DISSIPATIVE MHD SOLUTIONS FOR RESONANT ALFVÉN WAVES IN 1-DIMENSIONAL MAGNETIC FLUX TUBES

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Abstract. The present paper extends the analysis by Sakurai, Goossens, and Hollweg (1991) on resonant Alfvén waves in nonuniform magnetic flux tubes. It proves that the fundamental conservation law for resonant Alfvén waves found in ideal MHD by Sakurai, Goossens, and Hollweg remains valid in dissipative MHD. This guarantees that the jump conditions of Sakurai, Goossens, and Hollweg, that connect the ideal MHD solutions for $\xi_r$, and $P'$ across the dissipative layer, are correct. In addition, the present paper replaces the complicated dissipative MHD solutions obtained by Sakurai, Goossens, and Hollweg for $\xi_r$ and $P'$ in terms of double integrals of Hankel functions of complex argument of order $1/3$ with compact analytical solutions that allow a straightforward mathematical and physical interpretation. Finally, it presents an analytical dissipative MHD solution for the component of the Lagrangian displacement in the magnetic surfaces perpendicular to the magnetic field lines $\xi_\perp$ which enables us to determine the dominant dynamics of resonant Alfvén waves in dissipative MHD.

1. Introduction

Resonant absorption is an efficient means for dissipating energy of MHD waves in nonuniform plasmas. It was first studied as a means for the supplementary heating of fusion plasmas (see, e.g., Tatarchois and Grossmann, 1973; Grossmann and Tatarchois, 1973; Chen and Hasegawa, 1974; Hasegawa and Chen, 1974) and subsequently proposed as a mechanism for heating magnetic flux tubes in the solar corona by Ionson (1978). Since the original suggestion, resonant absorption has remained a popular mechanism for explaining the heating of the solar corona (see, e.g., Kuperus, Ionson, and Spicer, 1981; Ionson, 1985; Hollweg, 1990, 1991; Goossens, 1991). Recently, resonant absorption has been considered as a possible explanation of the observed loss of power of acoustic oscillations in sunspots (Hollweg, 1988; Lou, 1990; Goossens and Poedts, 1992; Stenuit, Poedts, and Goossens, 1993).

The study of driven Alfvén waves involves forced oscillations in dissipative MHD. This means that the time dependent nonlinear equations of dissipative MHD
have to be integrated in the presence of a time varying force term. Most studies have used linear theory of wave motions superimposed on an ideal equilibrium state. Even in the context of linear MHD, it is common practice to circumvent the time integration by considering the asymptotic or steady state of the Alfvén waves. In the asymptotic state all the perturbed quantities oscillate with the same frequency as the incoming wave so that the time dependency can be removed out of the mathematical formulation. Detailed results based on large-scale numerical simulations of the asymptotic state of Alfvén wave heating were obtained by Grossmann and Smith (1988) in ideal MHD and by Poedts, GoosSENS, and Kerner (1989a) and (1989a) and Poedts, GoosSENS, and Kerner (1989b, 1990a) in resistive MHD. Time-dependent computations of Alfvén wave heating in linear dissipative MHD have been carried out by Poedts, GoosSENS, and Kerner (1990b), and Poedts and Kerner (1992). In what follows we shall exclusively focus on the asymptotic state in linear MHD.

The physical basis of resonant absorption as a dissipative mechanism can be understood from spectral theory of linear ideal MHD. The linear ideal MHD spectrum of a nonuniform plasma contains an Alfvén continuum and a slow continuum in addition to discrete eigenfrequencies. The spatial solutions that correspond to these continuum frequencies are non-square integrable (see, e.g., Goedbloed, 1983). Consider now a MHD wave with given wave numbers and a frequency within the Alfvén continuum of the flux tube. If this wave impinges on the flux tube it excites an Alfvén wave on the magnetic surface where the local Alfvén frequency equals the frequency of the incoming wave. Part of the energy of the wave will be absorbed and the energy fluxes of the incoming and outgoing waves will be different. In ideal MHD this driven problem is governed by a set of differential equations that are singular at the point where the local Alfvén frequency equals the frequency of the wave. The spatial solutions are non-square integrable, and ideal MHD cannot be used to describe the resonant behaviour. Non-ideal effects such as viscosity and electrical resistivity will remove the singularity but at the same time the order of the system of differential equations is raised. The steep gradients of the spatial solutions cause efficient damping in non-ideal MHD.

The large values of the viscous and magnetic Reynolds numbers in the solar atmosphere imply that the dissipative terms in the MHD equations are unimportant except in narrow layers. In the case of resonant absorption of MHD waves the dissipative terms are only important in a narrow layer around the position where the frequency of the wave equals the local Alfvén frequency. Outside this narrow layer the MHD waves are accurately described by the equations of ideal MHD. This observation led Sakurai, GoosSENS, and Hollweg (1991a) to design a method for obtaining the solutions of resonant Alfvén waves which does not require solution of the full set of dissipative MHD equations in the whole of the flux tube. It suffices to solve the equations of ideal MHD in the regions to the left and the right of the dissipative layer. The solutions of the dissipative MHD equations are used to connect the ideal MHD solutions over the dissipative layer. The cumbersome part
is to solve the equations of dissipative MHD in the dissipative layer and in regions to the left and the right of the dissipative layer that overlap with the regions where ideal MHD is valid. In general the combination of the dissipative layer and the overlap regions is only a tiny fraction of the flux tube so that it is possible to use simplified dissipative MHD equations which allow analytical solutions. If one is only interested in the global behaviour of the solution, the dissipative solution can be seen as a means to connect the ideal solutions across the dissipative layer. As a consequence, once these connection formulae are found, it is not any longer necessary to solve the dissipative MHD equations.

In an early version of this approach, Hollweg (1987a, b) and Hollweg and Yang (1988) noted that in planar geometry the Eulerian perturbation of total pressure, $P'$, could be taken to be approximately constant across the dissipative layer. Thus $P' = \text{constant}$ represented the desired connection formula, and it allowed for a straightforward description of the structure of the velocity amplitude in the dissipative layer. However, $P' = \text{constant}$ was assumed \textit{ad hoc}, and has to be replaced with a different conservation law in cylindrical geometry. Sakurai, Goossens, and Hollweg (1991a) obtained the fundamental conservation law for cylindrical geometry at the Alfvén resonance point in ideal MHD. They assumed that this ideal conservation law remains valid in dissipative MHD and obtained the solutions to the dissipative MHD equations for the radial component of the Lagrangian displacement $\xi_r$ and the Eulerian perturbation of total pressure $P'$ in terms of double integrals of Hankel functions of order $\frac{1}{3}$ of an complex argument. An asymptotic expansion of these solutions enabled Sakurai, Goossens, and Hollweg to obtain the connection formulae for $\xi_r$ and $P'$. These connection formulae in combination with solutions of ideal MHD were then used by Sakurai, Goossens, and Hollweg (1991b) to study absorption of acoustic waves in sunspots and by Goossens and Hollweg (1993) to obtain conditions for maximal and also total absorption.

The dissipative MHD solutions obtained by Sakurai, Goossens, and Hollweg (1991a) are by far too complicated to give us any information about the solutions in the dissipative layer. The object of the present paper is to extend the analysis of Sakurai, Goossens, and Hollweg (1991a) in three ways. First, we aim to show that the fundamental conservation found in ideal MHD remains indeed a conservation law in dissipative MHD. Second we want to obtain simple analytical solutions to the dissipative MHD equations for $\xi_r$ and $P'$ which allow a straightforward physical interpretation. Finally we want to obtain the dissipative solution for the component of the Lagrangian displacement perpendicular to the magnetic field lines in the magnetic surfaces $\xi_\perp$ since ideal MHD predicts that for Alfvén waves the dominant dynamics resides in the components perpendicular to the magnetic field lines in the magnetic surfaces.

The paper is organized as follows. In Section 2 we list the basic set of linear differential equations that govern linear motions superimposed on a static equilibrium state in linear resistive MHD. In Section 3 we collect results obtained by Sakurai, Goossens, and Hollweg (1991a) on resonant Alfvén waves in ideal MHD which
will return in a modified form in resistive MHD. In Section 4 we simplify the linear resistive MHD equations of Section 2 by restricting the analysis to the subclass of linear displacements that correspond to resonant Alfvén waves in ideal MHD. In Section 5 we determine the solutions to these linear resistive MHD equations in an interval that embraces the dissipative layer and two overlap regions where ideal MHD is also valid. Section 6 summarizes the main results of the present paper.

2. Basic Equations of Linear Resistive MHD

In what follows we shall concentrate on the asymptotic state of Alfvén waves in linear MHD. All perturbed quantities are put proportional to

\[ \exp(-i\omega t) \]

with \( \omega > 0 \) the frequency of the incoming wave or the external forcing term. The asymptotic state is in principle only valid for \( t \to +\infty \), but in practice it gives an accurate description for \( t \gg t_{\text{crit}} \) where \( t_{\text{crit}} \sim (R_m)^{1/3} \) and \( R_m \) is the magnetic Reynolds number (see Kapraff and Tataronis, 1977; Poedts, Goossens, and Kerner, 1990b).

The present paper considers Alfvén waves in 1-dimensional magnetic flux tubes. It is obvious that 1-dimensional magnetic flux tubes are a simplification of reality. Sunspots and coronal loops are complicated structures with spatial variations of the physical quantities in two or even three directions. Macroscopic magnetic structures that exist over relatively long time spans in the solar atmosphere have to satisfy the equilibrium and stability limits set by ideal MHD. As a consequence an equilibrium state with well defined magnetic surfaces is very probably a good approximation of the global magnetic equilibrium of a sunspot and also of coronal loop. Spatial variations of the equilibrium quantities in different directions affect the waves in different ways. For example, radial stratification causes resonant absorption of acoustic oscillations in sunspots (see, e.g., Hollweg, 1988; Lou, 1990; Sakurai, Goossens, and Hollweg, 1991b; Goossens and Poedts, 1992), while axial stratification causes conversion of acoustic waves into slow waves producing a downward drainage of wave energy (Spruit and Bogdan, 1992; Cally and Bogdan, 1993). It is evident that these two effects should be combined and that there is a need for studying waves in 2-dimensional equilibrium states. However, the construction of realistic 2-D and 3-D equilibrium models for sunspots and coronal loops is a research area in its own right, and the study of waves in 2-D magnetic equilibrium configurations is a very ambitious undertaking. In this context it is necessary to have a clear understanding of waves in simple 1-D equilibrium configurations which contain part of the relevant physics.

The static equilibrium state of the flux tube is idealized as a cylindrically-symmetric column of plasma. We use a system of cylindrical coordinates \( r, \varphi, z \) with the \( z \)-axis coinciding with the axis of symmetry of the cylinder. The components of the equilibrium magnetic field \( B(0, B_\varphi, B_z) \) as well as the pressure \( p \) and
density \( \rho \) are functions of the radial coordinate \( r \) only. They satisfy the radial force balance equation

\[
\frac{d}{dr} \left( p + \frac{B^2}{2\mu} \right) = -\frac{B^2}{\mu r} .
\]  

The linear displacements superimposed on this background state are governed by the linearized versions of the MHD equations. Since the aim is to study the resonant behaviour of Alfvén waves in dissipative MHD, we have to include a dissipative term that removes the ideal singularity. Electrical resistivity is a prime candidate in this respect (very similar results are obtained if viscosity is taken to be the dissipation mechanism). The equations of linear resistive MHD are:

\[
\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho V') = 0 ,
\]

\[
\frac{\partial p'}{\partial t} + V'_r \frac{d\rho}{dr} = \frac{\gamma p}{\rho} \left( \frac{\partial \rho'}{\partial t} + V'_r \frac{d\rho}{dr} \right) ,
\]

\[
\rho \frac{\partial V'}{\partial t} = -\nabla p' + \frac{1}{\mu} \left[ (\nabla \times B) \times B' + (\nabla \times B') \times B \right] ,
\]

\[
\frac{\partial B'}{\partial t} = \nabla \times (V' \times B) + \eta \nabla^2 B' ,
\]

In these equations \( \eta \) is the coefficient of magnetic diffusivity and \( \gamma \) is the ratio of specific heats. These two quantities are assumed to be constant. The other notations are the same as in Sakurai, Goossens, and Hollweg (1991a). Equation (4) is the equation for the variation of internal energy not including non-ideal effects. In particular the term due to finite electrical conductivity is ignored in this equation (the adiabatic equation). The inclusion of finite electrical conductivity in the generalized form of Ohm’s law (Equation (6)) but not in the energy equation, is an approximation often made in MHD and based on the fact that finite electrical conductivity has its most important effect in Equation (6). Poedts, Goossens, and Kerner (1990b) have included the resistive term in the equation for the variation of internal energy in their numerical study of the temporal evolution of resonant absorption. They found that inclusion of the resistive term in the energy equation has virtually no effect. The adiabatic equation (4) is a very good approximation of the energy equation.

Since the equilibrium quantities depend on \( r \) only, we can Fourier-analyze the perturbed quantities with respect to \( \varphi \) and \( z \) and put them proportional to

\[
\exp[i(m\varphi + kz)] .
\]

Here \( m \) (an integer) and \( k \) are the azimuthal and axial wave numbers. As the time-dependence \( \exp(-i\omega t) \) has already been factored out, the perturbed quantities are
functions of $r$ only and the differential equations are reduced to ordinary differential equations.

3. Resonant Alfvén Waves in Linear Ideal MHD

Dissipative MHD is only necessary to treat the dissipative layer. Part of the basic physics of resonant Alfvén waves can be understood in the context of linear ideal MHD. The aim of the present section is to collect results obtained by Sakurai, Goossens, and Hollweg (1991a) in ideal MHD in such a way that expressions which return in a modified form in resistive MHD are both well-defined and well-distinguished.

All but two of the perturbed variables may be eliminated from the linear ideal MHD equations leading to a set of two first-order differential equations for the radial component of the Lagrangian displacement, $\xi_r$ and the perturbed total pressure $P'$ ($P' = p' + B \cdot B'/\mu$):

$$D \frac{d(r\xi_r)}{dr} = C_1 r \xi_r - C_2 r P',$$

$$D \frac{dP'}{dr} = C_3 \xi_r - C_1 P'. \tag{8}$$

Equations (8) govern the linear motions of a compressible cylindrical plasma. They were first obtained in this form by Appert, Gruber, and Vaclavik (1974). The coefficient functions $D$, $C_1$, $C_2$, and $C_3$ depend on the equilibrium quantities and on the frequency $\omega$. They take the form

$$D = \rho(c^2 + v_A^2)(\omega^2 - \omega_A^2)(\omega^2 - \omega_C^2), \tag{9}$$

$$C_1 = \frac{2}{\mu r} B_{\varphi}^2 \omega^4 - (c^2 + v_A^2)(\omega^2 - \omega_C^2) \frac{2m_f B}{\mu r^2} B_{\varphi}, \tag{10}$$

$$C_2 = \omega^4 - (c^2 + v_A^2)(\omega^2 - \omega_C^2) \left( \frac{m_f^2}{r^2} + k^2 \right), \tag{11}$$

$$C_3 = D \left[ \rho(\omega^2 - \omega_A^2) + \frac{2B_{\varphi}}{\mu} \frac{d}{dr} \left( \frac{B_{\varphi}}{r} \right) \right] +$$

$$+ \frac{4\omega^4 B_{\varphi}^4}{\mu^2 r^2} - 4\rho(c^2 + v_A^2)(\omega^2 - \omega_C^2)\omega_A^2 \frac{B_{\varphi}^2}{\mu r^2}. \tag{12}$$

The symbols have their usual meaning: $c$ is the adiabatic velocity of sound, $v_A$ is the Alfvén velocity, $\omega_A$ is the local Alfvén frequency, and $\omega_C$ is the local cusp frequency. The squares of these quantities are defined as
\[ c^2 = \gamma p/\rho, \quad v_A^2 = B^2/(\mu \rho), \quad \omega_A^2 = f_B^2/\mu \rho, \]
\[ \omega_C^2 = c^2 \omega_A^2/(c^2 + v_A^2), \quad f_B = k \cdot B, \quad k = (0, m/r, k). \]

In a nonuniform plasma \( c^2, v_A^2, \omega_A^2, \) and \( \omega_C^2 \) are functions of position. The variation of \( \omega_A \) with position plays a fundamental role in the present discussion of resonant Alfvén waves.

The other perturbed quantities \( (p', \rho', \text{ etc.}) \) can be computed once \( \xi_r \) and \( P' \) are known. For later use we note that the component of the displacement vector in the magnetic surfaces and perpendicular to the magnetic field lines \( (\xi_\perp = (\xi_\varphi B_z - \xi_z B_\varphi)/B) \) is related to \( \xi_r \) and \( P' \) as

\[ (\omega^2 - \omega_A^2)\xi_\perp = \frac{i}{\rho B} (g_B P' - 2 f_B B_\varphi B_z \xi_r/\mu r), \]  
(13)

where

\[ B = (B_\varphi^2 + B_z^2)^{1/2}, \quad g_B = \frac{m B_z}{r} - k B_\varphi. \]
(14)

Equations (8) define an eigenvalue problem with \( \omega^2 \) as eigenvalue parameter when they are supplemented with boundary conditions. In a driven problem \( \omega \) is prescribed. Equations (8) have regular singular points at the zeroes of the coefficient function \( D \). As a consequence we have mobile regular singular points at the positions \( r \) where

\[ \omega^2 = \omega_A^2(r), \quad \omega^2 = \omega_C^2(r). \]
(15)

Equation (15a) defines the Alfvén resonance point, while Equation (15b) defines the slow resonance point. Since \( \omega_A^2(r) \) and \( \omega_C^2(r) \) are functions of position, these two equations define two continuous ranges in the spectrum which are classically referred to as the Alfvén continuum and the slow continuum. In what follows we shall concentrate on the Alfvén resonance.

Sakurai, Goossens, and Hollweg (1991a) focus on a frequency in the Alfvén continuum and determine the spatial behaviour of the corresponding perturbation close to the singular point where the condition \( \omega^2 = \omega_A^2(r_A) \) is satisfied. It is convenient to introduce the new radial variable \( s \) defined as

\[ s = r - r_A. \]
(16)

Sakurai, Goossens, and Hollweg (1991a) use series expansions of the coefficient functions around \( s = 0 \) to obtain simplified versions of the relevant differential equations. These simplified versions are valid in the interval \([ -s_A, s_A] \) around the point of resonance where the linear Taylor polynomial is a valid approximation of \( \omega^2 - \omega_A^2(r) \); hence \( s_A \) has to satisfy

\[ s_A \ll \frac{2(\omega_A^2)'}{(\omega_A^2)''}. \]
(17)

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The simplified versions of Equations (8) close to the Alfvén resonance point are:

\[ s \Delta \frac{d \xi_r}{ds} = \frac{g_B}{\rho B^2} C_A(s), \]

\[ s \Delta \frac{d P'}{ds} = \frac{2 f_B B_\varphi B_z}{\mu r_A \rho B^2} C_A(s), \]  

(18)

where

\[ C_A(s) = g_B P' - \frac{2 f_B B_\varphi B_z \xi_r}{\mu r_A}. \]  

(19)

In Equations (18) all equilibrium quantities are evaluated at \( s = 0 \) \( (r = r_A) \), and

\[ \Delta = \frac{d}{dr} (\omega^2 - \omega_A^2). \]  

(20)

The fact that the right-hand members of Equation (18) have the common factor \( C_A(s) \) which is a linear combination of \( \xi_r \) and \( P' \) is a key point in the present analysis. It suggests that we should obtain a differential equation for \( C_A(s) \). This is readily done by taking the appropriate linear combination of Equations (18):

\[ s \frac{d C_A(s)}{ds} = 0. \]

(21)

Since the large solutions of \( \xi_r \) and \( P' \) (i.e., solutions containing logarithmic terms) have to be continuous, Equation (21) implies that

\[ C_A(s) \equiv g_B P' - \frac{2 f_B B_\varphi B_z \xi_r}{\mu r_A} = \text{constant}. \]  

(22)

Condition (22) is the fundamental conservation law at the Alfvén resonance point (Sakurai, Goossens, and Hollweg, 1991a).

The solutions for \( \xi_r \) and \( P' \) take the form

\[ \xi_r(s) = \frac{g_B}{\rho B^2 \Delta} C_A \ln(|s|) + \begin{cases} \xi_+ & s < 0, \\ \xi_- & s > 0, \end{cases} \]

\[ P'(s) = \frac{2 f_B B_\varphi B_z}{\mu r_A \rho B^2 \Delta} C_A \ln(|s|) + \begin{cases} P'_- & s < 0, \\ P'_+ & s > 0. \end{cases} \]

(23)

The jumps in \( \xi_r \) and \( P' \) are due to dissipative effects and will be specified later.

Let us now return to the conservation law at the Alfvén resonance point. In an equilibrium with a straight magnetic field \( (B_\varphi = 0) \), the conservation law reduces to

\[ [P'] = 0. \]  

(24)
The solution (23) for $\xi_r$ remains unchanged, while $P'$ can be considered to be constant across the point of resonance. The approximate constancy of $P'$ was the key ingredient in the physical discussions of resonance absorption given by Hollweg (1987a, b, 1988) and Hollweg and Yang (1988). Thus the discovery of the conservation law $C_A = \text{constant}$ puts the approach by Hollweg and Yang on a rigorous mathematical footing, and extends the approach to cylindrical geometry.

For an equilibrium with a curved magnetic field we can use Equation (13) to rewrite the conservation law in terms of $\xi_\perp$:

$$s\xi_\perp = i \frac{C_A}{\rho B \Delta}.$$

Equation (25) expresses that $s\xi_\perp$ is constant across the resonant layer, or that $\xi_\perp$ has a $1/s$-singularity and a $\delta(s)$ contribution which dominate the $\ln |s|$ singularity and the jump found for $\xi_r$ and $P'$. The dominant singularities in the solution reside in the components in the magnetic surfaces and perpendicular to the magnetic field lines (see also Goedbloed, 1983). The dominant dynamics of the wave is contained in $\xi_\perp$ and the wave is polarized in the magnetic surfaces perpendicular to the magnetic field lines. The nonuniform plasma supports an Alfvén wave that is confined to the magnetic surface where the dispersion relation for Alfvén waves in a uniform plasma is locally satisfied. The confinement of the Alfvén wave is not absolute. The Alfvén wave is linked to the outside world as it is coupled to a wave with components normal to the magnetic surfaces. From a dynamic point of view we can consider the wave as an almost pure Alfvén wave confined to its resonant magnetic surface and polarized perpendicular to the magnetic field lines. From the point of view of the energetics $\xi_r$, the component normal to the magnetic surfaces, is essential since it is this quantity which provides the unidirectional transfer of energy to the resonant surface.

The above discussion leads to a simple interpretation of the conservation law at the Alfvén resonant point. With the help of the induction equation, the radial component of the momentum equation may be written

$$\rho (\omega^2 - \omega_A^2) \xi_r = \frac{dP'}{dr} + \frac{2B_\varphi}{\mu r} \left( ik B \xi_\perp - \frac{d(B_\varphi \xi_r)}{dr} \right).$$

The left-hand side has terms varying as $s$ or $s \ln |s|$ near $s = 0$, and will be neglected. Since the motion near $s = 0$ is primarily Alfvénic, we can take $\nabla \cdot \xi \approx 0$ and $B_\varphi \xi_\varphi + B_z \xi_z \approx 0$. Taking $r = \text{constant}$ near $s = 0$ then gives

$$\xi_\perp \approx \frac{iB}{g_B} \frac{d\xi_r}{dr}$$

and the radial component of the momentum equation becomes

$$0 \approx \frac{dP'}{dr} - \frac{2f_B B_\varphi B_z}{\mu r A g_B} \frac{d\xi_r}{dr}.$$
if all equilibrium quantities are evaluated at $s = 0$ ($r = r_A$). Integration yields the conservation law (22). An alternative form is

$$0 \approx \frac{dP'}{dr} + \frac{2B_\varphi B_z}{\mu r_A} \frac{d\xi_\perp}{d\sigma},$$

(29)

where $\sigma$ is the distance along the helical equilibrium magnetic field line. The term involving $d\xi_\perp/d\sigma$ is simply the increased tension force associated with displacing an equilibrium magnetic field line by $\xi_\perp$. This is readily demonstrated by writing the position vector for the displaced field line as

$$\mathbf{R}(\sigma) = r_u_r(\sigma) + \frac{B_z \xi_\perp}{B} u_\varphi(\sigma) + \left[ \frac{B_z \sigma}{B} - \frac{B_\varphi \xi_\perp}{B} \right] u_z(\sigma),$$

(30)

where the $u$'s are the unit vectors. The radius of curvature is given by

$$r_{\text{curv}} = |\mathbf{R}'|^3/|\mathbf{R}' \times \mathbf{R}''|,$$

where the primes denote differentiation with respect to $\sigma$. Keeping only linear terms, we find

$$\frac{1}{r_{\text{curv}}} = \frac{B_\varphi^2}{r B^2} + \frac{2B_\varphi B_z}{r B^2} \frac{d\xi_\perp}{d\sigma}.$$  

(31)

The tension force associated with the first term is balanced in equilibrium, while the tension force associated with the second term is balanced by $dP'/dr$. Thus the conservation law (22) expresses the balance between the total pressure gradient and the inward tension force generated by the displacement, $\xi_\perp$, of the equilibrium magnetic field.

4. Resistive MHD Equations for Resonant Alfvén Waves

The aim is to find out how the singular solutions for the resonant Alfvén waves found in ideal MHD are modified by dissipation. For the present purpose it suffices to consider non-zero electrical resistivity since this removes the singularity present in the ideal equations. We are not interested in the equations that govern arbitrary linear displacements of a cylindrical plasma in resistive MHD. Our interest goes exclusively to the subclass of linear displacements that correspond to resonant Alfvén waves in ideal MHD. The restriction to this subclass of perturbations makes a significant simplification of the equations of resistive MHD possible. The effects of dissipation are generally small and only important in the vicinity of the ideal resonance. In this dissipative layer the derivatives of the perturbed quantities with respect to $r$ are much larger than those with respect to $z$ and $\varphi$. In addition the derivatives of equilibrium quantities can be neglected in comparison with the derivatives of the perturbed quantities with respect to $r$. Therefore we retain in the dissipative terms only the $r$-derivatives of the perturbed quantities while we
neglect those of the equilibrium quantities. We write $\partial^2 b / \partial r^2$ instead of $\nabla^2 b$. We rewrite Equations (3)–(6) as

$$\rho' = -\xi_r \frac{d\rho}{dr} - \rho \nabla \cdot \xi ,$$

(32)

$$p' = -\frac{d\rho}{dr} \xi_r + \frac{\gamma \rho}{\rho} \left( \rho' + \frac{d\rho}{dr} \xi_r \right) ,$$

(33)

$$\rho \omega^2 \xi_r = \frac{dP'}{dr} - \frac{1}{\mu} \left( i f_B B'_r - \frac{2}{r} B_\varphi B'_\varphi \right) ,$$

(34)

$$\rho \omega^2 \xi_\varphi = \frac{im}{r} P' - \frac{1}{\mu} \left( \frac{1}{r} \frac{d}{dr} (r B_\varphi) B'_r + i f_B B'_\varphi \right) ,$$

(35)

$$\rho \omega^2 \xi_z = ik P' - \frac{1}{\mu} \left( \frac{dB_z}{dr} B'_r + i f_B B'_z \right) ,$$

(36)

$$\left( 1 - \frac{i \eta}{\omega} \frac{d^2}{dr^2} \right) B'_r = i f_B \xi_r ,$$

(37)

$$\left( 1 - \frac{i \eta}{\omega} \frac{d^2}{dr^2} \right) B'_\varphi = i f_B \xi_\varphi - B_\varphi \nabla \cdot \xi - \frac{1}{r} \frac{d}{dr} \left( \frac{B_\varphi}{r} \right) \xi_r ,$$

(38)

$$\left( 1 - \frac{i \eta}{\omega} \frac{d^2}{dr^2} \right) B'_z = i f_B \xi_z - B_z \nabla \cdot \xi - \frac{dB_z}{dr} \xi_r .$$

(39)

We now substitute $B'$ from Equations (37)–(39) into Equations (34)–(36) and apply the operator $1 - (i \eta / \omega) \partial^2 / \partial r^2$ to Equations (34)–(36). We keep in mind that the dissipative terms are only important close to the ideal resonant position and that the (radial) derivatives of equilibrium quantities are to be neglected in comparison to the radial derivatives of perturbed quantities in dissipative terms. Straightforward algebra yields the following components of the equation of motion:

$$\rho \left( \omega_\eta^2 - \omega_\Lambda^2 \right) \xi_r = \frac{\omega_\eta^2}{\omega^2} \frac{dP'}{dr} +$$

$$+ \frac{2}{\mu} \left[ \frac{i}{r} f_B B_\varphi \xi_\varphi - \frac{1}{r} B_\varphi^2 \nabla \cdot \xi - B_\varphi \frac{d}{dr} \left( \frac{B_\varphi}{r} \right) \xi_r \right] ,$$

(40)

$$\rho \left( \omega_\eta^2 - \omega_\Lambda^2 \right) \xi_\varphi = \frac{im \omega_\eta^2}{r \omega^2} P' + \frac{i f_B}{\mu} B_\varphi \left( \nabla \cdot \xi - \frac{2}{r} \xi_r \right) ,$$

(41)

$$\rho \left( \omega_\eta^2 - \omega_\Lambda^2 \right) \xi_z = \frac{ik \omega_\eta^2}{\omega^2} P' + \frac{i f_B}{\mu} B_z \nabla \cdot \xi ,$$

(42)
where $\omega^2_\eta$ stands for the differential operator

$$
\omega^2_\eta = \omega^2 \left(1 - \frac{i\eta}{\omega} \frac{d^2}{dr^2}\right). \quad (43)
$$

We have deliberately used the notation $\omega^2_\eta$ for this differential operator in order to have equations which are formally as similar as possible to the ideal equations.

Just as in ideal MHD all but two of the perturbed variables may be eliminated from the linear resistive MHD equations leading to a set of two differential equations of third order for $\xi_r$ and $P'$. One key step is the introduction of $P'$ as an unknown variable and the other key is the derivation of the compression term $\nabla \cdot \xi$.

Now from (32)–(33) and (38)–(39) we derive with the aid of (2)

$$
\left(1 - \frac{i\eta}{\omega} \frac{d^2}{dr^2}\right) P' = \frac{i f}{\mu} \mathbf{B} \cdot \xi + \left(\frac{2}{\mu r} B^2_\varphi + \frac{d \rho}{d r} \frac{i\eta}{\omega} \frac{d^2}{dr^2}\right) \xi_r - \\
- \rho \left(c^2 + V^2_\varphi + c^2 \frac{\omega^2}{\omega} \frac{d^2}{dr^2}\right) \nabla \cdot \xi. \quad (44)
$$

Elimination of $\mathbf{B} \cdot \xi$ from Equation (44) with the aid of Equations (41) and (42) gives

$$
\rho \left(c^2 + V^2_\varphi\right) \left[\omega^2_\eta - \omega^2_c - \frac{i\eta}{\omega} \frac{c^2}{c^2 + V^2_\varphi} \left(\omega^2_\eta - \omega^2_\varphi\right) \frac{d^2}{dr^2}\right] \nabla \cdot \xi = \\
= -\frac{\omega^4_\varphi}{\omega^2} P' + \left[\frac{2}{\mu r} B^2_\varphi \omega^2_\eta - \frac{i\eta}{\omega} \frac{d \rho}{d r} \left(\omega^2_\eta - \omega^2_\varphi\right) \frac{d^2}{dr^2}\right] \xi_r, \quad (45)
$$

This is the desired expression for the compression term. In ideal MHD it reduces to Equation (7) of Sakurai, Goossens, and Hollweg (1991a); the resistive terms introduce additional derivatives of $\xi_r$ and $P'$. On the other hand with the aid of Equations (41) and (42) we easily get

$$
\rho \omega^2_\eta \nabla \cdot \xi = \rho \frac{\omega^2_\eta}{r \omega^2} \frac{d}{dr} (r \xi_r) - \\
- \left(\frac{m^2}{r^2} + k^2\right) \frac{\omega^2_\varphi}{\omega^2} P' + \frac{2mf_B}{\mu r^2} B_\varphi \xi_r. \quad (46)
$$

Let us recall that dissipation is only important close to the ideal singularity. Away from the position of the ideal singularity electrical resistivity can be dropped and we need to keep terms proportional to $\eta$ only if they appear in the expression $\omega^2 - \omega^2_\varphi - (i\omega\eta)(d^2/dr^2)$, i.e., $\omega^2 - \omega^2_\varphi$.

The two differential equations for $\xi_r$ and $P'$ are obtained by elimination of $\nabla \cdot \xi$ from Equations (45) and (46) on the one hand and by elimination of $\xi_\varphi$ and $\nabla \cdot \xi$.
from Equation (40) with the aid of Equations (41) and (45) on the other hand. The equations are

\[ D_\eta \frac{d(r\xi_r)}{dr} = C_1 r\xi_r - C_2 r P', \]

\[ D_\eta \frac{dP'}{dr} = C_3 \xi_r - C_1 P', \]  

where \( D_\eta \) is the differential operator

\[ D_\eta = \rho(c^2 + V_A^2)(\omega_\eta^2 - \omega_A^2)(\omega^2 - \omega_C^2). \]  

Equations (47) are the equations that govern the resonantly driven linear displacements of a cylindrical plasma in resistive MHD. They are a set of two differential equations of third order. They are formally the same as Equations (8), but the coefficient function \( D \) is replaced by the differential operator \( D_\eta \), in agreement with Sakurai, Goossens, and Hollweg (1991a). The singularities are removed from the equations, but the order of the set of differential equations is raised from 2 (in ideal MHD) to 6 (in non-ideal MHD), and in addition the coefficient of the derivative of highest order is proportional to \( \eta \).

The final equation that we want to obtain is the equation for \( \xi_\perp \). From Equations (41) and (42) we easily obtain

\[ (\omega_\eta^2 - \omega_A^2) \xi_\perp = \frac{i}{\rho B} \left( g_B P' - \frac{2}{\mu r} f_B B_\rho B_z \xi_r \right). \]  

Equation (49) is the resistive generalization of Equation (13). Equation (49) is formally the same as Equation (13), but \( \omega^2 \) is now replaced by the second-order differential operator \( \omega_\eta^2 \). As a consequence the resistive equation for \( \xi_\perp \) is a differential equation of second order for which \( s = 0 \) is not a singular point.

5. Resistive MHD Solutions for Resonant Alfvén Waves Close to the Ideal Resonance Position

The aim is to determine the solutions to the set of dissipative MHD equations (47) and (49) in the vicinity of the critical point \( r_A \) defined by the condition \( \omega_A^2(r_A) = \omega^2 \). We use the radial variable \( s \) defined by Equation (16) and use series expansions of the coefficient functions around \( s = 0 \) to obtain simplified versions of the dissipative MHD differential equations. Just as in ideal MHD these simplified versions are valid in the interval \([-s_A, s_A]\) around the point of resonance where the linear Taylor polynomial is a valid approximation of \( \omega^2 - \omega_A^2(r) \); hence \( s_A \) has to satisfy again the inequality (17). All the remaining equilibrium quantities
are replaced by their values at \( s = 0 \). The simplified versions of Equations (47) are

\[
\left( s \Delta - i \omega \eta \frac{d^2}{ds^2} \right) \frac{d \xi}{ds} = \frac{g B}{\rho B^2} C_A(s),
\]

\[
\left( s \Delta - i \omega \eta \frac{d^2}{ds^2} \right) \frac{d P'}{ds} = \frac{2 f_B B_\varphi B_z}{\mu r_A \rho B^2} C_A(s),
\]

where \( C_A(s) \) is again given by Equation (19). Equations (50) are the resistive generalizations of the ideal equations (18). The ideal factor \( s \Delta \) is now replaced by the second order differential operator \( s \Delta - i \omega \eta (d^2/ds^2) \). The set of two differential equations (18) is replaced by a set of two differential equations of third order. The ideal singularity at \( s = 0 \) is obviously absent from the resistive equations (50). As in Equations (18) all equilibrium quantities are evaluated at \( s = 0 (r = r_A) \).

Again we have the common factor \( C_A(s) \) in the right hand sides of equations (50). It is straightforward to obtain the differential equation for \( C_A(s) \) in dissipative MHD by taking a linear combination of Equations (50):

\[
\left( s \Delta - i \omega \eta \frac{d^2}{ds^2} \right) \frac{d C_A(s)}{ds} = 0.
\]

Equation (51) has Equation (21) as ideal counterpart. Equation (49) reduces to

\[
\left( s \Delta - i \omega \eta \frac{d^2}{ds^2} \right) \xi_\perp = i \frac{C_A(s)}{\rho B}.
\]

This is the resistive generalization of Equation (25). Equations (50)–(52) have no singularities at \( s = 0 \) in contrast to their ideal counterparts (18), (21), (25).

Dissipation is important when the terms \( s \Delta \) and \( \omega_\eta (d^2/ds^2) \) in the left hand sides of Equations (50)–(52) are comparable. This results in a dissipative layer with a thickness measured by the quantity \( \delta_A \):

\[
\delta_A = \left( \frac{\omega \eta}{|\Delta|} \right)^{1/3}.
\]

The thickness of the dissipative layer therefore scales as \( (\eta/|d\omega_A/dr|)^{1/3} \), a result already obtained by Kapraaff and Tatoronis (1977) and Hollweg and Yang (1988) and numerically verified by Poedts, Goossens, and Kerner (1990a). In view of the very large values of the magnetic Reynolds number in the solar corona we have that

\[
\frac{s_A}{\delta_A} \gg 1.
\]

Inequality (54) is important for the present discussion. It implies that the interval of validity of the simplified versions of the dissipative MHD equations embraces
the dissipative layer and in addition contains two overlap regions to the left and the right of the dissipative layer where ideal MHD is valid. As in Sakurai, Goossens, and Hollweg (1991a) we now introduce a new scaled variable

$$\tau = \frac{s}{\delta_A}$$  \hspace{1cm} (55)

which is of order 1 in the dissipative layer, but in view of inequality (54) $s \rightarrow \pm s_A$ corresponds to $\tau \rightarrow \pm \infty$.

With this new variable Equations (50)–(52) take the form

$$\left\{ \frac{d^2}{d\tau^2} + i \text{sign}(\Delta) \tau \right\} \frac{d\xi_r}{d\tau} = i \frac{g_B}{\rho B^2|\Delta|} C_A,$$

$$\left\{ \frac{d^2}{d\tau^2} + i \text{sign}(\Delta) \tau \right\} \frac{dP'}{d\tau} = i \frac{2f_B B_0 B_z}{\rho B^2 \tau_A |\Delta|} C_A,$$

$$\left\{ \frac{d^2}{d\tau^2} + i \text{sign}(\Delta) \tau \right\} \frac{dC_A}{d\tau} = 0,$$

$$\left\{ \frac{d^2}{d\tau^2} + i \text{sign}(\Delta) \tau \right\} \xi_\perp = \frac{-C_A}{\delta_A |\Delta| \rho B}.$$

(56)

Sakurai, Goossens, and Hollweg (1991a) did not obtain the differential equations (56c) and (56d) for $C_A$ and $\xi_\perp$ in dissipative MHD. They focussed on the dissipative equations for $\xi_r$ and $P'$ in an attempt to find the jumps in these quantities across the dissipative layer. They assumed that the ideal conservation law (22) remains valid in dissipative MHD. This assumption implies that the right-hand sides of Equations (56ab) are constant. Sakurai, Goossens, and Hollweg (1991a) then obtained solutions of the dissipative equations (56ab) in terms of a double integral of Hankel functions of order \(\frac{1}{3}\) of a complex argument. Asymptotic expansions of these solutions enabled them to obtain the jumps in $\xi_r$ and $P'$.

A related approach was used by Hollweg (1987b) and Hollweg and Yang (1988) in a discussion of the viscous dissipative layer for planar geometry. In essence, their approach was to take $P'$ to be constant across the dissipative layer at the outset. This was equivalent to neglecting the inertia in the dissipative layer. Since $C_A = g_B P'$ for planar geometry, they thus started with the constancy of $C_A$, and then derived an equation equivalent to (56d), which was solved in terms of Airy functions with imaginary argument. The advantage of this approach was the reduction of the 6th-order dissipative problem to the second-order equation (56d). They were also able to explicitly show that the total viscous heating (equivalent to the 7-integral of the volumetric heating rate) in the dissipative layer was independent of the viscosity coefficient, and that this heating was equal to the corresponding energy lost from the waves by resonant absorption. The heating, and the energy lost from the waves
by resonant absorption, was found to be adequate to heat active region loops. But
the difficulty is that the coronal viscosity is so small that all the heating takes
place in a very thin layer. If the same amount of energy could be distributed over
the diameter of an active region loop, then one would have a reasonable model of
coronal heating.

In the present paper we use a more compact and straightforward method for
obtaining the solutions to Equations (56). We shall first show that $C_A$ is also
a constant in dissipative MHD for $|s| \leq s_A$ so that the ideal conservation law
continues to be a conservation law in dissipative MHD. This proves the assumption
by Sakurai, Goossens, and Hollweg (1991a) and guarantees that the jump conditions
found by Sakurai, Goossens, and Hollweg (1991a) are correct. Then we shall obtain
compact analytical solutions for $\xi_r$, $P'$, and $\xi_\perp$ in the dissipative and the overlap
regions which allow a very simple mathematical and physical interpretation.

Inspired by the techniques used by Boris (1968) and Mok and Einaudi (1985)
for incompressible perturbations we look for solutions in integral form. In order
to obtain the solutions to Equations (56) it is instructive to consider the following
two differential equations of second order:

$$\left\{ \frac{d^2}{d\tau^2} + i \operatorname{sign}(\Delta) \tau \right\} \Psi(\tau) = 0,$$

$$\left\{ \frac{d^2}{d\tau^2} + i \operatorname{sign}(\Delta) \tau \right\} F(\tau) = -1. \quad (57)$$

It is straightforward to show that

$$F_1(\tau) = \int_0^\infty \exp(iu\tau \operatorname{sign}(\Delta) - u^3/3) \, du,$$

$$F_2(\tau) = \int_{\Gamma_2} \exp(iu\tau \operatorname{sign}(\Delta) - u^3/3) \, du, \quad (58)$$

$$F_3(\tau) = \int_{\Gamma_3} \exp(iu\tau \operatorname{sign}(\Delta) - u^3/3) \, du$$

are three solutions of Equation (57b). Here $\Gamma_2$ and $\Gamma_3$ are the two rays in the complex
plane that start at the origin and go to $\infty \exp(2\pi i/3)$ and to $\infty \exp(-2\pi i/3)$
respectively.

It immediately follows that

$$\Psi_1(\tau) = F_3(\tau) - F_1(\tau), \quad \Psi_2(\tau) = F_3(\tau) - F_2(\tau) \quad (59)$$

are two linearly independent solutions of Equation (57a). It is easily shown that
$\Psi_1(\tau) = -i\pi(Ai(\xi) - iBi(\xi))$, while $\Psi_2(\tau) = -2\pi iAi(\xi)$, where the argument
of the Airy functions is $\zeta = i\tau \text{sign}(\Delta)$. The general solutions of the second order differential equations (57a) and (57b) then take the form

$$
\Psi(\tau) = A_1(F_3(\tau) - F_1(\tau)) + A_2(F_3(\tau) - F_2(\tau)),
$$

$$
F(\tau) = F_1(\tau) + A_1(F_3(\tau) - F_1(\tau)) + A_2(F_3(\tau) - F_2(\tau)),
$$

(60)

where $A_1$, and $A_2$ are two yet undetermined constants. It is obvious that we are only interested in the solutions of Equations (57) that are bounded, in particular they have to remain bounded for $\tau \to \pm \infty$. The asymptotic behaviour of $F_1(\tau)$, $F_2(\tau)$, and $F_3(\tau)$ is determined in Appendix A. In this Appendix it is shown that out of the three functions $F_1(\tau)$, $F_2(\tau)$, and $F_3(\tau)$ only the function $F_1(\tau)$ is bounded everywhere. The functions $F_2(\tau)$ is unbounded at $\tau \to -\infty \text{sign}(\Delta)$, and the function $F_3(\tau)$ at $\tau \to +\infty \text{sign}(\Delta)$. This implies that both $\Psi_1(\tau)$ and $\Psi_2(\tau)$ become unbounded, and that we have to take

$$
A_1 = A_2 = 0
$$

(61)
to obtain bounded solutions for $\Psi(\tau)$ and $F(\tau)$ so that

$$
\Psi(\tau) \equiv 0,
$$

$$
F(\tau) \equiv F_1(\tau).
$$

(62)

Equation (62a) implies that the bounded solution of Equation (56c) satisfies

$$
\frac{dC_A(\tau)}{d\tau} = 0,
$$

$$
C_A(\tau) = \text{constant}.
$$

(63)

The ideal conservation law obtained by Sakurai, Goossens, and Hollweg (1991a) continues to hold in dissipative MHD. Equation (62b) also implies that

$$
\frac{d\xi_r}{d\tau} = -i \frac{g_B}{\rho B^2|\Delta|} C_A F(\tau),
$$

$$
\frac{dP'}{d\tau} = -i \frac{2f_B B_x B_z}{\mu r_A \rho B^2|\Delta|} C_A F(\tau),
$$

(64)

and

$$
\xi_\perp = \frac{C_A F(\tau)}{\rho B \delta_A |\Delta|}.
$$

(65)
Integration of Equations (64a) and (64b) gives the solutions in dissipative MHD for $\xi, P'$, and $\xi_\perp$ that remain finite for $|\tau| \to \infty$ as

$$\xi = -\frac{gbCA}{\rho B^2\Delta} G(\tau) + C_\xi,$$

$$P' = -\frac{2f_BB_\phi B_zCA}{\rho B^2\mu \tau \Delta} G(\tau) + C_P,$$

$$\xi_\perp = \frac{CA}{\delta A\rho B} F(\tau),$$

where $C_\xi$ and $C_P$ are constants of integration and

$$G(\tau) = \int_0^\infty \frac{e^{-u^3/3}}{u} \{\exp(iu\tau \sign(\Delta)) - 1\} \, du. \tag{67}$$

From the definition of $C_A$ (Equation (19)) it follows that the constants $C_\xi$ and $C_P$ cannot be independently chosen, but are related by $C_A = gbCP - 2f_BB_\phi B_zC_\xi/\mu \tau_A$.

Straightforward Mac Laurin expansions give absolute convergent power series for all $\tau$ for $G(\tau)$ and $F(\tau)$:

$$F(\tau) = \sum_{n=0}^\infty a_n \tau^n,$$

$$G(\tau) = i\tau \sign(\Delta) \sum_{n=0}^\infty \frac{a_n}{n+1} \tau^n,$$ \tag{68}

where

$$a_n = \frac{3^{n/3}}{3^{2/3}} \Gamma \left(\frac{n+1}{3}\right) \frac{[i \sign(\Delta)]^n}{n!} \tag{69}$$

and $\Gamma$ is the gamma-function. These power series can be used to obtain solutions for $\xi, P'$, and $\xi_\perp$. In particular for $|\tau| \leq 1$, they show that in resistive MHD all the physical quantities take finite values in the dissipative layer and at the ideal resonance position where the ideal MHD solutions diverge. From Equation (69) it immediately follows that the coefficients $a_{2k}$ with even indices are purely real, while the coefficients $a_{2k+1}$ with uneven indices are purely imaginary. This implies that $\text{Re} F(\tau)$ and $\text{Re} G(\tau)$ are even functions of $\tau$, while $\text{Im} F(\tau)$ and $\text{Im} G(\tau)$ turn out to be uneven functions of $\tau$.

The power series given in Equations (68) are not well suited to find out how the resistive MHD solutions have to be connected to the ideal MHD solutions. To this end we determine the asymptotic behaviour of the resistive MHD solutions for $\tau \to \pm \infty$. The asymptotic behaviour of $G(\tau)$ for $\tau \to \pm \infty$ is calculated.
in Appendix B (Equation (95)). With the aid of Equation (95) we easily get the following asymptotic expansions for $\xi_r$ and $P'$ for $\tau \to \pm \infty$:

$$\begin{align*}
\xi_r & \approx \frac{g_B C_A}{\rho B^2 \Delta} \left( \ln |\tau| + \frac{2\nu}{3} + \frac{1}{3} \ln 3 - \frac{i\pi}{2} \text{sign}(\Delta \tau) \right) + C_\xi, \\

P' & \approx \frac{2f_B B_\varphi B_z C_A}{\mu r_A \rho B^2 \Delta} \left( \ln |\tau| + \frac{2\nu}{3} + \frac{1}{3} \ln 3 - \frac{i\pi}{2} \text{sign}(\Delta \tau) \right) + C_P,
\end{align*}$$

(70)

where $\nu$ is the Euler constant. These results provide us with the asymptotic behaviour of $\xi_r$, and $P'$ in the overlap regions where ideal MHD is also valid. In the overlap regions the asymptotic versions of the dissipative solutions (70) and the ideal solutions (23) represent the same solutions. The asymptotic versions recover the logarithmic behaviour of $\text{Re}(\xi_r)$, and $\text{Re}(P')$ already found in ideal MHD in Equation (23), but show that this logarithmic behaviour is only valid away from the ideal resonance position for large $|\tau|$.

Comparison of the asymptotic versions of the dissipative MHD solutions with the ideal MHD solutions enables us to obtain expressions for $\xi_\pm$, and $P'_\pm$ in Equations (23). We obtain

$$\begin{align*}
\xi_\pm & = \frac{g_B C_A}{\rho \Delta B^2} \left( \frac{2\nu}{3} + \frac{1}{3} \ln 3 - \ln \delta_A \mp \frac{i\pi}{2} \text{sign}(\Delta) \right) + C_\xi, \\

P'_\pm & = \frac{2f_B B_\varphi B_z C_A}{\mu r_A \rho \Delta B^2} \left( \frac{2\nu}{3} + \frac{1}{3} \ln 3 - \ln \delta_A \mp \frac{i\pi}{2} \text{sign}(\Delta) \right) + C_P.
\end{align*}$$

(71)

In particular they allow us to obtain the jumps in $\xi_r$ and $P'$:

$$\begin{align*}
[\xi_r] & = -i\pi \frac{g_B C_A}{\rho B^2 |\Delta|}, \\
[P'] & = -i\pi \frac{2f_B B_\varphi B_z C_A}{\rho B^2 \mu r_A |\Delta|}.
\end{align*}$$

(72)

These jumps and the conservation law were first derived by Sakurai, Goossens, and Hollweg (1991a). An important property of resonant Alfvén wave heating is that the jumps are independent of $\eta$. This implies that the amount of absorbed wave energy and the total amount of resistive heating in the dissipative layer are also independent of $\eta$.

The asymptotic behaviour of $F(\tau)$ for $\tau \to \pm \infty$ is calculated in Appendix A (Equation (79)). With the aid of Equation (79) we easily get the following asymptotic expansions for $\xi_\perp$:

$$\xi_\perp \approx \frac{i C_A}{\tau \rho B \delta_A \Delta}.$$  

(73)

This asymptotic version recovers the $\tau^{-1}$ behaviour of $\text{Im}(\xi_\perp)$, already found in ideal MHD in Equation (25), but it shows that this behaviour is only valid away
from the ideal resonance position for $\tau \to \pm \infty$. In order to understand fully the relation between the ideal and resistive solution for $\xi_\perp$ we still have to find out what has happened to the ideal $\delta(s)$ contribution to $\xi_\perp$. To this end we determine the limit of the resistive solution of $\xi_\perp$ for $\delta_A \to 0$, as a function of $s$. We first calculate the Fourier transform $\hat{\xi}_\perp(q)$ of $\xi_\perp(s)$, we then determine the limit of $\hat{\xi}_\perp(q)$ for $\delta_A \to 0$, and we finally calculate the reciprocal Fourier transform to get the limit of $\xi_\perp(s)$.

We have the sequence of equalities

\[
\frac{1}{\delta_A} \hat{F}_1(q) = \frac{1}{\delta_A} \int_{-\infty}^{\infty} e^{-isq} ds \int_0^{\infty} \exp(i \text{sign}(\Delta)us/\delta_A - u^3/3) du = \\
= \frac{1}{\delta_A} \int_0^{\infty} e^{-u^3/3} du \int_{-\infty}^{\infty} \exp(is(\text{sign}(\Delta)u/\delta_A - q)) ds = \\
= \frac{2\pi}{\delta_A} \int_0^{\infty} \delta(\text{sign}(\Delta)u/\delta_A - q) e^{-u^3/3} du = \\
= 2\pi H(\text{sign}(\Delta)q) \exp(-\text{sign}(\Delta)\delta_A^3 q^3/3),
\]

where $\delta$ is the delta-function and $H$ the Heaviside function. We see that

\[
\lim_{\delta_A \to 0} \frac{1}{\delta_A} \hat{F}_1(q) = 2\pi H(\text{sign}(\Delta)q) = \pi \{1 + i \text{sign}(\Delta q)\}.
\]

The reciprocal Fourier transform of 1 and of $\text{sign}(q)$ are $\delta(s)$, and $(i/\pi)\mathcal{P}(1/s)$ respectively, where $\mathcal{P}$ denotes the principal Cauchy value. These results enable us to find the desired limit for $\xi_\perp$,

\[
\lim_{\delta_A \to 0} \xi_\perp = \frac{C_A}{\rho B} \left[ \frac{\pi}{|\Delta|} \delta(s) + \frac{i}{\Delta} \mathcal{P}\left(\frac{1}{s}\right) \right].
\]

Thus the $\delta(s)$ contribution to $\xi_\perp$ can be thought of as arising from $\text{Re} \ F(\tau)$, which is an even function of $\tau$. The amplitude of $\text{Re} \ \xi_\perp$ at $s = 0$ is proportional to $1/\delta_A$, while $\text{Re} \ \xi_\perp$ becomes small if $|s| \geq \delta_A$. The area under $\text{Re} \xi_\perp(s)$ is thus independent of $\delta_A$, leading to the $\delta$-function as $\delta_A \to 0$.

The real and imaginary parts of $F(\tau)$ and $G(\tau)$ are shown in Figures 1 and 2. Except for the normalization, the curves for $F(\tau)$ appear to be identical to results obtained previously by Hollweg (1987b) and Hollweg and Yang (1988). Hollweg considered the structure of the viscous dissipative layer in an incompressible plasma, while Hollweg and Yang considered the viscous dissipative layer around the Alfvén resonance in a compressible plasma. In both cases the geometry was planar. Our present work shows that the same behaviour occurs also in cylindrical geometry for the resistive layer.
In Section 3 we have seen that in ideal MHD the dominant dynamics of resonant Alfvén waves resides in the perpendicular component of the displacement $\xi_\perp$. The logarithmic singularity and the jump contribution to the radial component of the displacement $\xi_r$ are overruled by the $s^{-1}$ singularity and the $\delta$-function contribution to $\xi_\perp$. In resistive MHD all these singularities disappear and all physical variables take finite values. To determine the dominant dynamics in resistive MHD we
need to make an estimate of the relative importance of $\xi_r$ and $\xi_\perp$ close to the ideal resonance position. For $|\tau| \approx 1$ both $F(\tau)$ and $G(\tau)$ are nonzero and finite numbers, so that from Equation (66) we obtain $\xi_\perp \sim |k_\perp \delta_A|^{-1} \xi_r$ for $|\tau| \sim 1$, where $k_\perp = g_B/B$. Let us now introduce $L$ as a typical length scale for the variations of the equilibrium quantities so that $|\Delta| \approx \omega^2/L$ and let us assume that the length scale of the perturbation in the magnetic surfaces perpendicular to the magnetic
field lines is comparable to the length scale of the equilibrium quantities so that \( k_\perp \approx L^{-1} \). We then find that

\[
\frac{|\xi_\perp|}{|\xi_r|} \sim R_m^{1/3},
\]

(77)

where \( R_m = \omega L^2/\eta \) is the magnetic Reynolds number. Since \( R_m \) is very large, of the order of \( 10^{10} - 10^{12} \) in the solar atmosphere, Equation (77) implies that the dominant dynamics continues to reside in the perpendicular components as was found by Poedts, Kerner, and Goossens (1989a) and Poedts, Goossens, and Kerner (1989b, 1990a).

These analytical results provide us with the spatial solutions in the dissipative layer and in the two overlap regions where ideal MHD is valid. As such they enable us to understand the basic physics of resonant Alfvén waves. They also help us with the interpretation of the results of large-scale numerical simulations. Finally the jump conditions and the conservation law make it possible to determine the absorption of Alfvén waves without having to solve the dissipative MHD equations. This procedure was used by Sakurai, Goossens, and Hollweg (1991b) for studying the absorption of acoustic oscillations in sunspots. It was generalized to stationary equilibrium states by Goossens et al. (1992). Goossens and Hollweg (1993) used this scheme to obtain conditions for maximal and total absorption and to explain the variation of the spatial solutions with frequency.

6. Conclusions

This paper has extended the analysis of Sakurai, Goossens, and Hollweg (1991a) on resonant Alfvén waves on nonuniform magnetic flux tubes. The equations of linear resistive MHD for linear displacements superimposed on a static equilibrium state have been simplified for displacements that correspond to resonant Alfvén waves in ideal MHD. We have used series expansions of the coefficient functions around the ideal Alfvén resonance to obtain simplified versions of the linear resistive equations that govern the Alfvén waves in a region which contains the dissipative layer and two overlap regions where ideal MHD is also valid. We have solved this set of linear resistive MHD equations in a consistent manner. We have first shown that the fundamental conservation in ideal MHD found by Sakurai, Goossens, and Hollweg (1991a) remains valid in dissipative MHD. This implies that the total resistive heating in the dissipative layer, and the amount of absorbed wave energy, are independent of the resistivity. We have obtained compact analytical solutions for \( \xi_r, P' \), and \( \xi_\perp \) which allow a straightforward mathematical and physical interpretation of resonant Alfvén waves in the dissipative layer. But we have also shown how to determine the absorption of Alfvén waves without having to solve the dissipative MHD equations.
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Appendix A

In this appendix we calculate asymptotic behaviour of the functions $F_1(\tau)$, $F_2(\tau)$, and $F_3(\tau)$. Integration by parts gives

$$F_1(\tau) = \frac{i \text{sign}(\Delta)}{\tau} - \frac{i \text{sign}(\Delta)}{\tau} \int_0^\infty u^2 \exp(iu\tau \text{sign}(\Delta) - u^3/3) \, du. \quad (78)$$

Integration by parts then enables us to show that the second term in (78) is of the order of $\tau^{-2}$, so that the asymptotic behaviour of $F_1(\tau)$ is

$$F_1 \approx \frac{i \text{sign}(\Delta)}{\tau} \quad \text{for} \quad |\tau| \to \infty. \quad (79)$$

In particular this implies that

$$\lim_{\tau \to \pm \infty} F_1(\tau) = 0. \quad (80)$$

Next we consider $F_2(\tau)$. Let us first rewrite $F_2(\tau)$ as

$$F_2(\tau) = \exp(2\pi i/3) \int_0^\infty \exp \left[ -\text{sign}(\Delta) u\tau \exp(\pi i/6) - u^3/3 \right] \, du. \quad (81)$$

In order to calculate the asymptotic behaviour of $F_2$ at $\tau \to \infty$ we consider the countour that starts at the origin and runs along the real axis up to $(R, 0)$
(path 1), then runs along the circle with the origin as centre and with radius $R$ up to $(R \cos(\pi/12), R \sin(\pi/12))$ (path 2), and finally runs from there along a straight line back to the origin. It is easy to see that for $R \to \infty$ the integral with integrand

$$
\exp \left[ -\text{sign}(\Delta) u \tau \exp(\pi i/6) - u^3/3 \right]
$$

along the path 2 goes to zero. Since the integral along the complete contour is equal to zero this means that the integral along infinite path 1 is equal to the integral along the infinite path 3. As a consequence we can write the expression for $F_2$ as

$$
F_2 = \sqrt{-\tau \text{sign}(\Delta)} \exp(3\pi i/4) \times
$$

$$
\times \int_{0}^{\infty} \exp \left[ (-\tau \text{sign}(\Delta))^{3/2} \exp(\pi i/4)(u - u^3/3) \right] du .
$$

We now use the method of steepest descent (see, for instance, Nayfeh, 1981). The function $u - u^3/3$ has a maximum at $u = 1$. Then in accordance with the method of steepest descent we make the substituion $u = 1 + \bar{u}$, and we use the approximation $u - u^3/3 \approx 2/3 - 2\bar{u}^2$, and we take the lower integration limit to be $-\infty$ instead of $-1$. As a result we have

$$
F_2 \approx \sqrt{-\tau \text{sign}(\Delta)} \exp \left( \frac{2}{3} (-\tau \text{sign}(\Delta))^{3/2} \exp(\pi i/4) + \frac{3\pi i}{4} \right) \times
$$

$$
\times \int_{-\infty}^{\infty} \exp \left( -2\bar{u}^2 (-\tau \text{sign}(\Delta))^{3/2} \exp(\pi i/4) \right) d\bar{u} .
$$

(83)

Calculation of the integral in (83) leads to the required formula for the asymptotic behaviour of $F_2(\tau)$ for $\tau \to -\infty \text{sign}(\Delta)$,

$$
F_2 \approx \sqrt{\pi} (-\tau \text{sign}(\Delta))^{-1/4} \times
$$

$$
\exp \left( \frac{2}{3} (-\tau \text{sign}(\Delta))^{3/2} \exp(\pi i/4) + \frac{5\pi i}{8} \right) ,
$$

(84)

so that

$$
\lim_{\tau \to -\infty \text{sign}(\Delta)} |F_2(\tau)| = \infty .
$$

(85)

In addition it is easy to see that

$$
\lim_{\tau \to +\infty \text{sign}(\Delta)} |F_2(\tau)| = 0 .
$$

(86)

To calculate the asymptotic behaviour of $F_3(\tau)$ for $\tau \to +\infty \text{sign}(\Delta)$ we simply note that $F_3(\tau) = F_2(\tau)$, where the asterisk denotes the complex conjugate. Then we immediately obtain

$$
F_3 \approx \sqrt{\pi} (\tau \text{sign}(\Delta))^{-1/4} \times
$$

$$
\times \exp \left( \frac{2}{3} (\tau \text{sign}(\Delta))^{3/2} \exp(-\pi i/4) - \frac{5\pi i}{8} \right) .
$$

(87)
This implies that
\[ \lim_{\tau \to +\infty \text{ sign}(\Delta)} |F_3(\tau)| = \infty . \]  
(88)

In addition it is easy seen that
\[ \lim_{\tau \to -\infty \text{ sign}(\Delta)} F_3(\tau) = 0 . \]  
(89)

In summary this appendix shows that out of the three functions \( F_1(\tau), F_2(\tau), \) and \( F_3(\tau), \) only \( F_1(\tau) \) remains bounded for \( \tau \to \pm \infty. \)

**Appendix B**

In this appendix we determine the asymptotic behaviour of \( G(\tau) \) for \( \tau \to \pm \infty. \) We can rewrite \( G(\tau) \) as
\[ G(\tau) = \int_0^\infty \frac{\cos(u\tau) - 1}{u} e^{-u^3/3} du + i \text{ sign}(\Delta) \int_0^\infty \frac{\sin(u\tau)}{u} e^{-u^3/3} du . \]  
(90)

Let us first consider the asymptotic behaviour of the imaginary part of \( G(\tau). \) We introduce the new variable \( u|\tau| = \bar{\tilde{u}} \) and we omit the bar in what follows. We obtain
\[ \int_0^\infty \frac{\sin(u\tau)}{u} e^{-u^3/3} du = \text{ sign}(\tau) \int_0^\infty \frac{\sin u}{u} e^{-u^3/3|\tau|^3} du = \]
\[ = \text{ sign}(\tau) \int_0^\infty \frac{\sin u}{u} du + o(1) = \]
\[ = -\frac{i\pi}{2} \text{ sign}(\tau) + o(1) , \]  
(91)

where \( o(1) \) means value vanishing at \( |\tau| \to \infty. \)

We now proceed to the real part. It is straightforward to obtain the following sequence of equalities:
\[ \int_0^\infty \frac{1 - \cos(u\tau)}{u} e^{-u^3/3} du = \]
\[ = \left( \int_0^{\tau^{-2}} + \int_{\tau^{-2}}^\infty \right) \frac{1 - \cos(u\tau)}{u} e^{-u^3/3} du = \]
\[
\frac{1}{|\tau|} \int_0^1 \frac{1 - \cos u}{u} e^{-u^3/3|\tau|^3} du + \int_{|\tau|}^{\infty} e^{-u^3/3} du - \int_{|\tau|}^{\infty} \frac{\cos u}{u} e^{-u^3/3|\tau|^3} du = \\
\frac{1}{3} \int_{-\infty}^{-1/3|\tau|^6} e^u du - \int_{1/|\tau|}^{\infty} \frac{\cos u}{u} du + o(1) = \\
= -\frac{1}{3} \text{Ei}(-1/3|\tau|^6) + \text{Ci}(1/|\tau|) + o(1) ,
\]

where \text{Ei} is integral exponent and \text{Ci} integral cosinus. These functions have the following asymptotic expansions (see, for instance, Korn and Korn, 1961):

\[
\text{Ei}(y) = \ln(-y) + \nu + O(y) \quad \text{for} \quad y \to -0 , \\
\text{Ci}(y) = \ln y + \nu + O(y) \quad \text{for} \quad y \to +0 ,
\]

where \nu is Euler’s constant. With the aid of (93) we obtain

\[
\int_0^1 \frac{1 - \cos u}{u} e^{-u^3/3} du = \ln |\tau| + \frac{2\nu}{3} + \frac{1}{3} \ln 3 + o(1) .
\]

When we substitute (91) and (94) into (90) we finally get the desired asymptotic expression for \(\tau \to \pm \infty\)

\[
G(\tau) = -\ln |\tau| - \frac{2\nu}{3} - \frac{1}{3} \ln 3 + i \frac{\pi}{2} \text{sign}(\Delta \tau) + o(1) .
\]

References