The mixing-length parameter for solar-type convection zones inferred from hydrodynamical models of the surface layers

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Abstract: Based on detailed 2D numerical radiation hydrodynamics calculations of time-dependent compressible convection, we have studied the dynamics and thermal structure of the convective surface layers of stars representing the Sun in different evolutionary stages. We demonstrate that our hydrodynamical models can be used to determine the value of the mixing length parameter $\alpha$ which gives the correct depth of solar-type convection zones when used with a conventional stellar evolution code. We find that $\alpha$ is approximately constant ($\approx 1.65$) during the main-sequence evolution of the Sun while it increases significantly during its ascent towards the red giant branch.

Introduction

Standard evolutionary model sequences of low-mass stars are constructed by calibrating the mixing-length parameter $\alpha$ and the initial helium content $Y$ such that a solar model fits the radius and luminosity of the present Sun. Once fixed, it is assumed that $\alpha$ remains the same during all evolutionary stages as well as for stars of different mass. Clearly, one can pose the question whether this procedure is justified and convection in the solar main-sequence phase adequately represents all other situations. In evolutionary models $\alpha$ serves mainly as a technical parameter to calculate the entropy jump between the stellar surface and the deep, nearly adiabatic layers in the convective envelope. In this paper we demonstrate that under certain assumptions concerning the topology of solar-type convection zones, hydrodynamical models of stellar surface convection provide direct information about this entropy jump, offering the possibility of an independent determination of $\alpha$. Using this method, we have derived the value of $\alpha$ during the evolution of the Sun from the ZAMS through its ascent towards the RGB.

Hydrodynamical models of solar-type surface convection

We have obtained detailed 2-dimensional models of the surface layers of solar-type stars from extensive numerical simulations solving the time-dependent, non-linear equations of hydrodynamics for a stratified compressible fluid. The calculations take into account a realistic equation of state (including the ionization of H and He as well as formation of H\textsubscript{2}-molecules) and use an elaborate scheme to describe multi-dimensional, non-local, frequency-dependent radiative transfer. Similar to classical model atmospheres, the hydrodynamical models are characterized...
Figure 1: Depth dependence of the entropy in the solar surface layers as obtained from hydrodynamical simulations (L71D07) performed on a 140 (x) by 71 (z) grid with frequency-dependent radiative transfer. The mean entropy (horizontal and temporal average) is shown in panel (a), spatially resolved entropy profiles in panel (b). Geometrical height zero corresponds to $\tau_{Ross} = 1$.

by effective temperature $T_{eff}$, acceleration of gravity $\log g$, and chemical composition. They include the photosphere as well as part of the subphotospheric layers, with an open lower boundary, allowing a free flow of gas out of and into the model. A fixed specific entropy $s^*$ is (asymptotically) assigned to the gas entering the simulation volume from below. The value adopted for $s^*$ uniquely determines the effective temperature of the hydrodynamical model. For details about the physical assumptions, numerical method and characteristics of the resulting convective flows for different types of stars see Ludwig et al. (1994) and Freytag et al. (1995).

From the surface to the base of a convection zone

Fig. 1a shows the mean entropy as a function of depth obtained from a hydrodynamical surface model of the Sun by averaging over horizontal planes and over time. As in this example, our models in general do not extend deep enough to include those layers where the mean stratification of the convection zone becomes adiabatic. While the mean entropy stratification of the hydrodynamical models does not permit a direct determination of the entropy corresponding to the adiabat of the deep convection zone, the spatially resolved entropy profiles contain additional information. Fig. 1b displays the entropy profiles for an arbitrary instant of the sequence from which the mean stratification in Fig. 1a was computed. The granular convection pattern at the surface of solar-type stars is formed by broad hot upflows accompanied by concentrated cool downdrafts. Fig. 1b shows a remarkable entropy plateau in the subsurface layers, indicating that — in contrast to the narrow downdrafts — the gas in the central regions of the broad ascending flows is still thermally isolated from its surroundings. Neither radiative losses nor entrainment by material of low entropy can produce significant deviations from adiabatic expansion until immediately below the radiating surface layers. The height of the entropy plateau is essentially independent of time and corresponds to $s^*$.

We suggest that $s^*$ may be identified directly with the entropy of the deep, adiabatic convective layers. This idea has been put forward by Steffen (1993) and is based on the qualitative picture of solar-type convection zones proposed by Stein and Nordlund (1989) which is fundamentally different from MLT assumptions. According to this scenario the downdrafts continue all the way from the surface to the bottom of the convection zone, merging into fewer and stronger currents at successively deeper levels. The flow closes only near the base of the convective envelope. Most of the gas elements starting from the bottom of a deep convection zone overturn into neighboring downflows before reaching the surface. Only a small fraction of gas
continues to the surface, reaching the layers corresponding to the location of the lower boundary of our hydrodynamical models essentially without entropy losses, following an adiabat almost up to the visible surface. Hence, $s^*$ obtained from the simulations is the entropy of the warm, ascending gas throughout the convection zone. This, in turn, is very nearly equal to the mean (horizontally averaged) entropy near the base of the convection zone because (i) the downflows are markedly entropy-deficient only near the surface and become continuously diluted by overturning entropy-neutral gas as they reach greater depths, and (ii) the fractional area occupied by the downdrafts decreases with depth.

The entropy $s^*$ can be converted to an equivalent $\alpha$ by constructing envelope models (not subject to central boundary conditions) based on MLT which match $s^*$ in the deep, adiabatic layers. It should be stressed that the $\alpha$'s derived this way are well defined only in the context of the specific formulation adopted for MLT (here we adopt the formulation given in Kippenhahn & Weigert, 1990). For the inward directed integration of the envelope model one needs the radius of the star as an additional piece of information. Since here we are primarily interested in the Sun, we calculated the radius for given log $g$ assuming a one solar-mass star. In the envelope models we have adopted a chemical composition as similar as possible to that used in the hydrodynamical simulations: $X=0.7035$, $Y=0.2795$, $Z=0.017$ (by mass). OPAL opacities for the Grevesse 1991 heavy element mixture (Rogers & Iglesias, 1992) supplemented by low temperature opacities according to Weiss et al. (1990) have been used. In low density regions (<0.01 g/ccm) the MHD (Mihalas et al., 1988), at higher densities a simpler equation of state has been adopted that accounts for partial ionization according to the Saha-Boltzmann equations as well as for partial electron degeneracy.

**Helioseismology, Hydrodynamics, and Stellar Evolution**

The idea described above can be tested by comparison with helioseismological data of the present Sun. In addition one can check what $\alpha$ results from an evolutionary calibration. In order to perform such an absolute comparison, we have repeated the study by Steffen (1993) using the data from our latest hydrodynamical solar convection model computed with frequency-dependent radiative transfer. The results are summarized in the first row of Tab. 1. The uncertainty of $s^*$ reflects the variation over several hydrodynamical models differing in numerical details. It translates into the given uncertainties of $\alpha$ and depth of the convection zone. The second row of the table shows the results of a calibrated solar model from standard evolutionary calculations. The evolutionary model is based on the same input physics as the envelope models, which are used to determine $\alpha$ hydrodynamically, so all values of $\alpha$ given in Tab. 1 are directly comparable. The last row of the table gives the values of the surface abundance of helium and the depth of the convection zone as determined by Helioseismology. The helioseismological data were taken from Hernandez & Christensen-Dalsgaard (1994) and Christensen-Dalsgaard et al. (1991). Within the errors, the theoretical models give a consistent picture while there is a discrepancy with helioseismological measurements. Additional checks have shown that remaining uncertainties in the equation of state — primarily the neglect of Coulomb interaction — are of minor importance here. However, the discrepancy is rather small so that the basic scenario appears to be correct and deserves further investigation. In view of the improved microphysics used in the present investigation, the almost perfect agreement found by Steffen (1993) is now considered as a coincidence.

When taking the helioseismologically measured helium abundance at face value, the abundance in the theoretical models is too high. While the surface abundance of helium is a free parameter in the hydrodynamical models, its value is fixed in the evolutionary models by the
Figure 2: Mixing-length parameter $\alpha$ derived from hydrodynamical model atmospheres in the $\log T_{\text{eff}} - \log L$ plane. For each model a diamond indicates its position together with the derived values of the entropy of the adiabat ($s^*$ in $10^8$ erg/g/K, upper number) and the corresponding mixing-length parameter ($\alpha$, lower number). $s^*$ has been translated into $\alpha$ assuming a one solar-mass star, lines of constant $\log g$ are shown for this mass. The uncertainty of the determination of $\alpha$ amounts to ±0.02. An evolutionary track of the Sun is plotted with tick marks indicating the age from the ZAMS; the first mark corresponds to 0.1 Ga followed by marks in steps of 0.1 Ga. The first long mark corresponds to 1.0 Ga followed by marks in steps of 1.0 Ga.
<table>
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<th></th>
<th>X</th>
<th>Y</th>
<th>$s^* [10^9 \text{erg/g/K}]$</th>
<th>$\alpha$</th>
<th>$R_{\odot}/R_{\odot}$</th>
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<td>Hydrodynamics</td>
<td>0.7035</td>
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<td>1.645 ± 0.06</td>
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<td>1.62</td>
<td>0.731</td>
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<td>Helioseismology</td>
<td>-</td>
<td>0.242 ± 0.003</td>
<td>-</td>
<td>-</td>
<td>0.713 ± 0.003</td>
</tr>
</tbody>
</table>

Table 1: Results from hydrodynamical and evolutionary model calculations as well as helioseismological measurements. X: hydrogen abundance, Y: helium abundance, $s^*$: entropy of adiabat, $\alpha$: mixing-length parameter, $R_{\odot}/R_{\odot}$: location of the base of the convective envelope.

calibration procedure, so it can be changed only if additional physical effects are incorporated. Indeed, evolutionary models including diffusion of helium show a significant depletion of helium and deeper convective envelopes. Our first hydrodynamical model with a low helium abundance of $Y = 0.240$ (keeping $Z/X$ fixed) led to $R_{\odot}/R_{\odot} = 0.719$ which is still somewhat (0.08 $H_p$) too shallow. Nevertheless, it is interesting that the solar helium abundance can be predicted from such models offering a new theoretical method independent from evolutionary calculations.

**Is $\alpha$ a constant during the solar evolution?**

Figure 2 presents the results for the hydrodynamical determination of $\alpha$ near the evolutionary track of the Sun. To reduce the computational effort the radiative transfer has been treated in the grey approximation. This shortcoming is tolerable here since we intend to study the differential behavior of $s^*$ and $\alpha$ across the HRD. When the simulations have reached a statistically stable state, $s^*$ shows some remaining temporal variations. These variations have been taken as an estimate of the uncertainty of $s^*$ translating into an uncertainty of $\alpha$ to ±0.02. Figure 2 shows that the hydrodynamical calibration is marginally consistent with the assumption of a constant $\alpha$ during the main sequence evolution of the Sun. But we find a significant increase of $\alpha$ during the evolution towards the red giant branch.

As a final remark we want to point out that the $\alpha$'s derived above produce only the “right” entropy jump from the surface to the deep convection zone. This does not imply that the structure of the superadiabatic surface layers is also represented correctly in such mixing length models. Moreover, it is clear that our method does not allow for overshooting at the base of the convection zone, one of the possible reasons for the slight differences remaining between theoretical models and helioseismological measurements.

**References**


