PARTIAL REDISTRIBUTION IN MULTILEVEL ATOMS. I. METHOD AND APPLICATION TO THE SOLAR HYDROGEN LINE FORMATION

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ABSTRACT

We present a robust method for solution of multilevel non-LTE line transfer problems including the effects of partial frequency redistribution (PRD). This method allows the self-consistent solution for redistribution of scattered line photons simultaneously in multiple transitions of a model atom, including the effects of resonant Raman scattering ("cross-redistribution") among lines sharing common upper levels. The method is incorporated into the framework of the widely used non-LTE complete redistribution code MULTI. We have applied this method to the problem of transfer in hydrogen lines in a plane-parallel solar model atmosphere, including cross-redistribution between the \(\text{H}_\alpha\) and \(\text{L}\beta\), using general redistribution functions for the \(\text{L}_\alpha\) and \(\text{L}\beta\) lines which are not restricted by the impact approximation. The convergence properties of this method are demonstrated to be comparable to that of the equivalent complete redistribution problem. In this solar model, PRD in the \(\text{L}_\alpha\) line produces the dominant influence on the level populations. It changes considerably the populations of the excited states of hydrogen, as well as the proton number density, in the middle and upper chromosphere, owing to modification of the \(\text{L}_\alpha\) wing radiation.

The population of the hydrogen ground state undergoes only modest changes, however. The influence of cross-redistribution and PRD in \(\text{L}\beta\) has a much smaller influence on the level populations but a considerable influence on the wing intensity of the \(\text{L}\beta\) line.

Subject headings: atomic processes — line: formation — radiative transfer — Sun: chromosphere

1. INTRODUCTION

Coherent scattering of photons in the wings of strong resonance lines has long been known to significantly alter the shapes of their profiles emerging from stellar atmospheres. The profile shapes of strong resonance lines provide important and unique diagnostics of the thermodynamic properties of stellar chromospheres. When multiple scattering of line photons affects the line profile shapes, it is necessary to include partial frequency redistribution (PRD) of line photons into numerical procedures which simulate the non-LTE radiative transfer. Linsky (1985) and Hubeny (1985b) presented a historical review to that date of the implementation of these PRD effects into synthesis of stellar line profiles. Adding this additional physics into the line transfer problem allows one to make more meaningful comparisons of profiles predicted by models with those actually observed. Thus, the major application of PRD is to these resonance lines which are used for detailed line synthesis and modeling of stellar atmospheres.

For several reasons, the hydrogen PRD transfer problem in stellar atmospheres is at once important and challenging:

- **Effect of PRD on the model atmosphere.** — The effects of PRD on line fluxes, i.e., net radiative gains or losses within the atmosphere, are not typically very large, so that complete frequency redistribution (CRD) is usually a good approximation. However, multiple scattering in the wings of the \(\text{L}_\alpha\) line of hydrogen can lead to significant changes in the flux of that line, which in turn can affect the charge balance and gas pressure through a modification of the ionization balance of hydrogen. For this reason, it is prudent to incorporate PRD in the hydrogen problem when generating detailed models of stellar atmospheres.

- **Cross-redistribution in \(\text{L}\beta\).** — The \(\text{L}_\alpha\), \(\text{L}\beta\), and \(\text{H}\alpha\) lines of hydrogen are themselves important diagnostics for stellar atmospheres, so their synthesis must be treated with care.

Partial redistribution in the hydrogen atom is complicated by "cross-redistribution" (Milkey, Shine, & Mihalas 1975a; more precisely, resonant Raman scattering) among different transitions: the situation where a photon scattered in one line reappears in another line before the atom undergoes a redistributing collision. This effect is most prominent in other species such as \(\text{Ca}\) \(\Pi\), where the resonance lines connect, through their upper levels, to the infrared lines, which have metastable lower levels. In hydrogen, the \(\text{L}\beta\) line shares an upper level with the \(\text{H}\alpha\) line which effectively redistributes photons (Heinzel & Hubeny 1985). Owing to the large frequency difference between these two lines, the redistributed line photons in the bright far wings of \(\text{H}\alpha\) can be remitted in the inner wings of \(\text{L}\beta\), thereby affecting the \(\text{L}\beta\) emission profile locally in frequency. This effect is complicated further by the fact that Raman scattering is only possible for some of the fine-structure levels involved in the \(\text{H}\alpha\) and \(\text{L}\beta\) transitions.

- **Unique character of hydrogen redistribution functions.** — Further, the atomic redistribution function for hydrogen lines is unlike that of most other species in that the impact approximation for collisional redistribution is not applicable. This leads to frequency dependence of the broadening parameters (Cooper, Ballagh, & Hubeny 1989, hereafter CBH). A detailed description of this redistribution function should be used to

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compute detailed profiles of the hydrogen lines arising from a stellar atmosphere.

*Extreme coherence in Lz.*—The Lz transfer problem in solar-type stars presents an extreme test of computational methods for PRD transfer. Because Lz is extremely opaque owing to hydrogen being the dominant atomic species, the line formation region spans a wide range of optical depths in stellar chromospheres in which the densities are lower than in the formation regions of most other resonance lines. The opacity of Lz, along with its position in the hydrogen atom, leads to a situation where the scattering in the wings of this line is highly coherent.

For some time, “exact” techniques for calculation of partial redistribution in frequency and angle have been in existence. Methods such as that of Milkey, Shine, & Mihalas (1975a) treat the redistribution function in a general way and include multilevel interactions, both for the overall excitation of the model atom and for the cross-redistribution. The drawback of this approach is its computational inefficiency. Seeing the need for more computationally efficient methods to calculate PRD line transfer, Scharmer (1983) developed a method of solving the two-level PRD line transfer problem using operator perturbation techniques, and Uitenbroek (1989) generalized this method to multilevel atoms including the possibility of cross redistribution. Although this new method is much faster per iteration than the previous methods which treat the redistribution by brute-force linearization, it employs an approximate operator based on a frequency-independent source function. When PRD effects are extreme in strong, isolated resonance lines such as Lz or the Mg II resonance lines, the convergence of this method is very slow owing to the poor representation of the true operator by the approximate CRD operator. Therefore, we find a need for a robust and general method of solving the PRD line transfer problem in the context of multilevel atoms.

Our approach here is to implement the method of Hubeny (1985a) into the framework of the widely used Scharmer-Carlsson CRD method of solving the multilevel transfer problem (hereafter SC method; Scharmer & Carlsson 1985, Carlsson 1986). Hubeny (1985a) proposed an iterative procedure whereby those transitions within a model atom that are to be treated in PRD are computed using an equivalent two-level atom (ETLA) approach.

The effects specific to PRD are computed within the ETLA formalism and then communicated to the rest of the atom via a single frequency-dependent parameter: \( \rho(v) = \psi(v)/\phi(v) \), the ratio of the emission to absorption profile coefficients for this transition. Relatively minor modifications to the CRD multilevel code allow one to fix the frequency of the source function based on \( \rho(v) \), then recompute multilevel coupling coefficients which are used again in the equivalent two-level PRD code to update \( \rho(v) \). Thus, the influence of PRD on the global solution is lagged with respect to the iteration on the multilevel atom.

This approach works very well for several reasons: (1) The basic reason is that the radiative rates, which determine populations, primarily are sensitive to the radiation intensity in the line cores, while the PRD effects influence the radiation intensity mainly in the line wings and much less in the line cores. This indicates that, although the effects of PRD on level populations are not negligible, they are nevertheless small enough to justify “lagging” the PRD corrections in the iterative scheme. (2) The largest source of nonlinearity in the multilevel problem is contained within the nonlinear coupling among levels and transitions. This is treated in a fully linearized, self-consistent fashion. (3) The effects of PRD on the level populations (i.e., on the net radiative rates) are relatively small. (4) The frequency dependence of \( \rho(v) \) is modest compared with that of either \( \psi(v) \), \( \phi(v) \), or the redistribution functions themselves, and it exhibits only a slight dependence upon the level populations; therefore, it does not undergo large changes between iterations. (5) The nonlinear dependence of the equivalent two-level line transfer on the radiation field enters only through deviations of the stimulated emission profile from the absorption profile. In nearly every case of a stellar atmosphere, stimulated emissions are effectively negligible.

We will show that this new method converges rapidly, even for the difficult multilevel PRD problems described above, in spite of the fact that the PRD effects are not fully linearized within the multilevel context.

We also stress that the method is fully modular. It can be implemented easily in any global numerical scheme capable of treating a general multilevel line formation problem, such as either methods based on complete linearization (CL; Auer & Mihalas 1969, Auer 1973, Mihalas 1978, Heinzel, Gouttebroze, & Vial 1987), or accelerated lambda iteration (ALI; Scharmer & Carlsson 1985, Werner & Husfeld 1985), or a hybrid scheme (Hubeny & Lanz 1995). Likewise, any numerical method capable of treating a PRD problem in the equivalent two-level atom (ETLA) formalism can be used to determine \( \rho(v) \) in the formal solution step. Again, one may use either a direct method (e.g., a Feautrier scheme with a frequency coupling treated explicitly as we have done here), or a method based on ALI (Scharmer 1983; Auer & Paletou 1994). In this paper, we present an implementation of our scheme within the context of the multilevel program MULTI (Carlsson 1986), and using for a formal ETLA PRD solution a direct (Feautrier) method. We plan to replace the latter scheme by a faster ALI method in a near future.

The problem of formation of the hydrogen spectrum in the solar chromosphere poses a stringent test of any PRD computational scheme. We have noted above that the Lz line is an extremely opaque, isolated resonance line for which one expects the highest degree of coherent scattering of wing photons among all solar lines. The influence of cross-redistribution in the Lβ line noted above, along with the unique character of the hydrogen redistribution functions, represents perhaps the most complicated PRD situation one must face. On top of this, even the multilevel CRD hydrogen transfer in the solar atmosphere is among the most challenging non-LTE numerical problems to bring to convergence because the level populations change extremely rapidly where hydrogen is ionizing in the solar chromosphere.

From the viewpoint of using line profile synthesis to extract information about the structure of the solar atmosphere, the observed shapes of most strong solar lines forming in the chromosphere are not well represented in detail by profiles from synthetic models, even when PRD effects are included in the transfer computations. The solar Lyman lines are no exception. The likely source of most of this discrepancy must be found in the dynamic and inhomogeneous nature of the chromosphere. But, unlike profiles of lines from the minority species, the wings of the Lyman lines have also been difficult to match. Gouttebroze et al. (1978b), Basri et al. (1979), Vernazza, Avrett, & Loeser (1981), and Rousset-Dupré (1983) used various approximations to the PRD transfer, including partial
coherent scattering and ad hoc limits on the coherence fraction of Lz in attempts to match the observed profiles with those predicted by various models. All these approaches may be questioned on the basis of validity, both of their treatment of PRD, and of the redistribution functions that were used. Cooper, Ballagh, & Hubeny (1988) described the redistribution functions of hydrogen (later set forth in detail by CBH) which are essentially “complete” for the purposes of astrophysical plasmas and carried out illustrative computations of the Lz and Lf with these redistribution functions. However, they did not iterate the statistical equilibrium equations to full self-consistency. In each of these computations, the predicted wing intensity of Lβ line drops far more rapidly with distance from line center than does the observed profile. We note that the general redistribution functions of Cooper et al. (1988), including cross-redistribution, predict much larger wing intensities than those predicted from an isolated Lβ line. Our aim in this paper is then to carry out the hydrogen transfer problem for a solar model in full generality and self-consistency so that we may evaluate the magnitude of error introduced by neglecting some of these known PRD effects.

2. THE METHOD

The method has been described in detail by Hubeny (1985a). However, that paper does not consider cross-redistribution explicitly. We shall therefore present a brief description of the method here; the reader should consult Hubeny (1985a) for more details.

2.1. Emission and Absorption Coefficient

A general emission coefficient for a transition $j \rightarrow i$ ($E_j > E_i$, where $E_{ij}$ are the atomic level energies; we will denote this property simply as $j > i$) is given by Hubeny, Oxbiuuis, & Simonneau (1983, hereafter HOS) and CBH:

$$\frac{4\pi}{hv} \eta_{ji}(v) = n_j A_{ji} \phi_{ji}(v) + B_{ij} n_i \left[ \tilde{R}_{ij}(v) - \tilde{I}_{ij} \right] \phi_{ij}(v) + \sum_{k \neq i \neq j} B_{ik} n_i \left[ \tilde{P}_{kj}(v) - \tilde{I}_{kj} \right] \phi_{jk}(v),$$  

(1)

where $\eta_{ji}$ is the emission coefficient for the transition $j \rightarrow i$ as a function of frequency, $v$; $A_{ji}$ and $B_{ij}$ are the Einstein coefficients for spontaneous emission and absorption, respectively, $n_j$ and $n_i$ are atomic level populations, and $\phi_{ij}(v)$ is the absorption profile. Further,

$$I_{ij} = \int J(v) \phi_{ji}(v) dv$$  

(2)

is the frequency-averaged mean intensity of radiation for the transition $i \rightarrow j$. Similarly,

$$\tilde{R}_{ij}(v) = \int R_{ij}(v', v) J(v') dv' \phi_{ij}(v)$$  

(3)

is the so-called redistribution (or scattering) integral. $R_{ij}$ is the redistribution function for the scattering $i \rightarrow j \rightarrow i$, which represents a joint probability that a photon in the frequency range $(v', v' + dv')$ is absorbed, and another photon at $(v, v + dv)$ is reemitted. We note that in the case of complete redistribution (CRD), where $R_{ij}(v', v) = \phi_{ij}(v') \phi_{ij}(v)$, the redistribution integral is equal to the frequency-averaged mean intensity, $\tilde{R}_{ij} = I_{ij}$. We use here the angle-averaged redistribution functions. Strictly speaking, the redistribution integral is a function of both frequency and angle and should be given by

$$\tilde{R}_{ij}(v, n) = \int dv' \int (d\Omega/4\pi) R_{ij}(v', n', v, n) I(v', n') \phi_{ij}(v),$$  

(4)

where $R_{ij}(v', n', v, n)$ is a general angle-dependent redistribution function; $n'$ and $n$ are unit vectors in the direction of propagation of the absorbed and emitted photons, respectively. It can be shown (Milkey, Shine, & Mihalas 1975b) that equation (3) provides an excellent approximation for the exact redistribution integral, equation (4), at least for plane-parallel, horizontally homogeneous, static atmospheres. However, the approximation (3) should not be used for a treatment of moving atmospheres. We will defer a study of a more general angle-dependent scattering integral to a forthcoming paper.

Analogously,

$$\tilde{P}_{kj}(v) = \int P_{kj}(v', v) J(v') dv' \phi_{jk}(v)$$  

(5)

is the cross-redistribution integral for the process $\ell \rightarrow j \rightarrow i$. $P_{ij}$ is called a cross-redistribution function, or generalized redistribution function (Milkey et al. 1975a; Hubeny 1982: HOS). In the case of a resonance line originating between two lowest levels of an atom, such as hydrogen Lz, there is obviously no cross-redistribution contribution.

The first term of equation (1) represents a spontaneous emission $j \rightarrow i$, starting from a “naturally populated” (HOS) level $j$; i.e., an emission after the level $j$ was populated by a process that does not involve a correlation with a previously absorbed photon. Such a transition may be an inelastic collision, any type of recombination, or a spontaneous emission from a higher state. The second term describes a contribution of the ordinary resonance scattering process $i \rightarrow j \rightarrow i$ (i.e., absorption and reemission in the same transition); and the last term represents the resonant Raman scattering process $\ell \rightarrow j \rightarrow i$, starting at the same level $\ell < j$.

We also note that, in principle, one may include additional terms corresponding to a stimulated two-photon emission $u \rightarrow j \rightarrow i$ ($u > j$), but such contributions are completely negligible in virtually all cases of astrophysical interest.

It is customary to introduce the emission profile, defined by

$$\frac{4\pi}{hv} \eta_{ji}(v) = n_j A_{ji} \psi_{ji}(v).$$  

(6)

Both profile coefficients $\phi_{ij}(v)$ and $\psi_{ij}(v)$ are normalized to unity,

$$\int \phi_{ij}(v) dv = \int \psi_{ji}(v) dv = 1.$$  

(7)

The absorption profile is determined by the atomic properties and is usually given by

$$\phi_{ij}(v) = \frac{1}{\pi^{1/2} w_{ij}} H(a_{ij}, x),$$  

(8)

where $x = (v - v_{ij})/w_{ij}$; $v_{ij}$ and $w_{ij}$ are the line center frequency and the Doppler width, respectively; $a_{ij}$ is the damping parameter for the transition $i \rightarrow j$, given through the atomic damping parameter $\Gamma_{ij}$ as $a_{ij} = \Gamma_{ij}/(2\pi v_{ij})$; and $H$ is the Voigt function. Notice that in the absence of collisional broadening, $\Gamma_{ij} = (\sum a_{ki} A_{ik} + \sum a_{kj} A_{jk})/2$.

A basic quantity of our approach is the ratio of the emission and absorption profiles,

$$\rho_{ij}(v) = \psi_{ji}(v)/\phi_{ij}(v).$$  

(9)
As follows from equation (1), it is given by
\[ \rho_i(v) = 1 + \frac{\eta_i}{\eta_j} \left( \left[ \tilde{\rho}_{ij}(v) - J_{ij} \right] + \sum_j^{\nu} \frac{\eta_j}{\eta_i} \left[ \tilde{\rho}_{ij}(v) - J_{ij} \right] \right). \]  
(10)

It is clear from equation (10) that for CRD \( \rho_i(v) = 1 \).

Finally, the absorption coefficient is given by
\[ \frac{\alpha}{\hbar v} \kappa_{ij}(v) = \left[ n_i \chi_{ij} - n_j \chi_{ji} \right] \phi_i(v). \]  
(11)

The two terms represent the absorption and the stimulated emission, respectively. Notice that the stimulated emission term contains the emission profile \( \psi_i(v) = \rho(v) \phi_i(v) \) (Oxenius 1965).

2.2. Statistical Equilibrium

The statistical equilibrium equation (or rate equation) for level \( i \) is written as (see, e.g., Mihalas 1978)
\[ n_i \sum_{j \neq i} (R_{ij} + C_{ij}) - \sum_{j \neq i} n_j (R_{ji} + C_{ji}) = 0, \]  
(12)

where \( R_{ij} \) and \( C_{ij} \) are the radiative and collisional rates for the transition \( i \rightarrow j \), respectively (including continuum states). The upward radiative rate for a transition between two bound states \( i, j, i < j \) is given by
\[ R_{ij} = B_{ij} \tilde{J}_{ij}. \]  
(13)

The downward rate is more complicated. According to the semiclassical theory (Oxenius 1965; HOS), it is given by
\[ R_{ji} = A_{ji} + B_{ji} \int J(v) \psi_i(v) dv, \]  
(14)

because the stimulated emission rate should be described through the emission profile coefficient. However, from the quantum-mechanical point of view, the question is not so clear. One may argue that the expression
\[ R_{ji} = A_{ji} + B_{ji} \int J(v) \phi_j(v) dv = A_{ji} + B_{ji} \tilde{J}_{ij}, \]  
(15)

is only a good approximation for equation (14) but is in fact a more consistent expression. Indeed, when equation (15) is adopted, both the radiative transfer equation (i.e., the absorption and emission coefficients) as well as the rate equations are written in terms of the lowest order quantum-electrodymanical expansion of the radiation intensity. This question was discussed by Cooper et al. (1982) and in more detail by Cooper, Hubeny, & Oxenius (1983). Equation (15) applies in the case of a weak radiation field (i.e., when the stimulated emission rate is much smaller than the spontaneous one). If the stimulated emission becomes important, the whole astrophysical approach based on the two-photon redistribution function formalism breaks down anyway, because one would have to consider not only the redistribution function for the simplest photon correlation process \( i \rightarrow j \rightarrow i \), but also the generally infinite sequence of chains \( i \rightarrow j \rightarrow i \rightarrow j \rightarrow i \), \( i \rightarrow j \rightarrow i \rightarrow j \rightarrow i \), etc., corresponding to higher order expansion terms of the radiation intensity.

Concluding, equation (15) for the downward rate is not only a good approximation for the seemingly more exact equation (14), but it is in fact a more internally consistent form. Considering equation (14) and at the same time using only two-photon redistribution functions for processes of the type \( i \rightarrow j \rightarrow i \) is not consistent because it would take one aspect of the problem into account and neglect the other one. Therefore, we will use equation (15) in the subsequent development.

However, for most astrophysical applications, these subtle problems do not have any practical consequences, since the resonance lines of most species (and recall that it is primarily the resonance lines for which the PRD effects may be important) are located in the UV or optical region where the stimulated emission is indeed negligible. Nevertheless, we should bear in mind that the present approach, regardless of whether we consider equation (14) or (15) for the downward rate, breaks down in the case of a strong radiation field, i.e., when the stimulated emission is comparable to or dominant over the spontaneous emission.

The basic practical problem is to solve the set of statistical equilibrium equations, equation (12), together with the radiative transfer equation with the absorption and emission coefficients given in the previous section. It is well known that the problem is rather complicated due to a nonlinear and nonlocal coupling of atomic level populations and the radiation field. As we have pointed out in the introduction, there are several methods which are able to deal with this problem assuming CRD in all lines.

What are the main effects of departures from CRD on the solution of the multilevel transfer problem? In view of equation (15), the PRD effects do not enter directly the rate equations; they only enter through the emission coefficient in the radiative transfer equation. There are in principle two types of effects that arise as a result of the difference of the emission profile \( \psi_j \) from the absorption profile \( \phi_j \), namely, (1) a direct effect that involves changes of the radiation intensity, and therefore the radiation rate, in the line \( i \rightarrow j \) itself, and (2) an indirect effect on all the level populations owing to an overall switch of the population balance induced by the changes of the radiative rates in the transition \( i \rightarrow j \) (sometimes called the interlocking effect).

Nevertheless, under most astrophysically important circumstances, the direct effect is far more important than the indirect one. This is easy to understand: the PRD effects influence mostly the radiation intensity in the line wings, while the radiative rate is dominated by the line core, where the influence of PRD is much smaller. Consequently, the radiative rates, and therefore the overall population balance, are much less affected by PRD than the radiation field itself.

These considerations suggest that the multilevel PRD transfer problem may be solved by means of an iteration scheme that reflects the above mentioned dichotomy: the influence of the PRD emission profile, equation (1), on the transfer equation (direct effect) should be treated explicitly, while the interlocking effect (indirect effect) can be treated iteratively. Moreover, since the departures from CRD in the transition \( i \rightarrow j \) influence mainly the transition \( i \rightarrow j \) itself, and much less other transitions, this suggests that the radiation transfer in such PRD transitions can be best treated applying the equivalent two-level atom (ETLA) formalism.

The global solution of the multilevel transfer with partial redistribution is thus split into two parts: (1) solving a set of the ETLA problems for all transition selected for the PRD treatment. This gives the quantities \( \rho \) (eq. [10]) for these transitions. (2) Quantities \( \rho \) are then held fixed, and one solves the coupled multilevel problem, which is then formally analogous to the standard CRD multilevel problem. After one or more iter-
ations of the standard multilevel scheme, the values of \( \rho \) are updated by a new set of ETLA procedures. This process is iterated to convergence. The scheme will be described in detail in the subsequent section.

Finally, we stress that this method is reminiscent of another method which uses a lagged iteration of the ratio of two sharply varying functions of frequency, namely, the variable Eddington factors technique (VEFT) of Auer & Mihalas (1970). They iterate on the ratio of the moments of the specific intensity \( K_i/J, \alpha \), where the moments are obtained by a frequency-by-frequency solution of the transfer equation, and their ratio is held fixed during the next iteration of the global problem which involves a general kind of frequency coupling. The method thus numerically reduces a complex explicit frequency-angle coupling to a much simpler scheme where only the frequency coupling is treated explicitly, while the angle coupling is treated iteratively.

Our method is analogous. It also iterates on a ratio, in our case the ratio of the emission to the absorption profile, \( \rho \), which is obtained by a transition-by-transition solution of the PRD problem within the framework of the ETLA method.

This step is thus analogous to the frequency-by-frequency formal solution of the transfer equation of the VEFT method. The ratio \( \rho \) is then held fixed during the next iteration of the global, multilevel scheme, and it is updated after a completed global iteration. As was discussed extensively by Auer & Mihalas, it is numerically much more advantageous to iterate on a ratio of two quantities than on the quantities themselves, basically because the ratio changes much less from iteration to iteration. These considerations explain the motivation and the physical basis of our approach. We turn now to the formal presentation of the method.

### 2.3. Equivalent Two-Level Atom (ETLA) Approach

The line source function for the transition \( j \to i \) is given by

\[
S^L_{ij}(v) = \frac{n_i A_{ij}}{\kappa_{ij}(v)} \left[ 1 + \frac{(n_i B_{ij}/n_j A_{ji})(R_{ij}(v) - J_{ij} + \bar{P}_{ij}(v))}{1 - (n_j B_{ji}/n_i B_{ij})\rho_{ij}(v)} \right],
\]

where the term \( \bar{P}_{ij}(v) \) represents the contribution of all cross-redistribution transitions,

\[
\bar{P}_{ij}(v) = \sum_{\tau} \frac{n_{\tau} B_{ij}}{n_i B_{ij}} [P_{\tau j}(v) - J_{\tau j}] .
\]

The superscript \( L \) indicates the line source function; the total source function is given by (dropping indices \( i, j \))

\[
S(v) = \frac{\phi(v)}{\phi(v) + r} S^C(v) + \frac{r}{\phi(v) + r} S^L(v) ,
\]

where \( S^C \) is the continuum source function, and \( r \) is the ratio of the continuum opacity to the frequency-averaged line opacity.

The basis of the equivalent two-level atom (ETLA) approach is to eliminate the unknown populations \( n_i \) and \( n_j \) from the expression for the line source function, using the corresponding statistical equilibrium equations. The procedure is discussed extensively, e.g., by Mihalas (1978), Hubeny (1985a), and Skumanich & Lites (1986), and it need not be repeated here. The critical population ratio \( n_i/n_j \) is expressed as

\[
\frac{n_i}{n_j} = \frac{R_{ij} + \beta}{R_{ji} + \alpha} = \frac{B_{ij}J + \beta}{A_{ji} + B_{ji}J + \alpha} ,
\]

where we have dropped indices \( ij \) on \( J \) to simplify notation. Here \( R_{ij} \) and \( R_{ji} \) are the upward and downward radiative rate in the transition \( i \to j \), and \( \alpha \) and \( \beta \) represent all indirect rates \( j \to i \) and \( i \to j \), respectively, via any intermediate states. For detailed expressions, see Hubeny (1985a).

In the ETLA formalism for the line source function in the transition \( i \to j \), \( S^L_{ij} \), it is assumed that all populations except \( n_i \) and \( n_j \), and all the transition rates except \( R_{ij} \) and \( R_{ji} \), are known. In other words, the quantities \( \alpha \) and \( \beta \) in equation (19) are regarded as known. As discussed many times in the literature (see, e.g., Mihalas 1978), this approach may fail to provide a sufficiently robust scheme for solving the general multilevel problem because the multilevel transition interlocking is treated iteratively. Nevertheless, the ETLA technique has been used successfully in several multilevel transfer programs, like PANDORA (Avrett & Loeser 1987) and ALTAIR (Castor, Dykema, & Klein 1992).

In the present context, we use ETLA only for the treatment of PRD. General multilevel coupling is treated by means of a suitable global, self-consistent scheme, as for instance complete linearization, or an ALI-based method.

Substituting equation (19) to equation (16), we obtain a familiar expression for the line source function, viz.,

\[
S^L(v) = \frac{J + \eta + \chi [\bar{R}(v) - J + \bar{P}(v)]}{1 + \epsilon_v},
\]

where

\[
\eta = \frac{\beta}{B_{ij}},
\]

\[
\chi = (A_{ji} + B_{ji}J + \alpha)/A_{ji} ,
\]

\[
\epsilon_v = [\alpha + B_{ji}J(1 - \rho_s - \eta B_{ij} \rho_s)]/A_{ji} ,
\]

where we use a subscript \( v \) to indicate a frequency dependence of various quantities, as is customary in atmospheric radiative transfer. (Note that the quantity \( \chi \) was assigned the notation \( \mu \) in Hubeny [1985a]. Here we use the notation \( \chi \) to avoid confusion with the standard use of \( \mu \) as denoting the cosine of the zenith angle; e.g., in eq. [25].)

There are two important points to realize.

1. Unlike the CRD case, the source function is not a linear function of the mean intensity. There are two sources of nonlinearity, namely, an occurrence of \( J \) in the expression for \( \chi \); and then the presence of \( J \) as well as \( \rho_s \) (which is in fact an unknown function to be determined) in the expression for \( \epsilon_v \). Neither of these terms contributes in the CRD case, in which the source function is given by

\[
S^L_{CRD} = \frac{J + \eta}{1 + \epsilon_v} ,
\]

with \( \epsilon = (\alpha + B_{ji} \eta)/A_{ji} \), and is exactly linear in the mean intensity.

Both nonlinear PRD terms are connected with stimulated emission. However, as discussed above and by Hubeny (1985b), it is perfectly sound to treat the offending nonlinear terms in equations (22) and (23) iteratively, i.e., by using current values of \( J \) and \( \rho_s \) following from the previous iteration (or \( \rho_s = 1 \) and \( J \) based on a CRD formal solution in the case of the first iteration).

We note in passing that Milkey & Mihalas (1973) tried to handle the nonlinearity following from stimulated emission explicitly (using the idea of complete linearization), but as

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explained above such an approach is not necessary, and more- 
over it is not warranted in view of physical consistency 
(Hubeny 1985b).

2. Another important difference from the CRD case, and in 
fact the third potential source of nonlinearity of the ETLa 
source function, is the presence of the cross-redistribution 
term, $\tilde{P}$, which depends on the ratios of the (known) 
populations $n_i$ (starting levels of the cross-redistribution processes) 
and the (unknown) population $n_j$. However, in most cases of 
interest, $i$ is the ground state or a low-lying state, while $j$ is 
an excited state. Consequently, it is the population $n_j$, rather 
than $n_i$, which is influenced most when the transfer equation 
with the source function $S_{ij}$ is solved. Therefore, we take the ratios 
$n_j/n_i$ in the term $\tilde{P}$ as known and given by the current values of 
populations.

Finally, we stress that the above procedure does not neglect 
the stimulated emission, nor does it treat cross-redistribution 
approximately. The nonlinear terms are treated iteratively, 
using current values of state parameters. Therefore, they are 
exact at the convergence limit. The only feature that may arise 
from our treatment is a somewhat slower convergence rate, 
but since these terms are small anyway, the deceleration of 
convergence is rather small. The total computer time increase is 
certainly negligible compared to a possible increase in the case 
in which these nonlinearities were treated fully explicitly.

The method therefore proceeds in the following steps (let us 
assume that we have already computed a CRD multilevel 
solution):

(i) Using current values of populations and mean intensities, 
set up ETLa quantities $\eta$, $\chi$, $\epsilon_m$, and $\tau$ (eqs. [21]–[23]), for 
a selected transition $i \rightarrow j$.

(ii) A coupled system of transfer equations for the frequency 
points within the selected transition is solved, using the source 
function (eqs. [18] and [20]), i.e.,

$$
\frac{dI(v, \mu)}{dt} = I(v, \mu) - S(v),
$$

where $\tau$ is the monochromatic optical depth, and $I$ the specific 
intensity of radiation; $\mu$ is the cosine of the angle between the 
direction of propagation and the outward directed normal to 
the surface. Since, as explained above, the line source function $S(v)$ 
is linear in the mean intensity, equation (25) can be solved in 
by the appropriate direct solution involves inverting $NF \times NF$ matrices (if we use the variable 
Eddington factor technique; Auer and Mihalas 1970), 
or $(NF \times NA) \times (NF \times NA)$, without using Eddington factors); 
$NF$ and $NA$ are the number of frequency points within a 
given line and the number of angle points, respectively.

(iii) Once the solution of equation (25) is known, we can 
calculate the quantity $\rho_f$ from equation (10).

(iv) After completing analogous ETLa procedures for all 
transitions selected for a PRD treatment, we proceed with one 
or more iterations of a global CRD multilevel scheme, 
considering now the absorption and emission coefficients for the line $i \rightarrow j$ given by

$$
\frac{4\pi}{\nu} k_i(v) = \left[ n_i - n_j \frac{\partial_1}{\partial_1} \rho_i(v) \right] B_{ij} \phi_i(v),
$$

and

$$
\frac{4\pi}{\nu} \eta_{ij} = n_j A_{ij} \phi_i(v) \rho_i(v),
$$

where $\rho_i(v)$ is now viewed as a known quantity. In other words, 
the frequency coupling resulting from PRD is separated from 
the global multilevel coupling, and therefore any method 
capable of handling the multilevel problem can be used here.

2.4. Redistribution Functions

The formalism outlined in the preceding subsection has to 
be complemented by expressions for appropriate redistribution 
functions. As was already discussed, e.g., by Hubeny (1985a), 
the most appropriate redistribution function for resonance 
lines of nonhydrogenic species is the one given by Omont, 
Smith, & Cooper (1972, hereafter OSC), viz.,

$$
R(v', v) = \lambda R_{ii}(v', v) + (1 - \lambda) R_{ii}(v', v),
$$

where $R_{ii}$ and $R_{ii}$ are elementary redistribution functions in 
Hummel's (1962) nomenclature; $R_{ii}$ describes coherent 
scattering in the atomic rest frame, while $R_{ii}$ describes complete 
redistribution in the atomic rest frame. The quantity $\lambda$ is the 
so-called branching ratio, or coherence fraction, given by 
(OCS)

$$
\lambda = \frac{A_{ii} + C}{A_{ii} + C + \Gamma_{ii}},
$$

where $C$ and $\Gamma_{ii}$ are the inelastic and elastic collision 
rates, respectively. We note that the coherence fraction is usually 
denoted by $\gamma$ or $1 - \Lambda$ (with being the incoherence fraction); 
we adopted the notation $\lambda$ to avoid confusion with the notation 
for damping parameters (see eq. [36]). Also, we stress that the branching 
ration of the form of equation (29) corresponds to the 
normalized redistribution function (which is customarily used in 
atmospheric applications), while OSC derived the branching 
ration for an unnormalized redistribution function, which 
allows for a collisional destruction of a photon during a 
scattering process. In the astrophysical formalism, the collisional 
photon destruction is taken into account by means of the 
statistical equilibrium equations. This subtle point was carefully 
discussed, in a slightly different context, by Hubeny & Cooper 
(1986).

Replacing, as usual, $R_{ii}$ by complete redistribution in 
the laboratory frame, $R_{ii}(v', v) \approx \phi(v') \phi(v)$, we obtain 
(schematically)

$$
\tilde{R} - \tilde{J} = \lambda (R_{ii} - J),
$$

which is then substituted for all expressions containing the 
redistribution integral $R_{ii}$ is the redistribution integral, 
equation (3), calculated with the function $R_{ii}$.

The same procedure can be applied for subordinate lines, 
replacing $R_{ii}$ by another elementary redistribution function, $R_{ii}$ 
(Heinzel & Hubeny 1982).

Finally, the redistribution functions for Raman scattering 
$i \rightarrow j \rightarrow i$, called generalized redistribution functions 
(GRFs), or cross-redistribution functions, were studied systematically 
by Hubeny (1982). In the case of a process starting or ending 
at the ground or metastable state, the corresponding generalized 
redistribution function is given by

$$
P_{ij} = \lambda P_{ii} + (1 - \lambda) P_{ij},
$$

where $P_{ii}$ and $P_{ij}$ are the elementary GRFs (Hubeny 1982), 
parallel to the ordinary Hummel's redistribution functions. 
Function $P_{ii}$ was first introduced by Milkey et al. (1975a) and 
called $R_{ii}$.

Suitable expressions for the elementary GRFs are summarized 
by Hubeny (1982), and a useful approximation of $P_{ii}$ 
was developed by CBH (their Appendix II). Again, function $P_{ij}$
may be replaced by “complete redistribution,” i.e., complete noncorrelation, and the redistribution integral is replaced by
\[ \vec{p} - \vec{f} = \lambda (\vec{p}_n - \vec{f}) . \] (32)

3. PARTIAL REDISTRIBUTION IN HYDROGEN

Simple redistribution functions discussed above are valid in (i) the impact approximation, and (ii) the isolated line approximation. The impact approximation basically means that elastic collisions, responsible for line broadening and consequent destruction of photon coherence during scattering (the so-called collisional redistribution), are well separated in time. The isolated line approximation stipulates that the frequency difference of the atomic transitions of interest is large compared to the natural width of corresponding levels (given by the inverse of their lifetime).

Both these approximations break down for hydrogen line formation under typical astrophysical circumstances, as for instance in the solar chromosphere. Since the individual l-states of hydrogen are degenerate, the isolated line approximation can never be satisfied.

The region of validity of impact approximation for the hydrogen Lα line is very narrow. The limiting distance from the line center where the impact approximation is valid is given by (CBH)
\[ \Delta \nu_{\text{imp}} = \frac{1}{2\pi} \left( \frac{4\pi n_e e^2}{m_e} \right)^{1/2} \] (33)
for electron broadening, and
\[ \Delta \nu_{\text{ion}} = (m_e/m_i)^{1/2} \Delta \nu_{\text{imp}} \] (34)
for ions. For \( n_e = 10^{12} \), the impact approximation is valid for less than \( 10^{-3} \) nm from the line center, and the limit is even smaller for ion broadening.

One of the features of the L-degeneracy of the levels is that a new mechanism, namely, an absorption and emission of photon during collision, has to be taken into account. As shown by Burnett et al. (1980) and CBH, this mechanism gives rise to complicated correlation terms which represent collisional mixing of l-degenerate levels during a far wing emission. However, these correlation terms cancel out as the l-states are populated according to their statistical weights. It can be shown by detailed statistical equilibrium calculations of hydrogen in the solar chromosphere that this approximation is in fact well satisfied everywhere but in the transition region where only the line core of Lα is formed. The Lα core is, in turn, influenced only slightly by the far wing redistribution. Therefore, we will neglect the correlation terms here, which simplifies the CBH formalism significantly.

However, one has to retain the full nonimpact formalism. Fortunately, as was first shown by Yelnik et al. (1981), going beyond the impact regime is simply accomplished by replacing the original, impact, branching ratio \( \lambda \), equation (29), by a frequency-dependent \( \lambda(v) \).

Nevertheless, the matter is more subtle than that. It should be realized that the basic quantity which describes the redistribution of photons in a scattering process, and which is provided by detailed quantum–mechanical calculations, is not the redistribution function, but the emission coefficient. CBH derived the following expression for the emission coefficient in Lα in the rest frame of the atom neglecting collisional corre-
\[ j(\xi) = \frac{h \xi}{4\pi} \left\{ n_{1s} B_{1s2p} \frac{\Gamma_{2p1s}}{2 \Gamma_{2p1s}(\Delta \xi)} \int d\xi' J(\xi') f_{1s2p}(\Delta \xi') \right\} \times \left[ \delta(\xi - \xi') - f_{1s2p}(\Delta \xi') \right] + n_{2p} f_{1s2p}(\Delta \xi') \Gamma_{2p1s} \right\} , \] (35)
which was first given by Yelnik et al. (1981). Here, \( j(\xi) \) is the emission profile in the rest frame, and \( \xi \) and \( \xi' \) are, respectively, the frequencies of the emitted and absorbed photons measured in the atomic rest frame, \( \Delta \xi \) is the frequency difference from the line center, \( \delta(\xi) \) is the Dirac \( \delta \)-function, \( \Gamma_{2p1s} \) is the radiative decay rate for the transition 2p \( \rightarrow \) 1s (i.e., the Lα line), and \( \Gamma_{2p} \) is the total width of the 2p state.
\[ \Gamma_{2p}(\Delta \xi) = \gamma_{2p}(\Delta \xi) + \Gamma_{2p1s}/2 , \] (36)
where \( \gamma_{2p} \) is the collisional width of the 2p state. Notice that the collisional width, and consequently the total width, generally depend on frequency.

Finally, \( n_{1s} \) and \( n_{2p} \) are the populations of the 1s and 2p states, respectively, and \( f_{1s2p} \) is the atom's frame absorption profile coefficient,
\[ f_{1s2p}(\Delta \xi) = \frac{\Gamma_{2p}(\Delta \xi)}{\Delta \xi^2 + \Gamma_{2p}(\Delta \xi)^2} , \] (37)
which is formally given by the Lorenz profile, although with a frequency-dependent damping parameter, \( \Gamma_{2p}(\Delta \xi) \).

The laboratory–frame emission coefficient, such as the one given by equation (1), describes emission by an ensemble of atoms moving with random, thermal velocities. We assume here a Maxwellian velocity distribution for all particles, so the laboratory frame (also called the observer’s frame) coefficient is defined as
\[ n(v) \equiv \langle j(\xi) \rangle , \] (38)
where angle brackets represent the velocity and angular average,
\[ \langle F(v) \rangle \equiv \int \int du d\Omega \langle 4\pi \rangle W_d(u) F(v - \nu n \cdot u) , \] (39)
where
\[ W_d(u) = 4\pi \sum_{k,T} \frac{1}{(2kT/M)^{1/2}} , \] (40)
is the Maxwellian velocity distribution, and
\[ u = v/v_{th} , \quad v_{th} = (2kT/M)^{1/2} , \] (41)
and \( w \) is the Doppler width. Here \( v_{th} \) is the thermal velocity, and \( k, T, \) and \( M \) are the Boltzmann constant, temperature, and atomic mass, respectively.

The redistribution function \( R_{\text{th}} \) is defined (schematically) as (see Hummer 1962)
\[ R_{\text{th}}(v', v) = \langle f(\xi') \delta(\xi - \xi') \rangle , \] (42)
where \( f \) is the Lorenz profile, given by equation (37), but with frequency-independent damping parameter \( \Gamma \). We now see the basic problem we have to face when dealing with nonimpact redistribution. Instead of a simple frequency (and angle) average as in equation (42), we have to deal with a frequency average of a more complicated quantity, as follows from equation (35). CBH introduced the so-called nonimpact redistribution function, \( \tilde{R}_{\text{th}} \), defined by
\[ \tilde{R}_{\text{th}}(v', v) \equiv \langle \frac{1}{2\Gamma_{2p}(\xi)} \Gamma_{2p1s} f_{1s2p}(\Delta \xi') \delta(\xi - \xi') \rangle , \] (43)
which contains, in contrast to the ordinary function \( R_{\text{in}} \), the additional factor \( \Gamma_i/2\Pi(\xi) \) which depends on frequency and must therefore be included in the averaging procedure. Moreover, the absorption profile function \( f(\xi) \) contains a frequency-dependent term in the damping parameter.

The function \( R_{\text{in}} \) is thus in principle much more complicated than the ordinary \( R_{\text{in}} \). Fortunately, as was shown by CBH, this function can be approximated by (writing now \( A_{2p1s} \) for \( \Gamma_{2p1s} \))

\[
\tilde{R}(v', v) \approx \frac{A_{2p1s}}{A_{2p1s} + 2\gamma_{2p}(v)} R_{\text{in}}[v', v, a(v)] ,
\]

where \( a(v) \) is a frequency-dependent Voigt damping parameter

\[
a(v) = \frac{\Gamma_{2p}(v)}{2\pi v_0},
\]

and the Doppler width is given by

\[
w = (v_0/c)v_{\text{in}} ,
\]

\( v_0 \) being the line center frequency, and \( c \) the speed of light. In equation (44), we have specifically indicated a dependence of the ordinary \( R_{\text{in}} \) on frequencies and the damping parameter.

Equation (44) shows that one can use for the impact \( R_{\text{in}} \) the functional form of the ordinary \( R_{\text{in}} \), calculated with a different \( a(v) \) for each frequency, and multiplied by a frequency-dependent coherence fraction. Therefore, for \( L_{\alpha} \), the previously developed formalism remains valid, only the elastic collision rate \( \Gamma_i \) has to be considered as frequency-dependent, \( \Gamma_i = 2\gamma_{2p}(v) \). Notice also that the coherence fraction contains the Einstein coefficients \( A_{2p1s} \) and not \( A_{2p1} \), as is usually assumed.

In equation (29), \( C \) represents the total collisional rate out of the \( 2p \) state, which is dominated by the rates in transitions \( 2p \rightarrow 2s \) and \( 2p \rightarrow 1s \). Useful expressions for the collisional rates are given by CBH (their eqs. [4.1]–[4.17]). We note that since the \( \ell \)-states are assumed populated according to their statistical weight, one may consider the individual total populations of the \( n = 2, 3, \ldots \) levels, because \( n_{1s}B_{1s2s} = n_{1s}B_{1s2p} \), and \( n_{2s}A_{2s1s} = n_{2s}A_{2s1p} \).

For \( L_{\beta} \), equation (1) is modified slightly to read

\[
\frac{4\pi}{\hbar v} n_{3}(v) = \left\{ n_3 A_{31} + n_1 B_{13} \lambda_{3p} \left[ (\tilde{R}_{\mu} - J_{13}) + a_{2s} n_{2s} B_{23} \right] \right\} \phi_{13}(v) ,
\]

where

\[
\lambda_{3p}(v) = \frac{A_{3p1s}}{A_{3p1s} + A_{3p2s} + 2\gamma_{3p}(v)} ,
\]

and

\[
a_{2s} = \frac{n_{2s} B_{23}}{n_{2s} B_{23}} = 0.17 .
\]

The last term in equation (47) represents the Raman scattering (redistribution) contribution, namely, a correlation of photons absorbed in \( H_\alpha \) and reemitted in \( L_{\beta} \). The factor \( a_{2s} \) accounts for the fact that only the transition \( 2s \rightarrow 3p \rightarrow 1s \), not \( 2 \rightarrow 3 \rightarrow 1 \), is the true Raman scattering process. Similarly, the coherence fraction \( \lambda \) contains Einstein coefficients \( A_{3p1s} \) and \( A_{3p2s} \), and not \( A_{31} \) and \( A_{32} \). This point was not sufficiently realized in the astrophysical literature. An important consequence of this feature is that the limiting value of the coherence fraction is not \( A_{31}/(A_{31} + A_{32}) = 0.56 \), but rather \( A_{3p1s}/(A_{3p1s} + A_{3p2s}) = 0.88 \).

4. RESULTS AND DISCUSSION

We have calculated four different cases representing various approximations for the frequency redistribution. In all cases, we assume the atmospheric structure, i.e., the temperature and density, given by the VAL-C model (Vernazza et al. 1981). The hydrogen level populations given by the VAL-C model are used as a starting estimate of the populations. First, we calculate the case of CRD, in which we assume complete redistribution for all transitions.

Next, we solve the transfer problem \( L_{\alpha}-\text{PRD} \), where only \( L_{\alpha} \) is assumed to be formed with partial redistribution, whereas all the other transitions are formed in CRD. The next level of complexity is \( L_{\alpha}, L_{\beta}-\text{PRD} \), where both \( L_{\alpha} \) and \( L_{\beta} \) are formed in PRD, but we neglect the cross-redistribution, i.e., the term \( \tilde{R}_{\mu} - J_{13} \) in equation (47) is set to zero. Finally, the calculation \( L_{\alpha}, L_{\beta}-X-\text{PRD} \) contains full cross-redistribution between \( H_{\alpha} \) and \( L_{\beta} \). In all cases, all other lines are assumed to be formed with CRD. Our model atom contains five bound levels plus the continuum and comprises all radiative and collisional transitions among those levels. The transfer is solved for the Lyman continuum, but all other bound-free continua are treated by radiation temperatures (Carlsson 1986). We note that CBH derived an expression for the emission coefficient in \( H_{\alpha} \) in the case of full redistribution. However, as was shown by Hubeny & Heinzl (1984), the effects of redistribution in subordinate lines are negligible for the conditions characteristic of the formation of \( H_{\alpha} \) in the solar chromosphere. This is easily understood, since the wings of \( H_{\alpha} \) are formed deep in the photosphere where the density is high, and therefore collisional redistribution dominates.

We stress that the level populations are calculated self-consistently with the radiative transfer equation. Therefore, we may study the influence of redistribution in \( L_{\alpha} \) and \( L_{\beta} \) on hydrogen level populations and, importantly, on hydrogen ionization. We note that our present models are not fully self-consistent in a sense that we do not solve the charge conservation equation, which means that although we allow the proton number density to vary, the electron density is kept fixed. The present calculations are not meant to provide a new model solar chromosphere, but rather to study the effects of the PRD hydrogen line formation on the detailed spectroscopic diagnostics.

Figure 1 displays the basic parameter \( \rho \) as a function of the distance from the line center, \( \Delta \lambda \), and on the line center optical depth, \( \tau_0 \), for both \( L_{\alpha} \) and \( L_{\beta} \) for the \( L_{\alpha}, L_{\beta}-X-\text{PRD} \) case. The basic features are \( \rho \approx 1 \) for large optical depths, as well as for the line center. For \( L_{\alpha} \), \( \rho \) first exhibits a slight increase for all frequencies at optical depths of the order of \( 10^6-10^7 \), followed by a sharp drop for lower optical depths. This is explained by the decoupling of the core and the wing photons. Unlike the CRD case, photons are no longer effectively redistributed from the core to the wings, and therefore the emissivity (and thus \( \rho \)) drops (see also Fig. 2). The peaks of \( \rho \) at the near-wing frequencies (\( \Delta \lambda \approx 0.05 \) nm) are the result of an interplay between the chromospheric temperature rise and the transfer effects.

The behavior of \( \rho \) for \( L_{\beta} \) is different. The most conspicuous feature is a sharp rise of \( \rho \) at optical depths of the order of \( 10^6 \) for wing frequencies, which is the combined result of the chromospheric temperature rise, and of redistribution of photons from the \( H_{\alpha} \) line. The decrease of \( \rho \) at low optical frequencies is not \( A_{31}/(A_{31} + A_{32}) = 0.56 \), but rather \( A_{3p1s}/(A_{3p1s} + A_{3p2s}) = 0.88 \).
depths for the wing frequencies is much less pronounced than in \(L_x\), because the coherence fraction does not approach unity as for \(L_x\), but rather 0.88. So, while a photon in the wing of \(L_x\) can survive \(10^3\)–\(10^4\) consecutive scatterings before the coherence is destroyed (depending on the exact value of the incoherence fraction, \(1 - \lambda\)), the wing coherence in \(L_\beta\) is destroyed after \(1/(1 - 0.88) \approx 8\) scatterings. Therefore, the coherence effects cannot develop fully for \(L_\beta\). This consideration also shows that PRD effects must be even smaller for higher Lyman lines.

Figure 2 displays monochromatic source functions for the \(L_x, L_\beta-X\)-PRD case for four different frequencies corresponding to the line center, the emission peak, the near wing, and the far wing. The exact wavelength positions are also indicated by arrows in Figure 3. The behavior of the \(L_x\) source function exhibits several well-known features. First, the source function is identical for all frequencies at large depths. Then, progressively, frequencies closer and closer to the center begin to decouple from the core frequencies. Due to the above-mentioned lack of redistribution of photons from the core, the source function decreases below the core source function.

The source function for \(L_\beta\) exhibits an increase at the line wing frequencies, which is the result of an indirect effect of \(L_x\) on the populations of the \(n = 2\) and \(n = 3\) levels (see below), and also the result of the feeding of photons from \(H_\alpha\).

Let us explain the feeding mechanism in more detail. The usual approximation of the complete noncorrelation of the photons absorbed in \(H_\alpha\) and reemitted in \(L_\beta\) means that (see eq. [47])

\[
\bar{P}_{\text{in}}(v_{\text{in}}, v_{\text{out}}) \approx \int_{H_\alpha} P_{\text{out}}(v_{\text{in}}, v_{\text{out}}) J_\alpha(v_{\text{out}}) dJ_\alpha(v_{\text{in}}) \phi(v_{\text{out}}) - J_{H_\alpha} = 0.
\]

However, when the cross-redistribution is taken into account, the left-hand side of equation (50) is allowed to depart from zero. The function \(P_{\text{in}}(v_{\text{in}}, v_{\text{out}})\) exhibits a large degree of “coherence,” in a sense that the function \(P_{\text{in}}/\phi\) exhibits a sharp peak at \(x_{H_\alpha} = x_{L_\beta}\), where \(x\) is the frequency difference from the line center measured in units of Doppler widths (see CBH). Approximating \(P(x', x)/\phi(x') \approx \delta(x' - x)\), the left-hand side of equation (50) may be written as \(J(x_{H_\alpha} = x_{L_\beta}) - J_{H_\alpha}\).

However, the Doppler width for \(H_\alpha\) is 6.4 times smaller than that for \(L_\beta\) (because the line center frequency is 6.4 times smaller). This means that a photon reemitted at, say, three Doppler widths from the center of \(L_\beta\) was likely absorbed 19.2 Doppler widths from the center of \(H_\alpha\), i.e., in the continuum where the radiation intensity is larger than within the line, since \(H_\alpha\) is an absorption line. Therefore, the left-hand side of equation (50) is positive, or, in other words, there is an excess of emission in the near wings of \(L_\beta\) when the cross-redistribution is allowed for as compared to the case in which it is neglected.

The emergent flux for all four cases is displayed in Figure 3. The largest PRD effect is in the \(L_x\) wing starting at about 0.03
FIG. 2.—Line source functions $S_{\nu}$ as a function of column mass in the VAL-C model for $L\alpha$ and $L\beta$ for the full multilevel PRD solution with cross-redistribution are shown at various distances $\Delta \lambda$ from line center, as indicated. Also indicated are the line center opacities, $\tau_0$. The letters $a$–$h$ correspond to the positions in the line profiles indicated in Figs. 3 and 4.

This decrease of flux for the PRD case reflects the behavior of the monochromatic source function explained above. On the other hand, there is little difference between the three PRD models, which demonstrates that the most important mechanism for the $L\alpha$ line formation is the resonance scattering within the line itself, while the line profile is insensitive to what is assumed about the formation of $L\beta$.

The most interesting result is, however, a sensitivity of the predicted $L\beta$ line profile on model assumptions. Strikingly, all three PRD profiles lie above the CRD profile. The flux in the wing is highest for the $L\alpha$ PRD case, where $L\beta$ is still calculated assuming CRD! We see that the assumption of PRD in one line affects the formation of another line, $L\beta$. This effect is explained by the fact that PRD in $L\alpha$ increases the population of the $n = 2$ level in the upper chromosphere ($10^{-6} < m < 10^{-5}$) significantly—see Figure 6. Due to the collisional coupling, and due to the fact that $H_\beta$ is already optically thin (and therefore does not respond to the changes in populations), the same behavior is shared by the $n = 3$ level at well. On the other hand, the $n = 1$ level population does not change significantly. Since the CRD source function for $L\beta$ is

FIG. 3.—The emergent line flux is shown as a function of distance from line center $\Delta \lambda$ for both the $L\alpha$ and $L\beta$ lines. Curves are shown for the various assumptions regarding the scattering in the line (see text). Wavelengths $\Delta \lambda$ for which $S_{\nu}$ is given in Fig. 2 are indicated by the letters $a$–$h$.

FIG. 4.—Same as Fig. 3, except the emergent line intensity profiles near disk center ($\mu = 0.89$; one of the angle quadrature points of the CRD line transfer approximation) are given inside the line.

within the line itself, while the line profile is insensitive to what is assumed about the formation of $L\beta$.

The most interesting result is, however, a sensitivity of the predicted $L\beta$ line profile on model assumptions. Strikingly, all three PRD profiles lie above the CRD profile. The flux in the wing is highest for the $L\alpha$ PRD case, where $L\beta$ is still calculated assuming CRD! We see that the assumption of PRD in one line affects the formation of another line, $L\beta$. This effect is explained by the fact that PRD in $L\alpha$ increases the population of the $n = 2$ level in the upper chromosphere ($10^{-6} < m < 10^{-5}$) significantly—see Figure 6. Due to the collisional coupling, and due to the fact that $H_\beta$ is already optically thin (and therefore does not respond to the changes in populations), the same behavior is shared by the $n = 3$ level as well. On the other hand, the $n = 1$ level population does not change significantly. Since the CRD source function for $L\beta$ is

FIG. 5.—The relative limb darkening of both the $L\alpha$ and $L\beta$ lines is shown as a function of distance from line center $\Delta \lambda$. Curves for the transfer quadrature angles $\mu = 0.5, 0.11$ are shown relative to the quadrature point near disk center ($\mu = 0.89$). Note the extensive limb brightening of the outer core of the $L\alpha$ line.
proportional to \( n_1/n_2 \), we see that PRD in \( L_\alpha \) increases the CRD source function in \( L_\beta \). When we allow for PRD in \( L_\beta \), the flux in the wings decreases due to precisely the same mechanism as for \( L_\alpha \). Finally, when the cross-redistribution \( H_\alpha \rightarrow L_\beta \) is allowed for, the flux in the near wings increases due to the feeding mechanism explained above. The similarity of all \( L_\beta \) profiles, both CRD and PRD, is the result of the small number of possible consecutive coherent scatterings in the wing.

Figure 4 displays the emergent specific intensity near disk center (\( \mu = 0.89 \)). The behavior of profiles is analogous to that for the flux profiles. The difference of individual predicted profiles for \( L_\beta \) is now larger. The specific intensity for the model \( L_\alpha, L_\beta \) PRD decreases almost to the level of the CRD intensity. We stress that one could not conclude, based on this result, that PRD is unimportant for \( L_\beta \). A near agreement of the intensity is rather a consequence of near cancellation of two PRD effects: a decrease of intensity due to the resonance scattering in \( L_\beta \), and an increase of intensity due to the indirect effect of PRD in \( L_\alpha \). Finally, Figure 5 shows the limb darkening (brightening) for the model \( L_\alpha, L_\beta-X\text{-PRD} \) in detail.

Figure 6 displays the hydrogen level populations \( n = 1, 2, 3 \), and the proton number density. Interestingly, Figure 6 shows that the proton density is influenced significantly by PRD in \( L_\alpha \), while its sensitivity to the treatment of PRD in \( L_\beta \) is marginal. The difference between the CRD and PRD proton density is very similar to differences in the \( n = 2 \) and \( n = 3 \) level populations. This is readily explained by the fact that the dominant ionization mechanism is photoionization in the Balmer continuum. Since the number of photoionizations is proportional to \( n_2 \), it is larger for the PRD models.

Finally, Figure 7 illustrates the convergence of our code. We plot the maximum relative change in the populations as a function of iteration number. We show the convergence for the case \( L_\alpha, L_\beta-PRD \), initiated from the converged CRD solution, and followed by the full cross-redistribution case \( L_\alpha, L_\beta-X\text{-PRD} \).

5. Conclusions

We have presented a new numerical method for solving the partial redistribution line formation problem in multilevel atoms. We have demonstrated that that method works well for one of the most difficult and yet most important problems of this sort, hydrogen Lyman line formation in the solar chromosphere, taking into account partial redistribution in the \( L_\alpha \) and \( L_\beta \) lines, together with the cross redistribution between \( H_\alpha \) and \( L_\beta \).

We believe that our present computations incorporate all those aspects of frequency redistribution which are important to the formation of the hydrogen lines under conditions encountered in stellar atmospheres. Our results reiterate the importance of PRD for diagnostics of stellar atmospheres involving the wings of the Lyman lines. They also show that
PRD has a significant influence on the populations of the excited and ionized states of hydrogen. In the solar problem we have studied here, the influence of PRD on the population of the ground state of hydrogen is comparatively modest, but this conclusion cannot be transferred to atmospheres of stars differing considerably from that of the Sun without carrying out similar computations for such model stellar atmospheres.

Although the influence of partial frequency redistribution on the hydrogen transfer is significant, and especially so in the wings of Lα and Lβ when these lines are used as diagnostics, the effects of chromospheric fine structure and atmospheric dynamics surely have a much larger influence on the shapes and intensities of the line cores, hence the net flux radiated by these lines. With this in mind, our results suggest that a treatment of only the Lα line in PRD will carry most of the important PRD effects for dynamical models of the solar atmosphere. We caution again that our method must be modified to incorporate angular redistribution before dynamical computations with redistribution effects are to be meaningful. We intend to soon modify our code to account for angle redistribution.

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APPENDIX

MODIFICATIONS TO THE SCHARMER & CARLSSON (1985) FORMALISM

Here we present a list of expressions which differ from the original Scharmer & Carlsson (1985) scheme. Differences arise owing to the presence of the parameter $\rho$. Each equation is followed by the corresponding Scharmer & Carlsson (SC) equation number.

\[ \kappa_{\nu'\nu} = \kappa_{\nu \nu} + \tilde{z}_{j}(\nu', \mu)(n_{j} - G_{\nu'\nu} n_{j}) \]  
(\text{SC eq. [2.8]}),

\[ j_{\nu'} = j_{\nu} + (A_{\nu'\nu}/B_{\nu})\tilde{z}_{j}(\nu', \mu)G_{\nu'\nu} n_{j} \]  
(\text{SC eq. [2.9]}), with

\[ G_{\nu'\nu} = G_{\nu'\nu} \rho_{\nu'\nu} = (\tilde{g}_{\nu'\nu}/\tilde{g}_{\nu\nu})p_{\nu'\nu} \]  
(\text{SC eq. [2.10]}), where $\rho_{\nu'\nu}$ is the parameter $\rho$ defined by equation (9). Although our present formalism works in terms of the angle-averaged emission coefficient, and therefore $\rho$, we have introduced a general angular dependence of $\rho$ in equation (A3), in anticipating future extensions of our formalism.

\[ \epsilon_{\nu'\nu}^{(n)} = G_{\nu'\nu} \tilde{z}_{j}(\nu', \mu)(A_{\nu'\nu}/B_{\nu} + \tilde{I}_{\nu'\nu}^{(n)})/\kappa_{\nu'\nu}^{(n)} \]  
(\text{SC eq. [3.15]}),

\[ \delta_{ij}^{(n)} = \frac{1}{2} \int_{-1}^{1} \int_{0}^{\infty} \frac{\kappa_{\nu'\nu}}{\kappa_{\nu\nu}^{(n)}} \phi_{\nu'} dv d\mu + \frac{n_{i}}{2} \int_{-1}^{1} \int_{0}^{\infty} \frac{\tilde{z}_{j}(\nu'\mu)}{\kappa_{\nu'\nu}^{(n)}} \phi_{\nu'}(1 - \rho_{\nu'\nu}) dv d\mu \]  
(\text{SC eq. [3.18]}), where the first term is equal to the original Scharmer & Carlsson expression); and, finally,

\[ J_{ij} = \frac{1}{2} \int_{-1}^{1} \int_{0}^{\infty} \frac{\kappa_{\nu'\nu}}{\kappa_{\nu\nu}^{(n)}} I_{\nu'}^{(n)} \phi_{\nu'} dv d\mu + \frac{G_{ij}}{2} \frac{\eta_{\nu'}^{(n)} \delta_{ij}^{(n)} - \eta_{\nu}^{(n)} \delta_{ij}^{(n)}} {\delta_{ij}^{(n)} - G_{ij} \delta_{ij}^{(n)}} \int_{-1}^{1} \int_{0}^{\infty} \frac{\tilde{z}_{j}(\nu'\mu)}{\kappa_{\nu'\nu}^{(n)}} I_{\nu'}^{(n)} \phi_{\nu'}(1 - \rho_{\nu'\nu}) dv d\mu \]  
(\text{SC eq. [3.19]}), where again the first term is equal to the original Scharmer & Carlsson expression. All other equations from Scharmer & Carlsson remain unchanged. Finally, we stress that an important change concerns the frequency quadrature weights. Instead of using the original procedure of Carlsson (1986), which consists of a simple trapezoidal integration, we use the spline formalism of Adams, Hummer, & Rybicki (1971). This procedure is used not only for evaluating the so-called redistribution matrix, $R_{ij} = R(\nu, \nu)w_{\nu}$, but also for special quadrature weights for frequency integration with the absorption profile, $w_{\nu} = \phi(\nu)w_{\nu}$. Notice that the weights $w_{\nu}$ and $w'_{\nu}$ do not have to be the same, because in the spline integration the weights depend on the function over which we integrate. Notice also that both the redistribution matrix and the weights $w_{\nu}$ are generally depth dependent.
REFERENCES

———. 1970, MNRA, 149, 60

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