TIDAL EXCITATION OF MODES IN BINARY SYSTEMS
WITH APPLICATIONS TO BINARY PULSARS

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ABSTRACT

We consider the tidal excitation of modes in a binary system of arbitrary eccentricity. For a circular orbit, the modes generally undergo forced oscillation with a period equal to the orbital period (T). For an eccentric orbit, the amplitude of each tidally excited mode can be written approximately as the sum of an oscillatory term that varies sinusoidally with the mode frequency and a “static” term that follows the time dependence of the tidal forcing function. The oscillatory term falls off exponentially with increasing b (defined as the ratio of the periastron passage time to the mode period), whereas the “static” term is independent of b. For small-b, modes (b ≈ 1), the two terms are comparable, and the magnitude of the mode amplitude is nearly constant over the orbit. For large-b modes (b ≫ 1), the oscillatory term is very small compared with the “static” term, in which case the mode amplitude, like the tidal force, varies as the distance cubed. For main-sequence stars, p, f, and low-order g-modes generally have large b and hence small amplitudes of oscillation. High-overtone g-modes, however, have small overlap with the tidal forcing function. Thus, we expect an intermediate overtone g-mode with b ≈ 1 to have the largest oscillation amplitude. In addition, we find that the mode amplitude is independent of the dissipation rate except when the mode frequency is very close to orbital resonance or the damping time is less than T; both conditions are unlikely. Moreover, orbital evolution causes a resonant mode to move off resonance with time. This severely limits the amplitude of modes near resonance. Rotation of the star shifts the mode frequencies but otherwise has little effect on the mode amplitude (provided that the rotation rate is small). Hence, tidally excited modes have amplitudes and phases that are periodic with period T, making them readily distinguishable from oscillations excited by other mechanisms.

We apply our work to the SMC radio pulsar PSR J0045—7319, which is believed to be in a highly eccentric orbit with a 10 M⊙ B star. We find that the g1 mode (with period 1.07 days) of the B star has the largest oscillation amplitude, with a flux variation of 2.3 mmag and a surface velocity of 70 ms−1. The flux variation at periastron, summed over all modes, is about 10 mmag; in addition, we propose that the shape of the light curve can be utilized to determine the orbital inclination angle. The apsidal motion of this system, calculated without the usual static approximation, is larger than that predicted by the classical apsidal formula by about 1%. For the PSR B1259—63 system, the tidal amplitude of the B star companion is smaller by a factor of 70 because of its larger periastron distance. To understand the dependence of tidal excitation on stellar structure, detailed numerical calculations of modes of a general polytropic star are also presented.

Subject headings: binaries: close — pulsars: general — pulsars: individual (PSR J0045—7319) — stars: oscillations

1. INTRODUCTION

The study of tidal interactions in binary systems is, among other things, useful as a test of stellar structure (the apsidal motion test; cf. Schwarzschild 1958; Claret & Giménez 1993), as a mechanism for the formation of close binaries in globular clusters (Fabian, Pringle, & Rees 1975; Press & Teukolsky 1977; Lee & Ostriker 1986; Kochanek 1992), and for explaining the synchronization and circularization of orbits (Lecar, Wheeler, & McKee 1976; Zahn 1977, 1989; Hut 1981). The primary focus of this paper, however, is the study of the tidally induced oscillations of a star and their use as a possible probe of stellar or orbital parameters.

Our work is in part motivated by the recent discovery of two radio pulsars in binary systems with main-sequence stars (PSR J0045—7319 [see Kaspi et al. 1994a]; SR B1259—63 [see Johnston et al. 1994]). In the past, radio pulsar timing observations have been used to determine orbital parameters with unprecedented precision (Taylor & Weisberg 1989; Kaspi et al. 1994b). Since both stars in the binary systems were compact, tidal interactions could be neglected. Indeed, the absence of tidal effects in binary pulsar systems has allowed various general relativistic effects to be measured. The discovery of binary radio pulsars with normal star companions, however, has opened the possibility for measuring tidal interactions with great precision to probe the interior of normal stars. PSR J0045—7319, because of its high eccentricity and small periastron distance, is currently the best candidate for finding tidally excited oscillations. In fact, PSR J0045—7319 has one of the smallest values for the periastron distance (in units of stellar radius) of all the known detached binary systems.

Tidal deformation of a star can be modeled as a linear superposition of its nonradial pressure and gravity mode oscillations (Press & Teukolsky 1977). We investigate how individual mode amplitudes vary with modal and orbital parameters for a Keplerian orbit with arbitrary eccentricity. The dependence of

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the mode amplitude on the rotation of the star will also be discussed. We ignore, however, the new family of modes that appears in a rotating star (see Papaloizou & Pringle 1978 for a discussion of the rotation modes).

We calculate the surface oscillation amplitude of the star as well as the perturbation to the pulsar orbit as possible means for detecting these oscillations. We also calculate the detailed shape of the light curve near periastron, which can be used to determine the orbital inclination angle for eccentric binaries such as PSR J0045−7319.

The plan of this paper is as follows. In the next section we establish the main equation for the tidal excitation of modes and discuss, in general, the energy in tidally excited modes, the phenomena of tidal resonance, and the effects of rotation and orbital evolution on the mode amplitude. In § 3 we discuss the relation between the mode amplitude and the more observationally accessible quantities such as flux variation, surface velocity, and perturbation to the pulsar’s orbit. In § 4 we show how the light curve at periastron can be used to determine the orbital inclination angle. In § 5 we apply our work to two binary radio pulsars and to general polytropic stars. Our results are summarized in § 6.

2. FORMALISM

In this section we present our basic method for studying the tidal excitation of modes in rotating and nonrotating stars in orbits of arbitrary eccentricity, and discuss the significance of the various modal and orbital parameters. We further present order-of-magnitude estimates for the mode energy.

2.1. Basic Equation for Tidal Excitation and Its General Solution

Consider a binary system where the primary star, taken to be a point mass, has mass $M_\ast$, and its companion star, the secondary, has mass $M_0$. The position of the primary star in the orbit is specified by $R(t)$, the separation between the stars, and the azimuthal angle, $\phi(t)$. The secondary is assumed to be rotating as a solid body with angular speed $\Omega_\ast$ about an axis perpendicular to the orbital plane.

The perturbation to any physical quantity in the star—for instance, the Eulerian density perturbation due to the tidal gravitational field of the primary—can be written as a sum over its normal mode amplitudes

$$\delta \rho(t) = \sum_{n,m} A_{n,m} \delta \rho_{n,m}(t) Y_{n,m}(\theta, \phi),$$

(1)

where $\delta \rho_{n,m}$ is the normalized density eigenfunction (normalized in such a way that the energy in the mode is unity), $Y_{n,m}(\theta, \phi)$ is the spherical harmonic function, and $A_{n,m}$ is the mode amplitude. The pressure ($p$) and internal gravity ($g$) modes are uniquely specified by three numbers: the number of nodes in the radial direction $n$, the spherical harmonic degree $\ell$, and the azimuthal number $m$; the surface gravity mode, or the fundamental mode ($f$), has no nodes along the radial direction. We use a shorthand notation, $a$, to denote the collective index $(n, \ell, m)$.

The equation for the mode amplitude $A_a$ can be cast in the form of the following second-order ordinary differential equation for a forced, damped harmonic oscillator:

$$\frac{d^2 A_\ast}{dt^2} + 2\Gamma_\ast \frac{dA_\ast}{dt} + \omega_\ast^2 A_\ast = f_d(t),$$

(2)

with

$$f_d(t) = \frac{4\pi G M_\ast \omega_\ast^2 Q_{\ast}}{2\ell + 1} \frac{Y_{\ell+1/2}^\ell \left( \theta_{\text{orb}} = \frac{\pi}{2}, \phi_{\text{orb}}(t) - \Omega_\ast t \right)}{R(t)^{\ell+1}}$$

(3)

as derived for a nonrotating star in Press & Teukolsky (1977), where

$$Q_{\ast} = \int_0^{R_\ast} dr r^{\ell+2} \delta \rho_\ast(r)$$

(4)

is the overlap integral for the mode, $\Gamma_\ast$ is its dissipation rate, $R_\ast$ is the radius of the star, and the orbital plane is taken to be at $\theta_{\text{orb}} = \pi/2$. It should be noted that our $Q_{\ast}$ is equal to Press & Teukolsky’s overlap integral divided by $\omega_\ast$. This is due to the difference in the normalization of our eigenfunctions. Our normalized modes have unit energy, whereas Press & Teukolsky’s have energy equal to $\omega_\ast^2$.

It is useful to define the following reduced forcing function:

$$f_{\text{red}}(t) = \frac{Y_{\ell+1/2}^\ell \left( \theta_{\text{orb}} = \frac{\pi}{2}, \phi_{\text{orb}}(t) - \Omega_\ast t \right)}{[R(t)/a]^\ell+1},$$

(5)

where $a$ is the semimajor axis of the orbit. The function $f_{\text{red}}(t)$ contains all the time dependence of $f_d(t)$ and is independent of the mode properties. Since the driving term falls off as $1/R^\ell+1$, most of the energy is in the lowest degree modes that can be tidally excited, i.e., the quadrupole modes. We rewrite $f_d(t)$ in the following compact form:

$$f_d(t) = f_{\text{const}} K_\ast \Omega_\ast^2 f_{\text{red}}(t),$$

(6)

where

$$f_{\text{const}} = \left[ \frac{4\pi}{(2\ell + 1)a^{\ell+2}} \right] \left( \frac{M_\ast}{M_0} \right),$$

$$K_\ast = \omega_\ast^2 Q_{\ast},$$

(7)

$\Omega_\ast = (GM_\ast/a^3)^{1/2}$ is the orbital frequency, and $M_\ast$ is the total mass of the binary system. All the modal information is contained in the constant factor $K_\ast$. For quadrupole modes, considered in detail here, $f_{\text{const}}$ is equal to 4 for $1/5$ times the fractional mass of the primary star.

Equation (2) can be solved in terms of a Green’s function to yield

$$A_{\ast}(t) = f_{\text{const}} \left( \frac{K_\ast}{\omega_\ast^2} \right) \times \Omega_\ast^2 \int_0^t dt_1 \exp \left[ -\Gamma_\ast (t - t_1) \right] \sin \left[ \omega_\ast (t - t_1) \right] f_{\text{red}}(t_1),$$

(8)

where $\omega_\ast = (\omega_\ast^2 - \Gamma_\ast^2)^{1/2} \approx \omega_\ast$.

While equation (8) allows us to calculate the mode amplitude for a star with arbitrary rotation rate, it does not lend itself to a simple interpretation of the significance of various parameters. However, a simple interpretation is possible when the forcing function is periodic, such as when the star is nonrotating, and one can analyze the mode amplitude using a Fourier series. In the following discussion we assume that the star is nonrotating. The effects of rotation on the mode frequency and amplitude are addressed in § 2.3.

When rotation of the star is neglected, the reduced forcing function $f_{\text{red}}(t)$ is periodic with orbital period $T$, and we decom-
pose it in terms of its Fourier coefficients:

$$
\tilde{f}_{\text{en}}(t) = \sum_{n=1}^{\infty} C_{n}^{(m)} \sin(n \Omega_2 t) + \sum_{n=0}^{\infty} D_{n}^{(m)} \cos(n \Omega_2 t).
$$

(9)

The Fourier coefficients $C_{n}^{(m)}$ and $D_{n}^{(m)}$ depend only on the eccentricity $e$. The mode amplitude can easily be calculated using this series and is given below

$$
\tilde{A}_{d}(t) / \tilde{K}_{s} \equiv A_{d}(t) = \sum_{n=1}^{\infty} \frac{C_{n}^{(m)} \sin(n \Omega_2 t + \phi_{n})}{\left[ \left( n^2 - r_s^2 \right)^{1/2} + 4n^2 d_s^2 \right]^{1/2}} + \sum_{n=0}^{\infty} \frac{D_{n}^{(m)} \cos(n \Omega_2 t + \phi_{n})}{\left( n^2 - r_s^2 \right)^{1/2} + 4n^2 d_s^2} \right)^{1/2},
$$

(10)

where

$$
\tan \phi_{n} = - \frac{2nd_s}{r_s^2 - n^2},
$$

(11)

and the dimensionless numbers $r_s$ and $d_s$ are defined below:

$$
r_s \equiv \frac{\Omega_2}{\Omega_p}, \quad d_s \equiv \frac{\Gamma_s}{\Omega_p}.
$$

(12)

It is clear from equation (10) that resonances occur for integral values of $r_s$. Because of our choice of normalization for the eigenfunctions (to yield unit energy), the energy in the modes, $E_n$, is approximately equal to $|A_{d}(t)|^2$.

For highly eccentric orbits, the tidal force is appreciable only when the star is near periastron. Therefore, another relevant timescale is the periastron passage time. We define below a dimensionless parameter which expresses the mode frequency in terms of this timescale:

$$
b_s \equiv \frac{\omega_z}{\Omega_p},
$$

(13)

where $\Omega_p$ is the angular velocity of the star at periastron and is given by

$$
\Omega_p = \frac{(1 - e^2)^{1/2}}{(1 - e)^4} \Omega_2.
$$

(14)

The amplitude for a quadrupole mode is equal to $(4\pi/5)(M'_s/M_0)\tilde{K}_{s} \tilde{A}_{d}$ (see eq. [10]). The dependence of $A_{d}$ on the orbital parameters is entirely contained in $\tilde{A}_{d}$, which is a function of the dimensionless parameters $e$, $r_s$, and $d_s$; its properties are discussed below. In addition, $\tilde{A}_{d}$ depends on the structure of the star, via the parameter $\tilde{K}_{s}$, which will be calculated in § 5. The relation of $A_{d}$ to the observed quantities is nontrivial and is discussed in § 3.

The behavior of $\tilde{A}_{d}$ can be understood in terms of the Fourier coefficients. It is easy to show that for quadrupole tide the $D_{n}^{(m)}$ are real, the $C_{n}^{(m)}$ are imaginary, and $C_{n=0}^{(m)} = 0$. In addition, numerical calculations show that $|C_{n=0}^{(m)}| \sim |D_{n=0}^{(m)}|$ for large $n$ ($n \geq \Omega_p/\Omega_2$). Moreover, $D_{n=0}^{(m)}$ falls off as approximately $\exp \left( -\alpha z n \Omega_p/\Omega_0 \right) (1 - e)^{1/2}$ for $n \geq 5\Omega_p/\Omega_2$, where $\alpha_z \approx 1.3e^{-0.23}$. The Fourier coefficients for $m = \pm 2$ are larger compared with $m = 0$ by a factor of about $\exp (2\alpha_z)$. The exponential dependence arises because, for eccentric orbits, the characteristic timescale for the variation of the tidal force is the periastron passage time, $\Omega_p^{-1}$; therefore, the power spectrum for frequencies greater than $\Omega_p$ falls off exponentially.

Tidally excited modes of a star in a circular orbit exhibit oscillations with a period equal to that of the orbit and not their natural oscillation periods (so long as the orbital evolu-
that the amplitude of an oscillator driven off-resonance is independent of its damping. For \( d_s \gg \delta r_s \), the behavior is the same as the case of integer \( \delta r_s \) discussed above.

The oscillation amplitude is proportional to the product of \( A_0^0 \) and the overlap integral \( (Q_n) \). There are thus two opposing effects in the tidal excitation of modes. On the one hand, larger \( n \) g-modes have smaller frequencies and thus smaller values of \( b_s \); they therefore have larger values of \( A_0^0 \). On the other hand, \( Q_n \) is larger for modes with smaller \( n \); this condition favors low-order modes. For these competing reasons, the modes with the largest oscillation amplitude, for most binary systems, are some intermediate-order \( g \)-mode \( (n \sim 5) \) with \( \delta r_s \) close to zero (see § 2.2). It should be noted that the amplitude near periastron is dominated by the \( p_1, f_3 \), and \( g_1 \)-modes because of their large overlap integrals. Since the values of \( b_s \) are generally large for these modes, the time dependence of their amplitudes is the same as the tidal force. Thus we cannot infer much about the star from the observation of its light curve near periastron, although it can be utilized to determine the orbital inclination angle (see § 4).

To give a feel for the physically relevant range of parameters, we note that the periods of low-\( n \) modes for main-sequence stars are of order a few hours to a few days, and the damping times are of the order of 100 yr or greater. The energy in these modes due to tidal forcing is independent of damping unless their \( b_s \) values are of order unity and \( r_s \) happens to be extremely close to an integer \( (\delta r_s < d_s) \).

2.2. Order-of-Magnitude Estimate of Mode Energy

In this section we estimate mode frequencies and overlap integrals and use them to calculate the energy in tidally excited modes. The frequencies of quadrupolar \( f \)- and low-order \( p \)-modes are approximately equal to (Christensen-Dalsgaard & Berthomieu 1991)

\[
\omega_{n2} \approx 6^{1/4}(n + 1) \left( \frac{GM_\star}{R_\star^2} \right)^{1/2},
\]

where \( M_\star \) and \( R_\star \) are the mass and the radius of the star, and \( n = 0 \) for the \( f \)-mode. The \( f \)- and \( p \)-mode frequencies have a weak dependence on the structure of the star which is not included in the above expression. Gravity waves can propagate only in convectively stable media, and thus the \( g \)-mode fre-
quencies are sensitive to the stellar structure. For a polytropic star of index \( n_p \), and a constant ratio of specific heats \( \gamma \), the \( g \)-mode frequencies are given below (Christensen-Dalsgaard & Berthomieu 1991):

\[
\omega_n \approx f_n \left( \frac{G M}{R^3} \right)^{1/2},
\]

where \( f_n = 15\{n_p(\gamma - 1) - 1\}/(n_p + 1) \); the factor of 15 has been obtained by a fit of numerically calculated frequencies.

Since the eigenfunctions are normalized in such a way that there is unit energy in the mode, the overlap integral, \( Q_{a2} \), is proportional to \((R^2_n/G)^{1/2}\). For \( n \neq 0 \) the wave function is oscillatory in the radial direction, and this leads to cancellation of the tidal force experienced by the mode. Thus the overlap integral decreases with \( n \). We model this dependence crudely as \((n + 1)^{-\beta}\) and write \( |Q_n| \sim f_n(R^2_n/G)^{1/2}/(n + 1)^\beta \). Numerical calculation of \( Q_{a2} \) for polytropic stars with \( \gamma = 5/3 \) gives \( f_n \sim 1 \) for \( f \)- and \( p \)-modes and \( f_n \sim 1/2 \) for \( g \)-modes. The exponent \( \beta \) for \( g \)-modes varies from 1.5 to 2 as the polytropic index decreases from 3 to 2 (for an analytical derivation of these results see Zahn 1970), and \( \beta = 6 \) for \( f \)- and \( p \)-modes of \( n_p = 1.5 \) stars.

We estimate \( \tilde{A}_s(t) \) using equation (10) and find that the oscillation energy in \( m = 0 \) quadrupole modes is

\[
E_s = |A_s|^2 \sim 6 \left( \frac{M_p^2}{M_*^2} \right) \frac{D_s^2 Q_{a2}^2 \omega_s^2 \Omega_*^2}{(\delta r)^2 + d_*^2},
\]

where \( n_s \) is the integer nearest to \( r_s \). Making use of the above expressions for \( Q_s \) and \( \omega_s \) of quadrupole \( g \)-modes, and the expression for the Fourier coefficients given in the previous section, we find the oscillation energy in \( m = 0 \) \( g \)-modes to be

\[
E_{n20} \sim E_{orb} \left( \frac{M_p}{M_*} \right) \frac{R_p}{d_{\text{per}}} \frac{f_n^2}{(n + 1)^{2+2\beta}} \times \left( \frac{\delta r^2}{d_*^2} \right) \exp \left[ -2.666e^{-0.25} \right],
\]

where \( d_{\text{per}} = a(1 - e) \) is the periastron distance and \( E_{orb} = GM_p M_*/2a \) is the orbital energy of the binary system.

Since \( p \)-modes have larger frequencies, and thus larger values of \( b_p \), than \( g \)-modes, \( p \)-mode oscillation energies are generally much smaller than \( g \)-mode energies. Moreover, we see from equation (15) that the oscillation energy in \( g \)-modes falls off as approximately \((n + 1)^{-5}\) for polytropic stars of \( n_p = 3 \). Thus the mode with the largest oscillation energy is typically a \( g \)-mode of moderate \( n \)-value. The energy in \( m = \pm 2 \) modes is larger compared to \( m = 0 \) modes by a factor of approximately \( \exp(5e^{-0.25}) \).

If the periastron passage time is large compared to the periods of low-order \( p \)- and \( g \)-modes, the tide near periastron is almost static. In this case we can easily estimate the mode amplitude by discarding the time derivative terms in equation (2). We find that the energy in the tide near periastron, which is dominated by \( f \)- and low-order \( p \)- and \( g \)-modes, is equal to \( KE_{orb}(M_p/M_*)R_p/d_{\text{per}}^2 \), where \( k = (2\pi/5R_\odot)^2 \Sigma_n Q_{a2}^2 \) can be shown to be equal to the apsidal motion constant.

Thus far we have discussed the dependence of the mode amplitude and the average energy on the orbital parameters and the mode frequency (through \( e \), \( b_s \), and \( \epsilon \)) and provide simple order-of-magnitude estimates of the dependence on stellar structure. A more detailed discussion of the dependence on stellar structure will be given in § 5.2.

2.3. Effect of Stellar Rotation on the Mode Amplitude

Rotation of the star has been neglected in the previous sections. We now discuss the effect of rotation on the mode amplitude. Assuming that the rotation rate is small compared with the mode frequency, we can treat its effects using perturbation theory. To linear order in the rotation rate, the frequencies of the modes, as seen in the rotating frame of the star, are shifted due to the Coriolis force and are given by (Unno et al. 1989)

\[
\omega_* = \omega + \delta \omega,
\]

\[
= \omega - m\Omega_\ast \sum_s \int dr r \rho [2\chi_{s *}^2 \xi_{s*}^2 + (\phi_{s*}^2)^2] \]

\[
= \omega - m\Omega_\ast J_{s*},
\]

where \( \xi_{s*} \) and \( \phi_{s*} \) are the radial and transverse components of the displacement eigenfunction (see eq. [34]). Together with equation (8), it is clear that, to first order, stellar rotation has no effect on modes with \( m = 0 \). (For simplicity we have ignored the perturbation to the eigenfunctions, which could be substantial and would modify the overlap integral.) To find the effect of rotation on the amplitudes of tidally excited modes with \( m \neq 0 \), we substitute the Fourier series expansion of the reduced forcing function (eq. [9]) in equation (8) and carry out the integration, which yields the following expression for the reduced mode amplitude:

\[
A_s = \frac{\exp(\text{im}\Omega_* t)}{2}
\]

\[
\times \left[ \sum_{n=1}^{\infty} \frac{(D_{n}^{2m} + iC_{n}^{2m}) \exp (-i\Omega_* t \pm i\phi_*)}{\sqrt{[r^2 - (n - ms)^2]^2} + 4d_s^2(n - ms)^2} \right]^2
\]

\[
\times \left[ \Omega_\ast \left( \frac{r_s^2}{(r_s^2 - (n - ms)^2)^2} + 2imd_s \xi_{s*} \right) \right],
\]

where

\[
r_s = \frac{\Omega_*}{\Omega_\ast}, \quad s = \Omega_\ast /
\]

\[
tan \phi_s = \frac{2dn(2ms - n)}{(r_s^2 - (n - ms)^2)^2}, \quad tan \phi_* = \frac{2dn(2ms + n)}{(r_s^2 - (n + ms)^2)^2}.
\]

It is easy to show that for \( m = \pm 2 \), \( \text{Re} C_{n}^{2m} = 0 \) and for \( n \) greater than a few, \( \text{Im} C_{n}^{2m} \approx -\text{sign} m \text{ Re} D_{n}^{2m} \).

Making use of these in equation (17), we obtain

\[
A_{|m|} \approx \exp(\text{im}\Omega_* t)
\]

\[
\times \sum_{n=0}^{\infty} \frac{D_{n}^{2m} \exp [\pm i\Omega_* t \mp i\phi_*(|m|)]}{\sqrt{[r_s^2 - (n - |m| s)^2]^2} + 4d_s^2(n - |m| s)^2}.
\]
2.4. Effect of Orbital Evolution on Mode Amplitude near Resonance

Orbital evolution causes a resonant mode to move off resonance with time. Thus amplitudes of resonant modes do not diverge even in the limit of vanishing damping. We showed in § 2.1 that the mode amplitude is proportional to $1/\delta r_\text{e}$ if $d_\text{e} \ll \delta r_\text{e}$. Physically this is the result of the mode amplitude building up as the mode is kicked in phase at periastron over $1/\delta r_\text{e}$ orbits. If the change in the orbital period is $\Delta T$ in one orbit, then the phase difference of a mode at periastron in two consecutive orbits is $\omega_\text{e} \Delta T$. Therefore, the tidal force will be out of phase with the mode in $2m(\omega_\text{e} \Delta T)^{1/2}$ orbits. This implies that there is an effective lower limit, $\delta r_{\text{min}}$, to $\delta r_\text{e}$ for the purpose of calculating the tidal evolution of excited modes. A straightforward calculation yields

$$
\delta r_{\text{min}} \approx \left[ \frac{2 \pi}{2 m} \left( \frac{\Delta T}{T} \right) \right]^{1/2}.
$$

Thus, because of orbital evolution, it is not physically meaningful to consider resonance where the ratio of the orbital period to the mode period is any closer to an integer than the value $\delta r_{\text{min}}$. We calculate $\Delta T$ below, which we then substitute in equation (20) to determine $\delta r_{\text{min}}$.

The change in period $\Delta T$ is related to the decrease in orbital energy in one orbit, $\Delta E_{\text{orb}}$, as follows:

$$
\Delta T = \frac{3}{2} \frac{\Delta E_{\text{orb}}}{E_{\text{orb}}},
$$

(21)

The change in the orbital energy is equal to the work done on the star by the tidal force, which is

$$
\Delta E_{\text{orb}} = \int_0^T dt \int d^3 r \, \nu \cdot F_{\text{tide}},
$$

$$
= 4\pi GM_p \sum_{n_r, m} \frac{Q_{nm}}{2 l + 1} \int_0^T dt \, \frac{Y_m^*(n/2, \phi_{\text{orb}})}{R^{l+1}(t)} \frac{dA_{nm}}{dt},
$$

(22)

where $\nu$ is the velocity induced in the star by the tidal force and is equal to $\sum_n \xi_n A_{nm}/dt$. We calculate the time integral in the above equation using equations (9) and (10) and the result, for $b_\text{e} \gg 1$ for the low-order modes, i.e., the static tide limit, is given below:

$$
\frac{\Delta E_{\text{orb}}}{E_{\text{orb}}} = 32\pi k h(e) \left( \frac{\Gamma_{\text{tide}}}{\Omega_\text{e}} \right) \left( \frac{M_p M_*}{M_*^2} \right) \left( \frac{R_*}{a} \right)^8,
$$

(23)

where $k$ is the apsidal motion constant given by

$$
k = \frac{2\pi G}{5R_*^5} \sum_n Q_{nm}^2,
$$

(24)

$\Gamma_{\text{tide}}$ is the dissipation rate of the tide, which is in general different from the mode dissipation rate introduced earlier, and

$$
h(e) = \frac{3072 + 47616e^2 + 97920e^4 + 35520e^6 + 1200e^8}{2048(1 - e^2)^{15/2}}.
$$

(25)

In deriving equation (23) we took $\Gamma_{\text{e}}/\omega_\text{e}^2 = \Gamma_{\text{tide}}(R_*^2/GM_p)$. This is a crude description for the dissipation of static tides in terms of the normal modes of the star and corresponds to assuming that the lag angles of the star associated with different modes are equal. (It is easy to show that the lag angle associated with a mode, for slowly rotating stars, is $2\Gamma_{\text{e}}/\omega_\text{e}^2$.) We note that our expression for the change in orbital energy (eqs. [23]–[25]) is identical to that of Hut (1981) for nonrotating stars.

Combining equations (20), (21), and (23) we find the minimum possible value of $\delta r_\text{e}$ to be

$$
(\delta r_{\text{min}})^2 = 24r_\text{e}k h(e) \left( \frac{\Gamma_{\text{tide}}}{\Omega_\text{e}} \right) \left( \frac{M_p M_*}{M_*^2} \right) \left( \frac{R_*}{a} \right)^8.
$$

(26)

The lag angle of the tidal bulge, $\delta_{\text{tide}}$, is related to $\delta r_{\text{tide}}$: $\delta_{\text{tide}} \approx 2\Gamma_{\text{tide}} \Omega_* (R_*^2/GM_p)$. This can be used to express the above limit in terms of the lag angle.

The angular momentum of the orbit also changes as a result of tidal interaction, which causes a change in the orbital eccentricity. However, in the absence of energy transfer from the orbit to the star the orbital period remains unchanged. Therefore, a change in $e$ alone does not change the phase relationship of a mode and the forcing function at periastron. Thus we do not expect our result given in equation (26) to be significantly modified when angular momentum transfer is included.

Finally, we note that it can easily be shown that the apsidal motion of the system has the same effect on the mode amplitude and resonance as the stellar rotation discussed in § 2.3.

3. Observation of Pulsation

The pulsation of a star can be detected by one of the following three techniques: (1) by observing photometric or flux variations; (2) by spectroscopic observation, i.e., by measuring the time-dependent surface velocity; or (3), for binary pulsars, by observing the pulse arrival delay due to the periodic variation of the neutron star orbit caused by the time-dependent quadrupole moment of the oscillating star. The mode amplitude calculated in the last section needs to be transformed to yield these observational quantities. The details of these transformations are described below. It should be noted that one can distinguish tidally induced pulsation from the intrinsic pulsation of the star because the former has a definite phase relation with the orbital motion.

3.1. Flux Variation Associated with Pulsation

Associated with the tidally excited modes there are variations of the star's luminosity. The observed flux variation arises from the change in the surface temperature of the star as well as the change in its projected surface area. To first order, we can calculate these two effects separately and add them to obtain the total flux variation.

The flux change at the stellar surface associated with a mode $(n, l, m)$ is given by

$$
\delta F_{nm}(R_* \theta, \phi) = A_{nm} \delta F_{nm}(R_* Y_m(\theta, \phi),
$$

(27)
where $\delta F_{\ell}(\tau)$ is the flux variation eigenfunction. To calculate the observed flux variation, $\delta F_{\ell,\text{obs}}$, we need to integrate the above expression over the hemisphere of the star facing us. Thus the expression for the fractional observed flux variation is

$$
\frac{\delta F_{\ell,\text{obs}}}{F_{\text{obs}}} = \frac{\left\{ \int d\Omega \mu f_{\ell,\text{LD}}(\mu) Y_{\ell m}(\theta, \phi) \delta F_{n}(R_*) \right\}}{\int d\Omega \mu f_{\ell,\text{LD}}(\mu) F} ,
$$

where

$$
\mu = n_* \cdot n_{\text{obs}} ,
$$

$n_*$ is a unit vector normal to the stellar surface, $n_{\text{obs}}$ is a unit vector pointing to the observer (from the center of the star), $F$ is the unperturbed flux, and $f_{\ell,\text{LD}}$ is the limb-darkening function, which can be written as

$$
f_{\ell,\text{LD}}(\mu) = a_0 + a_1 \mu + a_2 \mu^2 ,
$$

where $a_0 \approx 0.3726$, $a_1 \approx 0.6500$, and $a_2 \approx -0.0226$ (see, e.g., Kopal 1959). The equation in expression (28) does not include the variation in the projected area of the star, which is calculated below.

Specializing to $\ell = 2$ modes and making use of the last four equations, we find the observed flux for the $m = 0$ modes to be

$$
\frac{\delta F_{\ell,\text{obs}}}{F_{\text{obs}}} = \frac{a_{\text{const}} f_{\ell,\text{LD}}}{4} \left( \frac{5}{16\pi} \right)^{1/2} \left[ K_{n_2} \frac{\delta F_{n_2}(R_*)}{F} \right] \tilde{A}_{n_20}(3 \cos^2 \theta - 1) ,
$$

and the observed flux for the sum of the $m = -2$ and $m = 2$ modes to be

$$
\frac{\delta F_{\ell,\text{obs}}}{F_{\text{obs}}} = \frac{a_{\text{const}} f_{\ell,\text{LD}}}{2} \left( \frac{15}{32\pi} \right)^{1/2} \left[ K_{n_2} \frac{\delta F_{n_2}(R_*)}{F} \right]
\times \left[ \text{Re} \left( \tilde{A}_{n_22} \right) \cos(2\phi_0 - 2\Omega_* t) - \text{Im} \left( \tilde{A}_{n_22} \right) \sin(2\phi_0 - 2\Omega_* t) \right] \sin^2 \theta ,
$$

where $i$ is the inclination angle of the orbit and $\phi_0$ is the azimuthal angle of the line of sight projected onto the orbital plane and is related to the longitude of periastron ($\omega$) by $\phi_0 = \omega - \pi/2$. The factor $a_{\text{LD}}$, which arises from limb darkening, is defined as

$$
a_{\text{LD}} = \frac{15a_0 + 16a_1 + 15a_2}{5(3a_0 + 2a_1 + 1.5a_2)} .
$$

Note that the observed flux variations for $m = 0$ and $m = \pm 2$ modes (eqs. [31] and [32]) have a different dependence on the inclination angle. This can be used to determine the orbital inclination angle (see § 4 for details).

The calculation of the flux variation eigenfunction, $\delta F_{n,\ell}$, requires solving coupled linear differential equations for nonadiabatic oscillations, which is quite involved. However, for most purposes it is reasonably accurate to use the relation $\delta F_{n}(R_*)/F \approx 4 \Delta T(\tau = \frac{3}{2})/T$, where $\Delta T(\tau = \frac{3}{2})$ is the Lagrangian temperature perturbation eigenfunction at optical depth $\tau = \frac{3}{2}$, which can be calculated using an adiabatic oscillation code. The product, $K_n \delta F_{n}(R_*)$, which contains the dependence of the observed flux variation on the structure of the secondary star, will be calculated for several different stars in § 5.

Next we calculate the change in the projected surface area of the distorted star associated with different modes. The change in the area also causes the observed flux to vary. The distortion of the surface due to one mode can be found from the displacement eigenfunction, which can be written as

$$
\xi_{n,m}(\theta, \phi) = \left\{ \xi_{n,m}^r(\theta, \phi) \frac{\partial}{\partial \theta} , \xi_{n,m}^t(\theta, \phi) \frac{\partial}{\sin \theta \partial \phi} \right\} Y_{\ell m}(\theta, \phi) ,
$$

where $\xi_{n,m}^r$ and $\xi_{n,m}^t$ are the radial and transverse displacement eigenfunctions, respectively. These eigenfunctions are normalized so that the energy in the mode is unity, i.e.,

$$
\omega_n^2 \int d^2 \xi \rho \xi_{n,m} \cdot \xi_{n,m} = 1 .
$$

The fractional change in the projected area of the star associated with a mode, $\delta A_{n,m}/A_{n,m}$, can be expressed in terms of the surface displacement. The flux variation arising from this change in projected area can be shown to be identical to the expression for $\delta F_{n,\text{obs}}/F_{\text{obs}}$ of equations (31) and (32), provided that we replace $a_{\text{LD}}[\delta F_{n_2}(R_*)/F]$ in these equations by

$$
\left[ 8 \xi_{n_2}^r(R_*) - 6 \xi_{n_2}^t(R_*) \right] .
$$

This allows us to define an effective flux variation eigenfunction $\delta F_{n,\text{eff}}$ as

$$
\frac{\delta F_{n,\text{eff}}}{F} = \frac{\delta F_{n_2}}{F} + \frac{2}{\omega_{m,\text{LD}}} \left[ 4 \xi_{n_2}^r(R_*) - 3 \xi_{n_2}^t(R_*) \right] .
$$

The total observed flux variation associated with the modes is obtained from equations (31) and (32) by replacing $\delta F_{n_2}/F$ in these equations with the effective flux eigenfunction as given by the above equation.

3.2. Surface Velocity Amplitude Associated with Modes

The calculation of the surface velocity proceeds in a similar manner to the flux variation calculation. The line-of-sight surface velocity, integrated over the stellar hemisphere facing the observer, is given by

$$
V_{\text{obs}}^{\text{obs}} = \frac{\omega_n A_{n,m}}{\pi} \int d\Omega \left| n_* \cdot n_{\text{obs}} \right| n_{\text{obs}} \cdot \xi_{n,m}(R_*) ,
$$

where the first factor in the above integral weighs the velocity by the projected area.

Carrying out the integral given in equation (37) for $\ell = 2$, we obtain the following expression for the observed velocity amplitude for an $m = 0$ mode,

$$
V_{n_20}^{\text{obs}} = \frac{1}{45\pi} \omega_n K_n \left[ 2 \xi_{n_2}^r(R_*) + 3 \xi_{n_2}^t(R_*) \right]
\times \sin^2 \theta \left[ \text{Re} \left( \tilde{A}_{n_2} \right) \cos(2\phi_0 - 2\Omega_* t) - \text{Im} \left( \tilde{A}_{n_2} \right) \sin(2\phi_0 - 2\Omega_* t) \right] ,
$$

and for the sum of the $m = \pm 2$ modes,

$$
V_{n_22}^{\text{obs}} = \frac{2}{15\pi} \omega_n K_n \left[ 2 \xi_{n_2}^r(R_*) + 3 \xi_{n_2}^t(R_*) \right]
\times \sin^2 \theta \left[ \text{Re} \left( \tilde{A}_{n_22} \right) \cos(2\phi_0 - 2\Omega_* t) - \text{Im} \left( \tilde{A}_{n_22} \right) \sin(2\phi_0 - 2\Omega_* t) \right] .
$$
TIDAL EXCITATION OF MODES IN BINARY SYSTEMS

These expressions have the same functional dependence as the flux variation derived in the previous subsection.

3.3. Perturbation to Pulsar Orbit Due to Pulsation

The contribution to the gravitational potential at the position of the primary star due to one mode of pulsation of the secondary can be easily calculated using equation (1) and is given below

$$\delta \Psi_* = -\frac{4\pi G Q_2}{(2^+)^{1+1}} Y_{\ell m}(\pi/2, \phi_{\text{orb}}(t)) A_{\ell m}(t).$$

(40)

The equation for the perturbation to the primary's position, R, due to this potential, to first order in R, is as follows:

$$\frac{d^2 R_1}{d t^2} = -\frac{Q_2}{(R_0/\alpha)^3} \left[ R_1 - 3\mathbf{R}_0 (\mathbf{R}_0 \cdot \mathbf{R}_1) \right] - \frac{M_*}{M_*} \nabla \delta \Psi_*, \quad (41)$$

where R_0 is the primary's unperturbed position with respect to the center of mass, R = R_0/|R_1|, and M_* is the total mass in the binary system. The first term in the above equation arises as a result of the change in the gravitational force on the primary star because of the perturbation to its position. In general the solution of equation (41) must be obtained numerically. However, for modes of small b, the mode amplitude A_{\ell m} is a sinusoidal function, and if r_0 \gg 1 (highly eccentric orbit), the first term on the right-hand side of the above equation can be ignored, and equation (41) can be solved to yield the ratio of the perturbed to the unperturbed velocity of the primary. The result below is for quadrupole modes:

$$\frac{|\omega_0 R_1|}{d \Omega_*} \approx \frac{16\pi^2}{25} \left( \frac{\omega_0 Q_2^2 M_*}{M_*} \right) A_{\ell m}(t) \frac{\Omega_0}{a^2 R_0^2} \left| Y_{\ell m}(\pi/2, \phi_{\text{orb}}(t)) \right|^2.$$

(42)

Using results from § 2.2, we find that the fractional perturbation to the primary's orbital velocity is typically very small.

It is useful to express the perturbation to the orbit as a delay in the pulse arrival time, \delta t_*, when the primary is a pulsar. This delay is a periodic function and is given below:

$$\delta t_* = (\sin i)(R_1 \cdot \mathbf{n}_*)/c,$$

where \mathbf{n}_*, a unit vector, is the projection of the observer's direction on the orbital plane.

4. DETERMINATION OF ORBITAL INCLINATION ANGLE

We describe a technique for determining the inclination angle of an eccentric orbit by fitting the observed light curve near periastron to theoretical light curves. The main physical idea behind this method is simple. In the static tide limit, when periods of the low-order modes are small compared to the periastron passage time, the star adjusts its shape to the instantaneous tidal force and thus the tidal bulge points in the direction of the other star irrespective of its rotation rate. Therefore, as the secondary star makes its passage through periastron, the tidal bulge continues to point toward the primary star, and the time dependence of the tidally distorted surface of the star projected onto the plane of the sky gives rise to the observed light curve. Since the observed part of the surface of the star depends on the inclination angle, the shape of the light curve also depends on the angle i.

For an orbital inclination angle i and a longitude of periastron \omega, the observed flux variation, \delta F_0, is the sum of the contribution from m = \pm 2 and m = 0 modes and can be written in the following form (see eqs. [31] and [32]):

$$\delta F_0(t) \propto \sin^2 i \left[ \sin (2\omega - 2\Omega_0) t \Psi_2(t) - \cos (2\omega - 2\Omega_0) t \Psi_1(t) \right] + (3 \cos^2 i - 1) \Psi_0(t), \quad (43)$$

where \Psi_1 and \Psi_2 are the real and imaginary parts of the mode amplitude summed over all m = 2 modes (weighted by K_{20} \delta F_{20}^2), as seen in the rotating frame of the star, and \Psi_0 is the sum over all m = 0 modes. In the static tide limit, the tidal bulge always points toward the other star; therefore, the observed flux variation is independent of the stellar rate \Omega_0 (see the derivation below). We see from equation (43) that the shape of the observed light curve in general depends on the orbital inclination angle because of the different relative contributions from m = 0 and m = \pm 2 modes. For the particular case of circular orbits, however, the amplitudes of m = 0 modes are time-independent so that the shape of the light curve is independent of i. Of course, the magnitude of the observed flux variation still depends on i, but because of the uncertainty in stellar mass and structure this is of limited use in determining i.

If the dimensionless mode frequency b for the first few p- and g-modes is greater than \sim 5, then the time derivative terms in equation (2) can be neglected, and the mode amplitudes are equal to f_{\ell m}/(a^3). In this static tide limit the function \Psi_0(t) is proportional to the reduced forcing function f_{10}, and \Psi_1(t) and \Psi_2(t) are proportional to the real and imaginary parts of f_{12} (it is easy to show that the tidal bulge points in the direction of the other star in this case). Thus, using equations (5) and (43), we obtain the following explicit expression for the light curve:

$$\delta F(t) = -F_c \left\{ \frac{1 - 3 \sin^2 i \sin^2 [\phi_{\text{orb}}(t) - \omega]}{[R(t)/a]^3} \right\}, \quad (44)$$

where F_c is a constant factor that depends on the stellar mass and structure and is independent of the orbital parameters. F_c can be calculated either by determining the normal mode eigenfunctions and the associated flux variation at the surface or by using von Zeipel's theorem (cf. Kopal 1959), according to which the emergent radiation flux, in the static tidal case, is proportional to the local gravity. We note that the rotation rate of the star has dropped out from the above equation for the observed flux variation as was claimed earlier. Thus a comparison of the shape of the observed light curve with the function given in equation (44) provides a robust method, which does not depend on the unknown mass, structure, and rotation rate of the star, for determining the orbital inclination angle.

Light curves, corresponding to several different inclination angles, for a 10 M_\odot main-sequence star in an orbit of eccentricity e = 0.4 and period of 100 days are shown in Figure 2. The value of \phi_0 is chosen to be zero. We emphasize that the magnitude of the flux variation in these graphs depends on the mass and structure of the star as well as the inclination angle. However, the shape, which is what we suggest should be used to compare with the observed light curve to determine the inclination angle, does not. We note that for small i, the contribution to the observed light curve is dominated by m = 0 modes. As i increases, the contribution from the m = \pm 2 modes increases, resulting in a significant change in the shape.
of the light curve. The shape of the light curve is most sensitive at intermediate values of \( i \) (20 \( \leq i \leq 70 \)); however, there are quite noticeable variations for all \( i \).

The order of magnitude of the flux variation can be estimated crudely from the change in the Lagrangian temperature at the point in the star’s atmosphere where the optical depth is \( \frac{3}{4} \). Since the tide is very nearly static, we can integrate the hydrostatic equilibrium equation from the surface over a fixed column of gas corresponding to an optical depth of \( \frac{3}{4} \), with and without the tidal force, to obtain the fractional change in the Lagrangian pressure, and find it to be \( \frac{\Delta P}{P} = 2 \frac{\delta R_*/R_*}{R_*} + 2(M_*/M_\odot)(R_*/d)^3 \), where \( g \) is the gravitational acceleration at the stellar surface, \( P \) is the tidal acceleration, \( d \) is the distance between the primary and the secondary, and \( \delta R_* \) is the change in the star’s radius, which is approximately equal to \( (M_*/M_\odot)(R_*/d)^3 R_* \). Thus the fractional change in the lumi-
5. APPLICATIONS

In this section we apply our formalism to two radio binary pulsar systems, PSR J0045—7319 and PSR B1259—63, and discuss the possibility of direct observation of the tidal excitation of modes in these systems. We also present results of numerical calculations for general polytropic stars in order to examine further the dependence of tidal excitation on stellar structure (§ 5.2).

5.1. Applications to PSR J0045—7319 and PSR B1259—63

Recently, Kaspi et al. (1994a) have reported the discovery of a radio pulsar (PSR J0045—7319) in the Small Magellanic Cloud. Measurements of the Doppler time delay have indicated that the pulsar is in a highly eccentric (e = 0.80798) orbit with a period of 51.169 days. Kaspi et al. (1994a) have optically identified the companion star as a 16th magnitude B star with mass ≈10 M\(_\odot\). In our calculations we also assume that the mass of the pulsar is 1.4 M\(_\odot\).

Using a B star model kindly provided to us by the Yale group, we have computed the eigenfunctions and the overlap integrals (Q\(_{ij}\)) for quadrupole oscillations. We find that the f-mode frequency for this system corresponds to a \(b_2\) value of approximately 21.7, and the g\(_3\)-mode has a \(b_2\) value of 3.96. The frequencies of \(m = \pm 2\) modes are shifted by approximately \(\pm 1\) \(\mu\)Hz, if the B star is rotating synchronously at periastron, which corresponds to a shift in the value of \(r_2\) of \(\pm 4.4\). The radiative damping time depends on the mode order: for g\(_2\) and g\(_3\)-modes we estimate it to be about 10\(^3\) yr, and for a g\(_{10}\)-mode to be about 1 year. The turbulent damping time for these modes is a factor of a few larger. The radiative damping time for the p\(_3\)-mode is about a year. Thus, for none of these modes do we expect the mode energy to have any dependence on damping.

For the 10 M\(_\odot\) ZAMS stellar model the product \(K_{\odot} \delta f_{\odot}(R_\odot)/F\) is very nearly independent of \(n\) for \(g\)-modes of order 4—10, beyond which this product starts to decrease. The probability that one of the approximately 25 low- to intermediate-order \(g\)-modes (counting \(m = 0\) and \(m = \pm 2\) as three distinct modes due to their different frequencies because of rotational splitting) has a value of \(r_2\) that is close to resonance (\(|\delta r_2| \leq 0.05\)) is very high. In fact, for the stellar model we use, the g\(_2\)-mode has \(\delta r_2\) of 0.02, which is the smallest value for the first 10 harmonic \(g\)-modes. The smallest physically meaningful value of [\(\delta r_2\)], resulting from the orbital evolution considerations (see § 2.4), is estimated using equation (26) to be 0.002. Thus, \(\delta r_2 = 0.02\) for the g\(_2\)-mode is physically acceptable. The g\(_7\)-mode, which has the largest amplitude, has a period of 1.07 days (corresponding to \(b_2 = 2.96\)). The stellar flux variation associated with this mode is 2.3 mmag, and the surface velocity amplitude is 70 m s\(^{-1}\) (calculated using eqs. [38] and [39]). For comparison, the flux variation associated with the f-mode, away from periastron, is about 10 mmag.

The perturbation to the pulsar's orbital velocity due to the time-dependent quadrupole moment of the g\(_7\)-mode has been calculated by integrating equation (41) and is found to be 10\(^{-3}\) cm s\(^{-1}\). The density fluctuation in the convection zone of the star also causes a random perturbation to the pulsar's orbital velocity that is estimated to be about 10\(^{-6}\) cm s\(^{-1}\) (see Appendix).

It should be stressed that the amplitudes of these tidally excited modes (shown in Fig. 1) have perfect phase coherence from one orbit to another, which should be helpful in observing these small-amplitude oscillations. On the other hand, since the mode with the largest observed amplitude is one with a frequency closest to resonance, it would be very difficult to make a precise identification of the order \(n\) of the observed mode, rendering its potential use in determining the properties of the star rather limited.

The light curve for PSR J0045—7319 near periastron, dominated by f- and low-order p-modes, is shown in Figure 3 for several different orbital inclination angles. The observed flux variation is about 10 mmag and is calculated assuming that the secondary star has a mass of 10 M\(_\odot\). The uncertainty in our estimate of the flux variation is about 20%, reflecting an uncertainty in the mass of the B star of 2 M\(_\odot\). The shape of the light curves in Figure 3, however, is independent of the mass of the star, and can be used to determine the inclination angle of the orbit.

We have calculated the periastron advance of the PSR J0045—7319 system, due to dynamical tides, by integrating equation (41) and summing the result over all modes. The advance is found to be 0.7 per orbit. This result is in excellent agreement with the classical apsidal motion formula (Claret & Giménez 1993); our result is about 1% larger compared to the classical formula. Thus the static tide approximation for calculating apsidal motion is an excellent approximation for this system.

The recent accurate observations of PSR B1259—63 (Johnston et al. 1994) indicate that its eccentricity and orbital period are approximately 0.8698 and 1236.79 days, respectively. The companion is a Be star with a mass of 10 M\(_\odot\). Thus the periastron distance is 20R\(_\odot\), and the \(b_2\) values for the modes of the Be star are about a factor of 13 larger than those for the SMC B star. Therefore, the amplitude near periastron, which falls off as the distance cubed, is expected to be a factor of about 70 smaller than that of the SMC system, and the oscillation amplitudes of even the g-modes, because of the large \(b_2\) values, are too small to have any observational consequences.

5.2. Application to Polytropic Stars

In this section we apply our discussion of mode amplitudes to a general polytropic star. In particular, we examine how the flux variation and surface velocity depend on the structure of the star. Our calculations include the f-mode as well as low-order p-modes and g-modes for polytropic stars with index \(n_p\) ranging from 1.5 to 3.

For a polytropic star, the eigenfunctions and overlap integrals can be expressed in nondimensionalized forms which
Fig. 3.—Light curve near periastron for the B star in the SMC pulsar system PSR J0045–7319 for several orbital inclination angles; $\phi_0$ is taken to be 25°:24, corresponding to a longitude of periastron of $\omega = 115°:24$, as quoted in Kaspi et al. (1994a). The shape of these curves can be used to determine the inclination angle for this system.

depend only on the index $n_m$. We have calculated these quantities without the Cowling approximation, and our results are in excellent agreement with values quoted in Robe (1968) and Lee & Ostriker (1986).

The dependence of the mode energy on the stellar structure is contained in the parameter $K^2_2$. We calculate this numerically as a function of the polytropic index $n_m$ for the quadrupole $f$-mode and low-order $p$-modes and $g$-modes. The results are plotted in Figure 4 in units of $GM^2_2/R_*$, Mode energy is obtained by multiplying $K^2_2$, $|\tilde{A}_{n2m}|^2$ (calculated using eq. [10]), and $f^2_{\text{const}}$. While the $f$-mode and the $p$-modes have comparable values of $K^2_2$, the $g$-modes are a few orders of magnitude smaller. This is a result of two separate effects. First, the overlap integral is largest for the $f$-mode and decreases with increasing mode order $n$. Second, the eigenfrequencies for the $p$-modes are much larger than those for the $g$-modes. Thus,
even though \( p \)-modes and \( g \)-modes could have comparable overlap integrals, \( K_g \) still differs significantly. Figure 4 shows that \( K_g^2 \) increases with \( n_{pl} \) for all modes. This might suggest that higher \( n_{pl} \) polytropes are more susceptible to tidal excitations. However, it is important to keep in mind that the mode energy is also proportional to \( |A_e|^2 \), which has a sensitive dependence on the mode frequency through \( b_\pi \); \( |A_e|^2 \) decreases exponentially with \( b_\pi \). Since, for a fixed mode, the mode frequency (and hence \( b_\pi \)) tends to increase with \( n_{pl} \), \( A_e \) tends to decrease with \( n_{pl} \). Therefore, depending on the orbital parameters, the actual mode energy might in fact decrease with \( n_{pl} \).

The fractional flux variation from the star depends on the dimensionless parameter \( K_g \delta F_{\omega}(R_\star)/F \), which is approximately equal to \( 4K_s \Delta T(\tau = \frac{3}{4})/T \), where \( \Delta T(\tau = \frac{3}{4}) \) is the Lagrangian temperature variation at optical depth \( \frac{3}{4} \). We calculate \( \Delta T \) numerically using an adiabatic oscillation code. This is then used to compute \( K_s \delta F_{\omega}(R_\star)/F \), and the results are shown in Figure 5 for several different quadrupole modes as a function of \( n_{pl} \) for a fixed stellar mass of 1.0 \( M_\odot \). The \( p \)-modes correspond to acoustical compression or expansion, while the \( g \)-modes are effectively incompressible. Hence, \( p \)-modes have correspondingly larger values of \( \Delta T/T \). For polytropic stars, the dimensionless parameter \( K_s \delta F_{\omega}(R_\star)/F \) is found to be almost independent of the mass of the star.

The surface velocity amplitude depends on the parameters \( K_s \omega_1 e_1^2 \) and \( K_s \omega_1 e_2^2 \) (see eqs. [38] and [39]). These parameters are shown in Figure 6 in units of \((GM_\odot/R_\star)^{1/2}\). The \( p \)-modes have predominantly radial displacement at the surface, while \( g \)-modes have predominantly transverse displacement.

The shift of the mode frequencies due to rotation can be calculated using the dimensionless number \( J_{n_2} \) (see eq. [16]), which is shown in Figure 7, for several different modes, as a function of \( n_{pl} \). For \( p \)-modes, \( J_{n_2} \) approaches zero as \( n \) increases, since the transverse displacement becomes negligibly small compared to the radial displacement. For \( g \)-modes, it is the radial displacement that becomes negligible as \( n \) increases, hence \( J_{n_2} \) approaches \( 1/(\ell + 1) \) or 0.1667 for quadrupole modes (see, e.g., Unno et al. 1989). Therefore, \( J_{n_2} \) is bounded above by the \( f \)-mode and is always less than 0.5.

In order to obtain the actual flux variation or surface velocity, one needs to multiply the relevant products (displayed in Figs. 5 and 6) by the reduced mode amplitude \( A_e \), which depends on the orbital and modal parameters through \( r_\pi \) and \( e \). As an example, consider a binary system with \( e = 0.7 \), consisting of a 1.0 \( M_\odot \) compact primary, and a 5.0 \( M_\odot \) secondary star of polytropic index 2. To find the flux variations associated with the \( p_\pi \) and \( g_\pi \)-modes of the secondary, we read off from Figure 5 that the values of \( K_s \delta F_{\omega}(R_\star)/F \) are approximately 100 and 0.1, respectively. Assume that the orbital period is such that the \( g_\pi \)-mode has \( r_\pi \approx 23.8 \), \( b_\pi \approx 3 \), and the \( p_\pi \) mode has \( r_\pi = 188.5 \), \( b_\pi = 23.8 \). Using the procedures outlined in § 2, we calculate \( A_e \) for the \( g_\pi \) (\( p_\pi \)) mode with \( m = 0 \) and find the peak value of the reduced amplitude at periastron to be 0.02 (3.3 \( \times 10^{-4} \)) and an oscillation amplitude of 0.01 (1.3 \( \times 10^{-7} \)) away from periastron. The product of \( A_e \), \( K_s \delta F_{\omega}(R_\star)/F \), and \( f_{\text{cool}} = 2.1 \) yields the flux variations for these modes to be 20 (33) mmag at periastron and 10 (0.013) mmag for the oscillations away from periastron.

For concreteness, we have calculated in Figure 8 the fractional luminosity variations \( \delta L_{\omega}(R_\star)/L_\star \) for several low-order \( g \)-modes in a \( n_{pl} = 3 \) polytrope as a function of orbital period. To prevent close resonances from confusing the general trend, we have modified the mode frequencies slightly in such a way that \( \delta \sigma = 0.1 \) for all the modes considered. The mass of the
secondary polytropic star is taken to be $5 \, M_\odot$, and the mass of the primary star is $2 \, M_\odot$. To illustrate the dependence on eccentricity, $e$ has been chosen to be 0.8 and 0.4 (Figs. 8a and 8b, respectively). Figure 8 shows that $\delta F_{2,1} / F$ falls off rapidly with orbital period and that it is significantly larger for more eccentric orbits. For larger orbital periods, variations due to higher overtone $g$-modes dominate because of their smaller $b_2$ values.

6. SUMMARY AND DISCUSSION

We have calculated the energy and amplitude of the tidally excited modes of a star in a binary system of arbitrary eccentricity. The effects of damping, resonance, orbital inclination, and rotation of the star on the observed mode amplitude have been investigated in detail. The important parameters for this problem are $b_2$, the ratio of the periastron passage time to the mode period; $\delta r_2$, the ratio of the difference between mode frequency and the nearest resonance frequency (resonance frequencies are integer multiples of the orbital frequency) to the orbital frequency; and $d_2$, the ratio of the orbital period to the mode damping time.

Modes of large $b_2$ couple poorly to the tidal force. The oscillation amplitude decreases with increasing $b_2$, as approximately $\exp \left(-1.3b_2 e^{-0.25}\right)$, where $e$ is the orbital eccentricity. The $p_1$, $f_1$, and low-order $g$-modes of main-sequence stars, for most binary systems, have values of $b_2$ that are much greater than unity; thus, their oscillation amplitudes are exponentially suppressed. Since the overlap integrals decrease with increasing mode order ($n$), the moderate-order quadrupole $g$-modes with $b_2 \sim 1$ and the smallest value of $\delta r_2$, i.e., the modes most nearly resonant, have the largest oscillation amplitudes.

The mode amplitude is independent of damping so long as $d_2$ is less than $\delta r_2$. This condition is expected to be satisfied for the low- to moderate-order $g$-modes of main-sequence stars which have damping times of at least a few years. In the limit of $d_2 \ll \delta r_2$, the mode amplitude is proportional to $1/\delta r_2$, as expected of an oscillator driven off resonance. Due to orbital evolution, a mode which is in perfect resonance with the orbit, i.e., $\delta r_2 = 0$, moves off resonance with time. This places a severe upper limit to the amplitudes of modes near resonance ($\delta 4$).

The mode amplitude and phase of tidally excited modes is the same from one orbit to another. This property should be useful in observational searches for tidally excited modes since it distinguishes them from intrinsically excited modes. Rotation of the star (in the limit of slow rotation rate) causes the frequencies of $m \neq 0$ modes to be shifted, but otherwise does not change the properties or time dependence of the mode amplitude.

We have applied this tidal excitation theory to the recently discovered pulsar in the SMC, PSR J0045–7319, which is in a binary system of eccentricity 0.81 and orbital period 51 days. The companion star, believed to be a $10 \, M_\odot$ B star, has a periastron distance of about 5 stellar radii. Thus this system is a good candidate for studying the tidal excitation of modes. Using a $10 \, M_\odot$ main-sequence stellar model, we find that the mode with the largest oscillation amplitude is the $g_{2,1}$ mode, which has $b_2 = 2.96$ and $\delta r_2 = 0.02$. We find that the flux variation associated with this mode is 2.3 mmag, and the surface velocity amplitude is $70 \, m \, s^{-1}$. The perturbation to the orbital velocity of the pulsar, as a result of the time-dependent quadrupole moment of the B star due to this mode, is about $10^{-3} \, cm \, s^{-1}$ at periastron. These amplitudes are independent of mode dissipation time because the value of $d_2$ is much less than $\delta r_2$ (the radiative and turbulent dissipation timescales are about 100 yr).

It should be pointed out that the small value of $\delta r_2$ for the
$g_7$-mode is a coincidence. A change of 0.2\% in the mode frequency changes $\delta r_s$ by 0.1. Thus $\delta r_s$ should be regarded as a random variable. Therefore, given 25 modes that are similar in terms of their coupling to the tidal force, we expect one of them to have $|\delta r_s|$ less than 0.02. In fact, modes $g_4-g_{10}$ of the Yale star model have very similar mode amplitudes for a given $\delta r_s$. Thus it is not surprising to find one of them, counting $m = 0$ and $m = \pm 2$ as three different modes because rotation lifts the $m$-degeneracy of the frequencies, with $\delta r_s = 0.02$. Orbital evolution places a lower limit on $|\delta r_s|$, which we estimate to be 0.002.

The tidal distortion of the star near periastron is dominated by low-order quadrupole $p$- and $g$-modes and the $f$-mode, for which $b_s$ is much greater than unity, and therefore the light curve near periastron follows the time dependence of the tidal force closely. Since the observed light curve depends on the
projection of the tidally distorted star onto the plane of the sky, which in turn depends on the inclination of the orbit, the shape of the light curve near periastron allows the determination of the inclination angle of the orbit for eccentric orbits. In particular, this technique should be applicable to the SMC pulsar, PSR J0045−7319, for which we estimate the flux variation near periastron to be ≈10 mmag and the surface velocity to be about 30 m s⁻¹.

For the SMC pulsar we further determine that the periastron advance due to the tidal distortion of the B star, treated dynamically, is about 0.7 orbit⁻¹. This is in excellent agreement (≈1.0%) with the static tidal calculation of the classical apsidal motion formula. The precession due to general relativity is estimated to be 2'0 orbit⁻¹ and that due to the quadrupole moment resulting from rotational distortion (for synchronous rotation) is about 2'' orbit⁻¹. Thus the total apsidal motion compares reasonably well with the current observational value of about 5'' orbit⁻¹ (Kaspi et al. 1994).

For the only other known binary radio pulsar with a main-sequence companion, PSR B1259−63, the flux variation near periastron is approximately 70 times smaller than that of the SMC system, and the oscillation amplitude of all g- and p-modes is negligibly small.

We have also calculated the stellar dependence of the mode energy, flux variation, and surface velocity amplitude by considering polytropic stars of index $n_p$ ranging from 1.5 to 3. These quantities can be written as a product of two terms: a term that depends only on the reduced mode amplitude ($\tilde{A}_n$)

**Fig. 7.** $I_{\alpha\beta}$ as a function of polytropic index $n_p$.Rotational splitting of the eigenfrequencies is, to first order in $\Omega_\ast$, proportional to $J_{\alpha\beta}$ (see eq. [16]). For $p$-modes, $I_{\alpha\beta}$ approaches zero as the order $\alpha$ increases. For $g$-modes, $I_{\alpha\beta}$ approaches $1/2$ as $\alpha$ increases.

**Fig. 8.** Fractional luminosity variation $\delta F_{\alpha\beta}/F$ for several low-order $g$-modes as a function of orbital period $T_{\text{orb}}$. The secondary star is taken to be an $n_p = 3$ polytrope with $M_{\ast} = 0.5 M_{\odot}$ and the primary star has $M_{\ast} = 2 M_{\odot}$. For the modes considered, $\delta_{\theta \varphi}$ is fixed artificially at 0.1, to avoid confusion due to close resonances, and $\delta_{\varphi} = 0.001$. The eccentricity of the orbit is taken to be 0.8 and 0.4 in (a) and (b), respectively. The lower limits for $T_{\text{orb}}$ of approximately 18 days (for $e = 0.8$) and 4 days (for $e = 0.4$) correspond to the onset of the overflow of the secondary's Roche lobe. For larger $T_{\text{orb}}$, the luminosity variation due to the higher overtone $g$-modes dominates because of their smaller $\delta_\varphi$ values. Moreover, the variations are significantly larger for the more eccentric orbit.
and a term that depends on the stellar structure (K, δF/F, etc.). The stellar structure–dependent term generally increases with n. However, this does not necessarily mean that higher n stars give rise to larger tidal effects. The mode frequency (and thus δn) increases with n, and consequently A decreases with n. Therefore, depending on the orbital parameters, tidal effects might in fact be larger for lower n stars.

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APPENDIX

In this appendix we estimate the perturbation to the primary star's orbital velocity due to density fluctuations in the convection zone of the secondary star.

We first calculate the quadrupole moment of the star due to fluctuating eddies in a 1 scale height thick layer (H) of fluid at a radial distance r from the center. Let the mean mass of this layer be M_H and the magnitude of the fluctuation be δM_H. Since the fractional density fluctuation, δρ/ρ, associated with turbulent flow is δρ/ρ ∝ δM_H/M_H, the Mach number of the turbulence, the fractional mass fluctuation of this layer is therefore δM_H/M_H ∝ H δρ/ρ/M_H. Let us assume that the luminosity carried by convection is fL_H, where L_H is the luminosity of the star. According to mixing-length theory fL_H ∼ 4πρr^2 C_H δρ/ρ ∝ M_H C_0^3 δM_H/H, where C_0 ∼ (gH)^1/2 is the sound speed and g is gravitational acceleration. Thus the rms fluctuation of the quadrupole moment is δQ_H ∼ r^2 δM_H ∼ rM_H^2 fL_H H^2/3/ρ. Since δQ_H ∝ (H^2 M_H)^1/3 ∝ (σH^2), the contribution to the quadrupole moment is dominated by eddies at the bottom of the convection zone. The perturbation to the gravitational potential due to the quadrupole moment δQ_H is

$$\delta \Psi \approx -\frac{4\pi G \delta Q_H}{5R^3}.$$  

We can determine the perturbation to the primary's orbital velocity by solving equation (41) with δΨ given in the above equation. If the convective timescale is much smaller than the orbital period, so that orbital modes are not resonantly excited by convection, we can discard the term in equation (41), and the solution of the resultant equation is given below:

$$\frac{1}{\Omega_3} \left| \frac{dR}{dt} \right| \sim f f' (\Omega_3 t_H)^2 \left( \frac{R_3}{R} \right)^2 \left( \frac{1}{\Omega_3 t_{KH}} \right)^2,$$

where f = r/R, t_H = H/V_H is the convective time, V_H is the convective velocity, and t_{KH} = GM_H^2/L_H R_H is the Kelvin-Helmholtz timescale of the secondary star. The dimensionless factors f and f' are roughly of order unity. Therefore, the random fluctuation to the primary's orbital velocity due to a 10 M_☉ main-sequence secondary star in an orbit with a/R_☉ ∼ 25 is a factor of 10^13 smaller than the mean orbital velocity.

REFERENCES


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