the acceleration site from their HXR bremsstrahlung spectra is considered for the case when they are injected into a thick target in which the plasma ionisation z varies with depth. The correction formulae derived will be useful and convenient for application to future high resolution HXR spectra.

1. Introduction

It is well known that HXR bremsstrahlung are an important diagnostic of flare electrons and hence of the acceleration processes associated with reconnection. In the case of a thick target, the HXR spectrum is related to the electron injection spectrum by a double integral with a kernel function involving the bremsstrahlung cross-section and the electron energy loss function in the target (Brown 1971, Brown and MacKinnon 1985). The inversion problems to recover the electron spectra from the HXR spectra were solved analytically by Brown(1971 etc) for the Bethe-Heitler and Kramer's bremsstrahlung cross-sections and for Coulomb energy losses in a fully ionised plasma. With the advent of high resolution Germanium spectrometry (Lin et al. 1987) it has proved possible to apply these ideas numerically to real data, as has been done by Thompson et al. (1992), Johns and Lin (1992) and Piana (1994) for various cross-sections, again for a fully ionised plasma.

Since the ionisation profile is a factor in the kernel of the integral equation and since inversion of such equations is often unstable to small changes in the kernel, as well as in the data (Craig and Brown 1986), this is an important issue to address if HESSI and similar instruments are to yield reliable information on flare electrons. In this paper we summarise the findings of a preprint in which this problem is solved analytically using Kramer's cross-section and neglecting directivity effects in the electron and photon distribution functions. Though these assumptions are only roughly satisfied, they are adequate here since we are considering only the relative effect of changing the ionisation profile.

2. Formulation of the Problem

Following Brown(1973) we consider only 1-D loss of energy $E$ of a beam electron along its path in a plasma of pure hydrogen, from initial energy $E_o$ at injection.
Then if the ionisation fraction is \( x(N) \) at total column density \( N \) (\( \text{cm}^{-2} \)) the energy loss rate is

\[
\frac{dE}{dN} = -\frac{2\pi e^4 \Lambda}{E} \left[ \lambda + x(N') \right],
\]

so that

\[
E^2(N) = E_o^2 - 2K \int_0^N (\lambda + x(N')) \, dN',
\]

where \( \Lambda = \Lambda_{\text{ee}} - \Lambda_{\text{eH}} \), \( \lambda = \Lambda_{\text{eH}}/\Lambda \) and \( K = 2\pi e^4 \Lambda \), while the bremsstrahlung emission rate \( J(\epsilon) \) for an electron injection rate \( F_o(E_o) \) and Kramer’s cross-section is

\[
J(\epsilon) = \frac{Q_o}{K \epsilon} \int_\epsilon^\infty F_o(E_o) \int_{E_1}^{E_o} \frac{dEdE_o}{\lambda + x(E,E_o)},
\]

where \( x(E,E_o) \) is the value of \( x \) at depth \( N \) where an electron of injection energy \( E_o \) has slowed to \( E \). Because the energy loss depends on \( x(N) \) it is very convenient to replace the depth measure \( N \) by

\[
M(N) = \int_0^N (\lambda + x(N')) \, dN',
\]

which can be computed given \( x(N) \) for any atmosphere model. Therefore expression (3) can be written as

\[
\int_\epsilon^\infty F_o(E_o) k \left( \frac{(E_o^2 - \epsilon^2)}{E_1^2} \right) dE_o = -H(\epsilon) = -\frac{K}{Q_o} \frac{d}{d\epsilon} (\epsilon J),
\]

where the kernel function is (with \( E_o = E_1 \), stopping at \( M = M_1 \))

\[
k = \frac{1}{\lambda + x \left( M = \frac{M_1(E_o^2 - \epsilon^2)}{E_1^2} \right)}. \tag{6}
\]

Integral equation (5) can be easily transformed into standard convolution form

\[
\int_{-\infty}^{\infty} f(\eta) k(\eta - \xi) \, d\eta = g(\xi) \tag{7}
\]

(Craig and Brown 1986) by setting \( \eta = E_o^2/E_1^2 \) and \( \xi = \epsilon^2/E_1^2 \) and so solved for \( F_o(E_o) \) by standard convolution methods such as Fourier transform as discussed in our full paper. Here we derive an analytic solution for a special case, close to the real one.

### 3. Solution for Step Function Ionisation

The steepness of the solar transition function in terms of \( x(M) \) allows us to approximate \( x(M) = 1 \) if \( M < M_1 \), and \( x(M) = 0 \) if \( M > M_1 \) where \( M_1 \) is the depth of the transition region. Then (7) reduces to the functional form for \( f(\xi) \)

\[
f(\xi) + \frac{1}{\lambda} f(\xi + 1) = -(\lambda + 1)g'(\xi), \tag{8}
\]
which suggests trying a recursive iteration which in fact proves to satisfy the
equation exactly when extended to infinity, provided the sum converges.

\[ f(\xi) = - (\lambda + 1) \sum_{j=0}^{\infty} (-\lambda)^{-j} g'(\xi + j). \]  

(9)

In terms of the original physical variables then

\[ F_o(E_o) = (\lambda + 1)^{\frac{K E_o}{Q_o E_1}} \sum_{j=0}^{\infty} \left( (-\lambda)^{-j} \right)^{\frac{E_1}{\epsilon}} \left[ \frac{d^2}{d\epsilon^2} (\epsilon J) \right]_{\epsilon = E_2 + j E_2} \]  

(10)

One useful form of equation (10) is to express it in terms of a fractional correction

\[ \Delta(E_o/E_1) = \frac{F_o(E_o)}{F_o(E_1)} - 1 \]  

where \( F_o^*(E_o) \) is the injection spectrum solution for \( F_o(E_o) \)
from \( J(\epsilon) \) in the case usually treated with \( x(M) = 1 \) everywhere. This is easily shown to be

\[ F_o^*(E_o) = (\lambda + 1)^{\frac{K E_o}{Q_o}} \left[ \frac{d^2}{d\epsilon^2} (\epsilon J) \right]_{\epsilon = E_o}, \]  

(11)

so that \( \Delta(E_o/E_1) \) is given for general \( F_o^*(E_o) \) (i.e. general \( J(\epsilon) \)) by

\[ \Delta(X) = \sum_{j=1}^{\infty} (-\lambda)^{-j} \left[ \frac{1}{X(1 + j/X^2)^{1/2}} \right] \frac{F_o^* \left( X E_1 (1 + j/X^2)^{1/2} \right)}{F_o^* \left( X E_1 \right)}, \]  

(12)

where \( X = E_o/E_1 \). Expression (12) can be used for any observed photon spectrum to correct, approximately, the thick target electron injection spectrum inferred for a fully ionised target to yield the result for the nonuniformly ionised target atmosphere, for any adopted value of \( M_1 \), while equation (7) can be used for more general forms of \( x(M) \). Alternatively \( F_o^*(E_o) \) can be expressed in terms of \( F_o(E_o) \) by

\[ F_o^*(X E_1) = F_o(X E_1) + \frac{F_o(\sqrt{(1 + X^2)E_1})}{\lambda \sqrt{(1 + 1/X^2)}} \]  

(13)

to show how much the inferred electron spectrum, assuming a fully ionised target, differs from the true spectrum for nonuniform \( x(M) \).

4. Results and Conclusion

Here we show results from the above equations for some illustrative cases. Firstly
when the true injection spectrum \( F_o(E_o) \) was a power law \( A E_o^{-\delta} \) with \( \delta = 3 \), for
the corresponding \( F_o^*(E_o) \), over a large dynamic range, the difference looked
minuscule. The fractional difference \( \Delta \) given by equation (12) was more appreciable though not large. The power-law case, however, involves an intrinsically very smooth form of \( F_o(E_o) \). In inverse problems we expect the largest effects of perturbations to be on sharp features in the solution. We therefore compared the forms of \( F_o(X E_1) \) and of \( F_o^*(X E_1) \) for the cases of a top hat function \( F_o(X E_1) = 0 \) outside the range (0.9,2.0) and a power-law injection spectrum cut off below \( X = 0.9 \), the latter being shown below.
Figure 1. Comparison of True Injection Spectrum with Inferred Spectrum assuming a Fully Ionised Atmosphere

From these we saw that inclusion of the ionisation jump across the transition zone in the thick bremsstrahlung inversions is easy to carry out, by a minor kernel change, and leads to significant changes in the inferred properties of the sharp features in the electron spectrum, in particular moving the positions of some discontinuities and enhancing their detectability somewhat. By contrast the ionised solution introduces electrons into \( F_0(E_0) \) where there are in fact none. This is because the near-discontinuity in \( x(E, E_0) \) acts to sharpen the kernel of the integral equation involved and recovery of detail with less smoothing.

References

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