LINE-DRIVEN INSTABILITY GROWTH RATES IN WOLF-RAYET WINDS
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ABSTRACT
We calculate the linear growth rate of small-scale radiative instabilities for Wolf-Rayet (W-R) wind models
driven by multiline scattering. Our approach involves a second-order extension of our previously developed
nonisotropic-diffusion treatment of the multiline transfer. We confirm that the isotropizing effect of multiple
scattering in such dense winds acts to suppress the instability, in comparison to the optically thin winds of OB
stars. However, we also show that the inherent sphericity of the wind expansion leads to a significant residual
instability.

More specifically, the instability growth rate in W-R winds is reduced relative to the OB case by the ratio
of the photon mean free path to the radius, \( \lambda/r \), which is characteristically of order the inverse wind-
momentum number, \( P_{\text{wind}}^{-1} = L_{\nu}/M v_{\infty} c \sim 0.1 \). Even with this reduction, the instability is generally still strong
enough for base wind perturbations to be amplified by a very large number of e-folds (typically at least 40) by
the time they exit the multiple-scattering region. This can be expected to lead to extensive wind structure,
including strong clumping, throughout the observable part of the wind. Such extensively clumped wind structure
could provide a natural explanation for the "moving bumps" commonly observed in optical emission-line
spectra formed in W-R winds. It would also imply a reduction in mass-loss rates inferred from diagnostics
to the square of the wind density.

Subject headings: instabilities --- line: formation --- scattering --- stars: mass loss --- stars: Wolf-Rayet

1. INTRODUCTION

Optical and UV line observations of Wolf-Rayet (W-R) winds show variability that has been interpreted as evidence
for wind structure over a wide range of scales (Moffat et al. 1988; MacCandless 1992; Robert 1992; Moffat & Robert 1992;
see recent review by Moffat 1994), and polarization variability also indicates the presence of transient inhomogeneities
(Robert et al. 1989). The physical origin of these structures is not well understood, but since W-R winds are expected to be at
least partially radiatively driven, it is reasonable to wonder whether they could arise from an instability analogous to the
strong line-driven flow instability that has been extensively investigated for the lower density winds from OB stars (see,
e.g., Owocki 1992, and references therein). However, the optically thick character of W-R winds gives rise to a much more
isotropic radiation field than the nearly free-streaming case of OB winds, and this can be expected to reduce or perhaps even
eliminate the instability in W-R winds. The fundamental aim of this paper is to calculate the linear growth rate of this instability,
which by steady state models of optically thick W-R winds.

The quantitative approach adopted here is to extend to second order the nonisotropic diffusion (NID) approximation,
as applied in Gayley, Owocki, & Cranmer (1995, hereafter Paper I) to the effectively gray line-scattering treatment of
Friend & Castor (1983, hereafter FC). This FC treatment represents a generalization of the usual line-driven wind theory for
OB stars (Castor, Abbott, & Klein 1975, hereafter C&K) to the multiline scattering case appropriate for W-R winds. By
solving the associated equation for nonlocal gray transfer, the FC method can treat both optically thin OB winds and optically
thick W-R winds on the same footing. In our Paper I, we showed how the mean dynamics of the thick W-R wind case
should be modeled with a local, diffusion approximation for the effectively gray transfer. This was a first-order diffusion treat-
ment, since the principal aim was to compute the mean line force, which depends on the flux, or first angle-moment of the
intensity. For the stability analysis here, however, it is necessary to extend this diffusion treatment to second order, since
the wind instability is set by the perturbed line force, which depends on the second angle-moment of intensity (see § 2.2).

Although we initially (§ 2) frame the problem of wind instability in a general way that applies to the OB as well as the
W-R case, much of the subsequent analysis specializes to a diffusion treatment that is simply not appropriate to thin OB
winds, for which the radiation is inherently free streaming, not diffusive. A diffusion approach involves first defining an infinite
power series for the intensity as an expansion about local conditions, with each higher order term representing more and
more nonlocal effects. For the optically thick case, each higher
term is smaller than the previous, roughly by the inverse of the optical depth, and so it is appropriate to truncate this to some
desired order. However, such a series simply would not con-
verge for an optically thin case, which is inherently nonlocal. In
this sense, OB and W-R winds define opposite extremes, for
which one uses different radiative transfer approaches.

The structure of this paper is first (§ 2) to quantify the growth rate of small-scale radial perturbations and discuss
how it is governed by the angular profile of the continuum radiation. In § 3, we apply the NID approximation to charac-
terize the radiation field, which is applied in § 4 to quantify the degree to which the instability is suppressed, and to calculate
the net amplification a perturbation undergoes as it advects
outward in the wind. In § 5, we analyze the physical source of

\[1\text{ In this context, "optically thick" refers to any case in which photons are deflected from free-streaming, whether by a continuum or overlapping lines.}\]
the instability and show that the local sphericity associated with the global spherical geometry plays a key role. Finally, § 6 summarizes our conclusions and outlines future work.

2. Perturbation Analysis for a Line-Driven Flow

Previous analyses of the line-driven instability for OB winds (e.g., Owocki & Rybicki 1984, 1985) show that the instability growth rate is essentially determined by the ratio of the perturbed line-driving force $\delta \eta_{\text{lines}}$ to the perturbed velocity $\delta v$. The focus of our analysis here will accordingly be to calculate this perturbed line force for the case of optically thick W-R winds driven by multiline scattering. As a prelude to this analysis of the perturbed force, let us first review the nature of the mean, unperturbed line force.

2.1. The Mean Line Force

The unperturbed line force per unit mass is given by

$$\eta_{\text{lines}} = \sum_k \frac{4\pi}{c} \left\langle \int_{-\infty}^{\infty} dx \phi(x) I_k(x, \mu) \right\rangle,$$

where $x$ is the absorption frequency in the comoving frame in Doppler units, $\phi(x)$ is the absorption profile, the $k$ are the mean cross sections per unit mass of the various lines, and the sum extends over all lines included. The angle brackets denote averaging over $\mu$, the direction cosine to the radial, and $I_k(x, \mu)$ is the intensity along $\mu$ per thermal Doppler bandwidth, at frequency $x$. Applying the Sobolev approximation (Sobolev 1960; Castor 1970) in complete redistribution gives

$$I_k(x, \mu) = I_k(\infty, \mu) e^{-\tau_k \phi(x)} + S_k \left[ 1 - e^{-\tau_k \phi(x)} \right],$$

where the Sobolev optical depth $\tau_k = \rho \kappa L_{\mu}$, with $\rho$ the mass density, $L_{\mu} = \nu_k r(1 + \alpha \mu^2)$ the Sobolev length along $\mu$, and $\sigma = \ln \nu / \nu / \ln r - 1$ the standard measure of the local anisotropy of the expansion. In addition, $S_k$ is the line source function, $I_k(\infty, \mu)$ is the intensity incident at the blue edge of the line, and $\Phi(x)$ is defined by

$$\Phi(x) = \int_{-\infty}^{\infty} dx \phi(x).$$

Defining next the escape probability along $\mu$,

$$p_k(\mu) = \frac{1}{1 + \delta v / v_{\text{th}}} \left[ 1 - e^{-\tau_k} \right],$$

we can then combine equations (1) and (2) into

$$\eta_{\text{lines}} = \frac{4\pi}{c} \sum_k \kappa \mu \left\langle I_k(\infty, \mu) p_k(\mu) + S_k \left[ 1 - p_k(\mu) \right] \right\rangle. \tag{5}$$

The diffuse force arising from the $S_k$ term averages to zero, but we leave it in here because it will play a key role in the instability, known as the line-drag effect (Lucy 1984).

The line force could now be determined by finding $I_k(\infty, \mu)$ and $S_k$ and summing over all lines. In practice, to obtain analytic results we replace the discrete line list by a statistical distribution, and in such a scheme it is not convenient to have any correlation between the angular profile of the incident continuum and the $\kappa$ of the line. Thus we further assume that $I_k(\infty, \mu)$ can be separated as

$$I_k(\infty, \mu) = \nu_k I(\mu),$$

which holds in both the free-streaming approximation (with prescribed limb darkening) and the effectively gray approximation (FC and Paper I). The discrete weighting function $w_k$ accounts for the wavelength dependence of the stellar continuum and will later be absorbed into the flux-weighted line distribution.

Assuming that the scattering in each line is conservative (as argued for hot-star winds by Abbott & Lucy 1985), the source function $S_k$ is given in Sobolev theory by

$$S_k = \nu_k \left\langle \frac{I(\mu)}{p_k(\mu)} \right\rangle. \tag{7}$$

This allows us to write the line force per unit mass in terms of the continuum intensity $I(\mu)$,

$$\eta_{\text{lines}} = \frac{4\pi}{c} \sum_k \kappa w_k \left\langle \mu \left[ I(\mu) - \left\langle \frac{P_k}{P_k} \right\rangle \right] \right\rangle. \tag{8}$$

Note that for conciseness we sometimes suppress explicit $\mu$ arguments inside angle-averaging brackets.

2.2. The Perturbed Line Force and Instability Growth Rate

Let us now consider how this line force is affected by a small-amplitude radial velocity perturbation $\delta v$. We assume here that this perturbation is on a short enough spatial scale to be optically thin, in the sense that one may ignore the effect of associated perturbations in the line intensity, i.e., $\delta I_k(x, \mu) \approx 0$. Strictly speaking, this requires that the perturbation scale must be smaller than the line-center mean free path of the strongest line. However, for the unstable, perturbed-force component that varies directly with the perturbed velocity, previous analyses for OB winds (Owocki & Rybicki 1984, 1985) indicate that the assumption holds pretty well over a much broader range of scales up to the Sobolev length $L = v_{\text{th}} / (dv/dr)$. This is because the comoving frequencies that control the unstable component of the perturbed force always lie near the blue edge of the lines (viz, $x_k$ defined in eq. [23] below) and, at such frequencies, all perturbation scales smaller than $L$ are optically thin and so yield no appreciable $\delta I_k$.

With $\delta I_k = 0$, the essential effect of the perturbation is to alter the absorption profile in the unperturbed wind frame by

$$\delta \phi(x) = \phi \left( x - \mu \frac{\delta v}{v_{\text{th}}} \right) - \phi(x) = -\mu \frac{\delta v}{v_{\text{th}}} \frac{d\phi(x)}{dx}. \tag{9}$$

Since $d\phi(x)/dx = -2\phi(x)$ for a Gaussian profile, we thus find from equation (1) that the perturbed line force is given in the unperturbed wind frame by

$$\delta \eta_{\text{lines}} = \frac{4\pi}{c} \sum_k \kappa w_k \left\langle \mu \left[ I(\mu) - \left\langle \frac{I(\mu) p_k(\mu)}{p_k(\mu)} \right\rangle \right] \right\rangle \times \int_{-\infty}^{\infty} dx \phi(x) e^{-\tau_k \phi(x)} \frac{\delta v}{v_{\text{th}}} e^{-\text{t}(\phi(x))}. \tag{10}$$

It is convenient to define the mean frequency of photon absorption on the line's blue edge,

$$x_k(\mu) = \frac{1}{p_k(\mu)} \int_{-\infty}^{\infty} dx \phi(x) e^{-\text{t}(\phi(x))}. \tag{11}$$

Then rescaling to a dimensionless growth rate $\Omega$ via

$$\frac{\delta \eta_{\text{lines}}}{\delta v} = \frac{\eta_{\text{lines}}}{v_{\text{th}}} \Omega,$$

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we may write
\[
\Omega = 2 \sum \frac{kw_\kappa \langle \mu^2 x_s(\mu)p_p(\mu)[I(\mu) - \langle I(\mu)p_p(\mu)\rangle]/\langle p_p(\mu) \rangle \rangle}{kw_\kappa \langle \mu I(\mu)p_p(\mu) \rangle}.
\]

(13)

Note that \( \Omega \) depends entirely on the angular dependence of \( I(\mu) \) if \( x_s(\mu) \) and \( p_p(\mu) \) are given. Thus we need to examine the processes that determine \( I(\mu) \).

2.3. Even/Odd Intensity Decomposition

The numerator in equation (13) exhibits a large degree of cancellation between the direct and diffuse force contributions, an effect termed "line drag" by Lucy (1984). Indeed, note that if \( I(\mu) \) were \( \mu \) independent, then \( I(\mu) = \langle I(\mu) \rangle \) and equation (13) would lead to zero growth. To circumvent the computational uncertainties associated with such a delicate cancellation, let us decompose the specific intensity into a form that will also prove useful in our diffusion analysis in the next section. We write \( I(\mu) \) as the sum of a constant component, \( \langle I(\mu) \rangle \), an odd component, \( I'(\mu) \), and a nonconstant even component, \( I''(\mu) \), so that
\[
I(\mu) = \langle I(\mu) \rangle + I'(\mu) + I''(\mu),
\]

(14)

where
\[
I'(\mu) = \frac{I(\mu) - I(-\mu)}{2}
\]

(15)

and
\[
I''(\mu) = \frac{I(\mu) - I(-\mu)}{2} - \langle I \rangle.
\]

(16)

Note that \( \langle I''(\mu) \rangle = 0 \).

Since \( p_p(\mu) \) is an even function of \( \mu \), the decomposition in equation (14) allows us to rewrite equation (13) as
\[
\Omega = 2 \sum \frac{kw_\kappa \langle \mu^2 x_s(\mu)p_p(\mu)[I'(\mu) - \langle I'(\mu)p_p(\mu) \rangle]/\langle p_p(\mu) \rangle \rangle}{kw_\kappa \langle \mu I'(\mu)p_p(\mu) \rangle},
\]

(17)

which along with equation (12) shows that
\[
\frac{\delta \theta_{\text{lines}}}{\delta u} = \frac{8\pi}{cv_{\text{th}}} \sum kw_\kappa \langle \mu^2 x_s(\mu)p_p(\mu) \rangle \left[ I'(\mu) - \langle I'(\mu)p_p(\mu) \rangle \right].
\]

(18)

This makes explicit the important fact that the growth rate depends entirely on \( I'(\mu) \). Also note that, while \( I'(\mu) \) characterizes the contrast between the inward and outward intensity streams and controls the mean force, \( I''(\mu) \) characterizes the contrast between the radial and azimuthal intensities and controls the instability.

The advantage of this viewpoint is that the subtraction in the numerator of equation (17) no longer corresponds to the direct force minus the diffuse force, and it generally does not exhibit a large degree of cancellation. [The line-drag effect is already incorporated into the first term when working with \( I'(\mu) \) instead of the full \( I(\mu) \).] For example, at the radius of local isotropic expansion (i.e., where \( \sigma = 0 \), the subtracted term in equation (18) is identically zero because \( \langle I'(\mu) \rangle = 0 \). This is useful because it means that the uncertainty in the growth rate is the same as the uncertainty in \( I'(\mu) \), and approximations can be applied in equation (17) without upsetting any delicate cancellation.

2.4. Summation over the Line List

Within our effectively gray treatment of the radiation field, it is possible to work entirely in terms of line-averaged quantities. One such quantity we will need is the average probability of escape from the resonance zone per emission along \( \mu \), which is
\[
\tilde{p}(\mu) = \frac{\sum kw_\kappa p_p(\mu)}{\sum kw_\kappa}.
\]

(19)

A second important quantity is
\[
\tilde{x}(\mu) = \frac{\sum kw_\kappa x_s(\mu)p_p(\mu)}{\sum kw_\kappa p_p(\mu)},
\]

(20)

which may physically be interpreted as the line-weighted average of the frequency of first absorption along \( \mu \).

We next make the approximation that
\[
\langle I'(\mu)p_p(\mu) \rangle \approx \frac{\langle \tilde{x}(\mu) \rangle p_p(\mu)}{\langle \tilde{p}(\mu) \rangle},
\]

(21)

which amounts to neglecting the \( k \) dependence of \( S_e/w_e \) (see § 3.3 and Appendix A). We can then write the line-summed growth rate as
\[
\Omega = \frac{2\langle \tilde{x}(\mu) \rangle \mu^2 [I'(\mu) - \langle I'(\mu)p_p(\mu) \rangle]/\langle \tilde{p}(\mu) \rangle}{\langle \tilde{p}(\mu) \rangle}.
\]

(22)

Use of the averaged form of \( p_p(\mu) \) in the determination of \( S_e \) (eq. [21]) typically introduces an error that is less than 10% when summed over the line list, and since there is no sensitive cancellation of the terms in equation (22), this introduces errors in \( \Omega \) of only a few percent, which are less than errors inherent in the diffusion treatment applied in § 3.

2.5. The Statistical Power-Law Distribution

Within the limitations of equation (21), formula (22) applies for any general line list. To allow for specific calculations, we will henceforth specialize to the standard case (equivalent to, e.g., Cak and Abbott 1980) of a line list with a power-law distribution over opacity. This distribution therefore obeys \( N(\kappa) \sim \kappa^{-\alpha} \), where \( N(\kappa) \) is the cumulative number of lines with cross section per unit mass greater than \( \kappa \), flux weighted by the factor \( w_\kappa \). If we assume there are many optically thick lines, as is required to drive massive winds, then the line summations can be carried out as integrals over the power law, yielding specific expressions for \( \tilde{x} \) and \( \tilde{p} \), which we shall denote in this case as \( x_s \) and \( p_p \), respectively.

In particular, by elementary integration we find that \( x_s \) is given by the angle-independent integral expression
\[
x_s = \langle 1 - \alpha \rangle \int_{-\infty}^{\infty} dx \phi(x) x \Phi(x)^{-\alpha},
\]

(23)

in the limit of many thick lines.

Figure 1 shows that \( x_s \) varies between roughly 0.5 and 1 over the range 0.5 < \( \alpha < 0.7 \) that is relevant to most hot-star winds. The line-averaged escape probability is furthermore found to obey the proportionality
\[
p_p(\mu) \propto (1 + \sigma \mu^2)^{\alpha},
\]

(24)
length. Since there are of order $v_0/b_{w}$ ~ 300 Sobolev lengths over the entire wind, when $\Omega \approx 1$ the potential for magnifying small perturbations is huge ($e^{300}$). Even a small fraction of this magnification would saturate the linear domain we treat, and nonlinear effects such as clumping must result. Indeed, though we find below (§ 4) that the instability growth rates for W-R winds typically have $\Omega \approx 0.1$, this is still quite sufficient to imply formation of a substantial degree of wind structure.

3. DIFFUSION TREATMENT FOR THE INTENSITY IN W-R WINDS

Our approach for treating Wolf-Rayet wind instability is to extend the NID approximation from Paper I. This diffusion treatment is applied throughout the multiple scattering volume, not just within the electron scattering photosphere. In the effectively gray approximation, electron opacity and line opacity are treated in a unified frequency-integrated fashion. Even after electron scattering becomes optically thin, the interline radiation field remains diffuse out to the radius at which line overlap begins to break down. This occurs when the flow comes within $P_{\mathrm{wind}}^{-1}$ of its terminal speed (see Paper I), where $P_{\mathrm{wind}} = M_{\infty} c/L_{\star}$ is the dimensionless momentum flux (derived the performance number by Springmann 1994). For a typical W-R wind with, e.g., $P_{\mathrm{wind}} \approx 10$, this holds until $v \approx 0.9 v_{\infty}$, and so it includes almost the entire wind acceleration region.

The essential feature of the diffusion approximation is that the decomposition of the intensity in equation (14) is developed as a power series in the (small) parameter $\lambda/r$, the ratio of the interline photon radial mean free path to the radius. This ratio was shown in Paper I to be of order $P_{\mathrm{wind}}^{-1}$ which for Wolf-Rayet is characteristically of order 0.1. Since only $I^\mu(\mu)$ is needed for the radiative force, our analysis of the mean wind dynamics in Paper I had only to consider the first two terms of the expansion. But as we now require $I(\mu)$ as well, we must carry this NID approximation to second order, involving considerably more complicated expressions. Fortunately, these expressions simplify at the characteristic point where $\sigma = 0$ (see § 4.3).

3.1. Effectively Gray Opacity and Source Function

The effectively gray opacity approximation is described in FC and in Paper I, so we review only the results here. The objective is to replace the complicated ensemble of line resonance zones with a single, frequency-averaged optical depth scale that treats the lines statistically. This radial optical depth scale $\tau$ is related to the effective total cross section per unit mass $\kappa(\mu)$ by

$$f(\mu) \, d\tau = -\rho \kappa(\mu) \, dr,$$

where

$$f(\mu) = \frac{\kappa_{\mu} + \kappa_{\nu}(1 + \mu^2 \alpha^2)/(1 + \alpha)}{\kappa_{\mu} + \kappa_r}$$

is the nonisotropic absorption profile (with $f(1) = 1$), $d\tau = -\rho \kappa_{\mu} + \kappa_{\nu} \, d\tau$, and $\kappa_{\mu}$ and $\kappa_{\nu}$ are the radial line and free-electron opacities defined in Paper I.

Along with the effectively gray opacity we can define an effectively gray emissivity, expressed in terms of the source function $S(\mu)$, which gives the specific intensity compiled over a mean free path along $\mu$. In FC's approach, resonance zone
emission is added to free electron scattering to form a total, frequency-integrated source function, and because electron scattering has a different angular response than the cumulative line scattering, this source function is angle dependent, i.e., \( S = S(\mu) \).

In their approximation of complete angular redistribution (CAR), FC's source function can be written concisely as

\[
S(\mu) = \frac{\langle I \rangle + \bar{Q}_p(\mu)/\langle p_x \rangle}{1 + \bar{Q}_p(\mu)},
\]

(30)

where

\[
\bar{Q}_p(\mu) \equiv \frac{\kappa}{\kappa_{es}} \left( \frac{1 + \sigma \mu^2}{1 + \sigma} \right)
\]

(31)

defines the proportionality constant left out of equation (24), although we will not need to specify the value of \( \bar{Q} \) here (but see Gayley 1995). Equation (31) can be rewritten in a form more conducive to our analysis,

\[
S(\mu) = \chi(\mu) \frac{\langle p_x I \rangle}{\langle p_x \rangle},
\]

(32)

where

\[
S(\mu) = \chi(\mu) \frac{\langle p_x I \rangle}{\langle p_x \rangle},
\]

(33)

and we have defined for conciseness the fraction of line opacity

\[
\chi(\mu) \equiv \frac{\kappa_{lines}(\mu)}{\kappa(\mu)} = \frac{\bar{Q}_p(\mu)}{1 + \bar{Q}_p(\mu)}.
\]

(34)

Equation (32) decomposes \( S(\mu) \) in a manner analogous to equation (14) with no odd component, and note that since \( I'(\mu) \) is second-order in \( \lambda/\tau \) in the diffusion approximation, the angle dependence of \( S(\mu) \) is also second-order. The expression for \( S(\mu) \) obeys radiative equilibrium, since \( \langle \kappa(\mu) S(\mu) \rangle = \langle \chi(\mu) I(\mu) \rangle \).

We note here the following subtlety in our assumption for \( S(\mu) \). In obtaining equation (30), FC assume that the resonance zone angular emission profile is uncorrelated with the initial absorption. Because the line optical depth connects the absorption and emission profiles, this can be strictly true only if the final emitting line is uncorrelated with the absorbing line. However, in writing \( S(\mu) \) in the form of equation (7), we are assuming that absorption and emission effectively occur in the same line. (Note that this assumption also underlies the Monte Carlo simulations of Lucy & Abbott 1993, hereafter LA.) Although this could, in principle, preserve some angular correlation, we show in Appendix A that this is a second-order effect. Thus it is not necessary for us to explicitly adopt FC's CAR approximation in order to use equation (30)—it is already consistent to the order we are working with the other assumptions we have made.

### 3.2. The Diffusion Form for \( \Omega \)

The form of equation (32) allows for a simplification in the expression for the growth rate \( \Omega \) that is useful in the diffusion approximation. Let us first define the transfer quantity

\[
h(\mu) \equiv I(\mu) - S(\mu),
\]

(35)

and note that

\[
h'(\mu) = I'(\mu)
\]

(36)

and

\[
h(\mu) = I(\mu) - S(\mu)
\]

(37)

are respectively first- and second-order quantities in \( \lambda/\tau \). After some manipulation of equation (25) using equation (32), we can then write the growth rate in the form

\[
\Omega = 2x_a \frac{\langle \mu^2 - \mu^2 \rangle p(x) h(\mu) \rangle}{\langle \mu p(x) h^2(\mu) \rangle},
\]

(38)

where

\[
\mu^2 \equiv \frac{\kappa}{\kappa_{es}} \langle \mu p(x) \rangle \chi(\mu)
\]

(39)

and we have defined

\[
\chi(\mu) \equiv \frac{\kappa_{lines}(\mu)}{\kappa(\mu)} = 1 - \chi(\mu)
\]

(40)

to be the free electron opacity fraction. Equation (38) demonstrates the central point of this subsection: the radial instability growth rate can be written entirely in terms of \( h(\mu) \), so effectively depends on \( I(\mu) - S(\mu) \) rather than just on \( I(\mu) \). This is convenient in the context of the diffusion approximation, whence \( I(\mu) - S(\mu) \) becomes a higher order quantity controlled by local gradients, which we now quantify.

### 3.3. The Second-Order NID Approximation

Our procedure for determining \( I(\mu) - S(\mu) \) is first to write the radiative transfer equation for the mean radiation field in spherical coordinates,

\[
\frac{\partial I(\mu)}{\partial \tau} - \frac{\lambda}{f(\mu)} \frac{\partial I(\mu)}{\partial \tau} - \frac{1 - \mu^2}{f(\mu)} \frac{\lambda}{\tau} \frac{\partial I(\mu)}{\partial \tau} = \frac{\mu^2}{f(\mu)^2} S',
\]

(41)

where \( \lambda = 1/(\kappa_{es} + \kappa) \) is the radial mean free path between continuum or resonance zone encounters. In the diffusion case considered here \( \lambda/\tau \ll 1 \), and \( I(\mu) \) can be approximated by a truncated sum of derivatives of \( S \). This sum can be conveniently generated from equation (41) by successively substituting the expression for \( I(\mu) \) into every appearance of \( I(\mu) \) on the right-hand side. As noted above, in order to derive a nonzero instability we must carry this sum to second-order in the quantity \( \lambda/\tau \), assuming every derivative is of this order. The resulting angular dependence of \( h(\mu) \) is

\[
h(\mu) \equiv \left\{ \frac{\mu}{f(\mu)} - \frac{1 - \mu^2}{f(\mu)^2} \frac{\lambda}{\tau} \left[ 1 - \frac{\partial \ln f(\mu)}{\partial \ln \mu} \right] \right. \]

\[
- \frac{\mu^2}{f(\mu)^2} \frac{\lambda}{\tau} \frac{\partial \ln f(\mu)}{\partial \ln \mu} \left[ S' + \frac{\mu^2}{f(\mu)^2} S'' \right]
\]

(42)

where \( S' \equiv \partial S(\mu)/\partial \tau \) and \( S'' \equiv \partial^2 S(\mu)/\partial \tau^2 \). The required derivatives of \( f(\mu) \) are approximated in Appendix B and yield

\[
h(\mu) \equiv \left[ \frac{\mu}{f(\mu)} + \frac{\lambda}{\tau} \psi(\mu) \right] S' + \frac{\mu^2}{f(\mu)^2} S'',
\]

(43)

where we have defined for convenience

\[
\psi(\mu) \equiv \left( \frac{1 - \mu^2}{f(\mu)} \right) \left[ 1 + \frac{3 \lambda^2}{f(\mu)(1 + \sigma \mu^2)} \chi(\mu) \right].
\]

(44)

We next evaluate \( S' \) and \( S'' \) by requiring both a constant luminosity and an effectively gray radiative equilibrium, which
respectively involve the $\mu$ and $f(\mu)$ moments of equation (42). We can express $S'$ in terms of the flux $H \equiv \langle \mu H(\mu) \rangle$, which is fixed in terms of the luminosity $L_\star$ by $H = L_\star / 16 \pi r^2$. Since $S(\mu)$ is even (eq. [32]), we have $H = \langle \mu H(\mu) \rangle = \langle \mu h(\mu) \rangle$, and so from equation (42), we find

$$S' = \frac{H}{\langle \mu^2 f(\mu) \rangle}.$$  (45)

To obtain $S''$, we apply the condition of radiative equilibrium in the effectively gray approximation,

$$\langle \kappa(\mu)[I(\mu) - S(\mu)] \rangle = \langle \kappa(\mu)h(\mu) \rangle = 0,$$  (46)

which, from equation (42), gives

$$S'' = \frac{\lambda}{r} \frac{H}{\langle \mu^2 f(\mu) \rangle^2},$$  (47)

where we have eliminated $S'$ using equation (45). Note that both $S'$ and $S''$ are angle independent to the order we consider here. This is a consequence of the fact that the angle dependence of $S$ is itself already second-order in $\lambda/r$.

Using equations (45) and (47) in equation (42) then gives the basic result of the second-order NID approximation, for which the odd and even components are, respectively,

$$h'(\mu) = \frac{\mu}{f(\mu)} \frac{H}{\langle \mu^2 f(\mu) \rangle}$$  (48)

and

$$h''(\mu) = \frac{\lambda}{r} \left( \frac{\mu^2 f(\mu)^2}{\langle \mu^2 f(\mu) \rangle^2} - \frac{\mu}{f(\mu)} \right) \left( \frac{H}{\langle \mu^2 f(\mu) \rangle} \right).$$  (49)

### 3.4. Finding the Opacity and Photon Mean Free Path

To complete the evaluation of the terms in equations (48) and (49), we need only to determine $\kappa_\sigma$, which will fix $\lambda$ and $f(\mu)$ at each $\sigma$ in the wind. Since $\kappa_\sigma$ depends on the length over which photons encounter optically thick resonance zones, we must consider the steady state force balance to find the density and velocity structure, using a CAK-type critical point analysis to solve for the smooth wind. This was detailed in FC and Paper I, so we merely outline the results here. The central limitations are the steady state assumption and the use of a Poisson-distributed line list with constant attributes over the wind.

Let us first define the quantity

$$w = \frac{r^2 v}{GM(1 - \Gamma)} \frac{dv}{dr},$$  (50)

where $GM(1 - \Gamma)/r^2$ is the gravitational acceleration corrected for free electron scattering. Then we have from the force balance equation that (see eq. [33] from Paper I)

$$\frac{\kappa_\sigma}{\kappa_{es}} = \frac{(1 - \Gamma) (1 + w)}{\Gamma F}.$$  (51)

Here $F$ is the actual line force relative to the point-star line force, given by

$$F = \frac{\langle (1 + \sigma \mu^2) \mu F(\mu) \rangle}{(1 + \sigma) \langle \mu F(\mu) \rangle}.$$  (52)

This leads to

$$\frac{\lambda}{r} = \frac{2}{\kappa_{es}} \left( \frac{v_{\infty}}{v_0} \right)^2 \frac{F}{v_0 \left[ w + 1 + \Gamma F/(1 - \Gamma) \right]} P^{-1}_{\text{wind}},$$  (53)

where $v_{\infty}$ is the escape speed. If we assume the dominant driving is by the line force, then the fact that $P_{\text{wind}} \gg 1$ for Wolf-Rayet winds implies that $\lambda \ll r$ except at large $r$. This is the justification for using a diffusion approximation over much of the wind.

The above expressions, in connection with equations (48) and (49), allow all quantities to be expressed as a function of $w$. The equations are closed by applying the CAK critical-point conditions, allowing $w$ to be found everywhere (Paper I). This in turn specifies $F$, $\kappa_\sigma$, $\lambda$, and $\sigma$ as a function of $r$, which allows us to specify $h'(\mu)$ and $h''(\mu)$. Figure 2 plots $h'(\mu)$ and $h''(\mu)$ at various $\sigma$ for the stellar parameters indicated. It is apparent that $h'(\mu)$ exhibits a radial excess, which induces radial instability. Also, the overall magnitude of $h'(\mu)/h''(\mu)$ is of order $\lambda/r$, and scales inversely with the mass-loss rate.

### 4. Quantitative Results

#### 4.1. The Growth Rate as a Function of Radius

We can finally quantify the instability in a Wolf-Rayet wind by using equation (38) with equations (48) and (49) to evaluate $\Omega(r)$. For two representative values of $\alpha$ and $\Gamma$, Figure 3 plots the radial variation of $\Omega$ for the canonical case $Mc^2/L_\star = 10^3$. As long as the wind is effectively optically thick ($r_{\text{wind}} \gg 1$), the velocity structure is independent of the mass-loss rate (Paper I). As a result, the mass-loss dependence of the instability growth rate is quite simple, scaling with the inverse of $Mc^2/L_\star$ through the proportionality with $\lambda/r$ (see eq. [53]). Growth rates for other mass-loss rates are thus readily deduced from Figure 3 by rescaling the ordinate by the factor $10^3 L_\star / Mc^2$.

#### 4.2. The Total Number of e-Folds

To quantify the total degree of instability in an absolute sense, it is useful to calculate the number of e-folds of amplifi-
cation a perturbation initiated at the subsonic wind base \((r \approx R_w)\) undergoes (in the linear regime) as it is advected out to some wind radius \(r\),

\[
N_{fold}(r) = \int_{R_w}^{r} \frac{d\Gamma}{v \Gamma_{th}} \Omega = \int_{0}^{\Gamma(r)} \frac{dv}{v_{th}} \left[ \frac{1 + w}{w} \right] \Omega , \tag{54}
\]

where the appropriate order-unity values of \(w\) can be taken from the corresponding steady state solution in Paper I. Table 1 assumes that \(M = 2.1 \times 10^{-5} \, M_\odot \, \text{yr}^{-1}\) is fixed and the stellar mass and radius are 10 \(M_\odot\) and 2 \(R_\odot\), respectively, and lists \(N_{es}\) and \(N_{diff}\), which denote \(N_{fold}\) at respectively the electron-scattering “photosphere” (where \(\tau_e = 1\)), and at the radius where \(\lambda/\rho = 0.5\) (beyond which the NID approximation completely fails). For simplicity, we adopt a fixed thermal speed that is roughly appropriate for CNO ions at a typical smooth-wind temperature of 50,000 K, so \(v_{th} \approx 7 \, \text{km} \, \text{s}^{-1}\). For other values, the instability growth rate \(\delta g/\delta t\) and the number of \(e\)-folds \(N_{fold}\) scale as \(v_{th}^{-1}\). Several values of \(\alpha\) and \(\Gamma\) are considered. The models we consider most relevant to Wolf-Rayet winds are given in the top two rows of Table 1, since the terminal speeds in the \(\alpha = 0.5\) cases are closest to those observed (Paper I), and \(\Gamma \approx 0.5\) at the surface appears to be theoretically favored (Langer 1989). The table shows that \(\Omega\) and \(\lambda/\rho\) at the \(\sigma = 0\) point are of comparable order to each other, and are also comparable to the inverses of the momentum number \(P_{wind}\) and the total effective radial optical depth \(\tau_{tot}\).

The large values of \(N_{diff}\) imply that even very small amplitude fluctuations should become amplified into nonlinear structures within the diffusion domain. As such, it is not necessary to consider the linear instability of the outermost wind, where the radiation begins to escape freely and our diffusion treatment breaks down. Determining the nature of these nonlinear structures is a very difficult problem in radiation hydrodynamics. By analogy with nonlinear simulations of unstable OB winds (see, e.g., Owocki 1992, and references therein), they might be expected to be dominated by high-speed rarefactions separating lower speed, dense clumps. Such structures may help explain the moving bumps commonly observed in optical lines formed in the outer portions of W-R winds (Robert 1992; Moffat & Robert 1992).

The lower, but still significant, values of \(N_{es}\) further suggest that instability might even lead to substantial structure at the wind’s optical photosphere. Such structure might conceivably give rise to variations in continuum polarization, such as has been observed for several W-R stars (Robert et al. 1989). As emphasized by Brown (1994), forming clumps at or below the optical photosphere is crucial to producing such polarization variability, since the polarization fluctuations from any clumps formed farther out tend to be offset by surrounding rared areas.

Figure 4 shows the dependence of \(N_{es}\) on mass-loss rate. The relative constancy is because the growth rate scaling with \(\Omega \propto M^{-1}\) is compensated by the tendency for the electron photosphere radius to vary in proportion to \(M\). For the characteristic W-R parameters chosen, the number of \(e\)-folds is large enough to suggest that substantial wind structure will emerge through this photosphere. Our fundamental conclusion is thus that small-scale, variable structure is likely to be ubiquitous throughout the observable portion of most W-R winds.

4.3. The Instability Growth Rate at the \(\sigma = 0\) Point

Another convenient way to characterize the overall wind instability is in terms of the growth rate at the point of local isotropic expansion (\(\sigma = 0\)), where the angle integrations required to compute \(\Omega\) become much simpler. Although Figure 3 indicates that \(\Omega\) varies with radius, the values at \(\sigma = 0\) near the center of the figure, are quite characteristic of the overall level. At this point, equation (38) for the growth rate simplifies to

\[
\Omega = 0 = \frac{2\chi \langle \mu^2 h'(\mu) \rangle}{\langle \mu h'(\mu) \rangle} = \frac{2\chi \langle \mu^2 - 1/3 \rangle}{\langle \mu \rangle} , \tag{55}
\]

where we have used that \(S\) is angle-independent and \(\langle h'(\mu) \rangle = 0\) when \(\sigma = 0\). The NID expressions for \(h'(\mu)\) and

![Figure 3](image3.png)

**Figure 3.** The unitless growth rate \(\Omega\) as a function of \(\epsilon/\Gamma_{es}\) for a dimensionless mass-loss rate \(\dot{M}c^2/L_c = 10^4\). Growth rates for other mass-loss rates are obtained by rescaling the ordinate by \(10^4 L_c/\dot{M}c^2\). The solid curves are for \(\alpha = 0.5\), and the dotted curves are for \(\alpha = 0.75\). The adopted \(\Gamma\) values are 0.3 and 0.5, with the smaller \(\Gamma\) in each pair corresponding to the higher growth rates.

![Figure 4](image4.png)

**Figure 4.** The number of \(e\)-folds of amplification for a small-scale radial perturbation adventing from the static surface to the radius of optical depth \(\frac{1}{2}\) in the free electron continuum, as a function of mass-loss rate. Here \(\sigma = 0.5\) with \(\Gamma = 0.3\) and \(\Gamma = 0.5\) for, respectively, the solid and dashed curves.

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\[ h'(\mu) = 3H \mu \]

and

\[ h'(\mu) = 3H \frac{2}{r} \left[ 1 - 3\mu^2 + \mu^2 \left( \frac{9}{5} - 3\mu^2 \right) \frac{\kappa_r}{\kappa_e + \kappa_r} \right]. \]

Applying these in equation (55), we then find that the growth rate at \( \sigma = 0 \) is

\[ \Omega_{NID} = \frac{8}{3} \frac{x_1}{\kappa_e + \kappa_r} \left[ 1 + \frac{3\alpha}{\kappa_e + \kappa_r} \right] \left( \frac{2}{r} \right)_{\sigma = 0} \approx 1.9 \frac{8}{3} \left( \frac{2}{r} \right)_{\sigma = 0}, \]

where the latter result holds to within the rough accuracy of the NID approximation for any typical values of \( \alpha \) and \( \kappa_r \). Its validity is also seen from Table 1, taking \( x_1 \) from Figure 1. Comparison with equation (27) shows that in the limit of multiple line overlap, the growth rate is reduced relative to the single-scattering case by roughly the factor \( \lambda/r \), which from equation (53) is of order the inverse momentum number \( P_{wind}^{-1} \).

Equation (58) can serve as a useful single-point characterization of the overall strength of Wolf-Rayet instability. The total number of e-folds over the wind can then be estimated simply by noting that an e-fold amplification is produced about every \( \Omega^{-1} \) Sobolev lengths, although there is additional dependence on \( \alpha \) due to the effects of \( w \) (see eq. [54]). Since there are \( v_{\infty}/v_{ib} \) Sobolev lengths over the wind, and \( \Omega \approx P_{wind}^{-1} = L_\mu/\alpha v_{ib} \), the total number of e-folds is of order \( L_\mu/\alpha v_{ib} \). For typical W-R winds this is quite large, as Table 1 indicates more quantitatively.

5. DISCUSSION OF THE PHYSICAL ORIGIN OF THE DERIVED INSTABILITY

Let us next examine the physical origin of the residual instability we infer for W-R winds. We have seen that the radiative instability is controlled by the \( \Gamma(\mu) \) component of the continuum intensity, so derives from the contrast between the radial and azimuthal intensities. For example, at the isotropic expansion radius (where \( \sigma = 0 \)), equation (55) shows that when the \( \mu^{2} - \frac{1}{2} \) moment of \( \Gamma(\mu) \) is positive—which implies that continuum radiation has a preference to be radially incident to the resonance zone—then positive radial growth results. In the single-scattering limit for OB winds, continuum radiation propagates directly from the static photosphere, and this introduces a strong radial asymmetry in the surrounding wind.

In Wolf-Rayet winds, however, the static star is embedded in an optically thick wind, which nearly isotropizes the radiation field and suppresses \( I' \) relative to the fixed flux-carrying \( I'' \), thus reducing \( \Omega \). This isotropization occurs through scattering in both the optically thick electron continuum, and in the effectively optically thick ensemble of resonance lines. Indeed, if we neglect sphericity and simply assume the wind outflow is plane-parallel, then applying the standard diffusion approximation for an atmosphere with isotropic opacity in radiative equilibrium would give \( I'' \propto \mu \) and \( I' = 0 \). Thus, without sphericity, there is no transverse escape, and the flux is carried solely by a contrast in the inward and outward photon streams, without requiring any contrast between the transverse and combined radial streams. This implies that a plane-parallel, optically thick flow would have essentially zero growth rate, given our assumptions.

We are thus led to examine the crucial role of large-scale curvature in yielding the instability in multiply scattered line-driven winds. Even in a spherically symmetric model, photons that propagate azimuthally over a distance \( l \) experience variations of order \( l/r \) in the local conditions, simply due to the global curvature. Furthermore, when flux diffuses outward through the convex surfaces of spherical symmetry, such curvature inherently supports a net escape of photons along azimuthal directions, which is not the case in plane-parallel geometry. To maintain radiative equilibrium, spherical symmetry thus requires net inflow along the radial direction, yielding just the type of radial/azimuthal contrast needed to give a positive radial growth rate. The upshot is that diffusive line-driving in spherical symmetry always tends to produce radial instability.

Since this sphericity correction is of order \( l/r \sim 1/P_{wind} \sim 0.1 \), the residual instability growth rate in W-R winds is reduced from optically thin OB winds by a factor of this order. But because the OB instability is extremely strong, this still represents an important effect. In summary, we have seen that sphericity effects are crucial for the growth of small-scale perturbations in high-density winds, and that this can be quantified using the diffusion approximation.

6. TECHNICAL ASPECTS OF THE APPROXIMATION AND ITS LIMITATIONS

We now turn to a discussion of several more technical issues relating to the photon transfer. These topics are included merely for completeness, and may be considered optional. Several limitations of the approach are discussed, and their impact is argued to be unimportant. It is possible to skip to the summary in § 7 at this time.

6.1. Effect of Breakdown in the Diffusion Approximation

In focusing on the role of sphericity, we have assumed that the radiation field is highly multiply scattered, so that the local mean free path determines its character. A source of instability we have therefore neglected comes from freely streaming photons that have not scattered in the wind, which retain their stellar core angular distribution and violate the diffusion approximation. These photons contribute to instability in the same manner as in OB winds.

However, the OB radiative instability is weak close to the static photosphere because the free-streaming radiation field is spread over the hemisphere, as quantified by Owocki & Rybicki (1985). Also, as photons stream through the narrow boundary between the isotropic opacity of the static photosphere and the nonisotropic effectively gray opacity of the wind model, they rapidly scatter in a W-R wind, and the NID approximation becomes applicable beyond this fairly narrow transition layer. In Paper I, we found that a wind with momentum number \( P_{wind} \approx 10 \) will scatter most of the stellar photons by the time the wind speed is roughly \( v_{ib}/10 \), which for a typical \( \beta \approx 1 \) velocity law occurs at about \( r \approx 1.1 R_\star \). Below this radius, the instability has only \( \lesssim 10 \) Sobolev lengths over which to act, and is weakened by the hemispheric spreading of the injected stellar photons. Using the Owocki & Rybicki (1985) growth rate at 1.05 stellar radii as characteristic, this would yield at most about one e-fold. Thus we conclude that if small perturbations are to evolve into dense clumps due to the radiative instability in W-R winds, this must occur primarily in
the region where most of the wind acceleration occurs, which is also where the diffusion approximation applies. Also, the linear instability at large radii \( r \gtrsim P_{\text{wind}} \), where free escape begins to occur, is not relevant if the diffuse instability at smaller radii is already sufficient to yield nonlinear structure.

Another way in which our diffusion treatment can break down as if the line distribution in frequency is not, as assumed, effectively gray, but instead contains significantly large gaps. In such gaps, radiation could freely stream over a much larger distance than assumed above, with potentially important consequences for both the mean wind dynamics (see Paper I), as well as for the instability. The recent Monte Carlo simulations of W-R winds by Lucy & Abbott (1993) suggest, however, that such spectral gaps tend to be filled as a result of ionization stratification. This will limit free streaming and help restore both the diffusive and effectively gray character of the transfer. Thus although quantitative calculations involving a realistic line list will be required to fully specify the adequacy of the NID approximation in each application, we feel that it is a powerful and effective tool for characterizing the radiative transfer in W-R winds.

6.2. Effect of a Breakdown in Radiative Equilibrium

Let us next consider how our assumption of radiative equilibrium in each line, invoked in equation (7), might alter the inferred instability. There are two levels at which this assumption can break down, each with different effects. The weaker type of breakdown occurs if radiative equilibrium still obtains over the entire line list, but not over each line individually. Thus energy is allowed to be transferred from line to line, but cannot be removed by thermal or continuum processes. This can affect instability in principle because the angular absorption profiles for thick and thin lines are different, so that the incident continuum feeds photons preferentially into thin lines, which then get transferred to thick lines, more diffuse scattering and greater line drag will result, reducing instability. Inversely, if photons are fed preferentially into the angular acceptance of thick lines rather than thin, reduced line drag and greater instability results.

However, we now show that a separate radiative equilibrium in each line is already implied by our basic assumption that \( S_{\kappa}/\omega_\kappa \) is independent of \( \kappa \). If we replace equation (7) with the weaker assumption of radiative equilibrium over the line list as a whole,

\[
\sum \kappa \langle p_\kappa (\mu) \rangle S_\kappa = \sum \kappa \omega_\kappa \langle p_\kappa (\mu) I(\mu) \rangle,
\]

then assuming \( S_{\kappa}/\omega_\kappa \) is nearly independent of \( \kappa \) (eq. [21]) yields

\[
S_\kappa \approx \frac{\langle 1 + \sigma \mu^2 \rangle \beta(\mu)}{\langle 1 + \sigma \beta \rangle},
\]

which gives rise to equation (25) as before. Thus our approach effectively neglects transfer between thick and thin lines (e.g., due to interlocking transitions), and assuming radiative equilibrium in the sum over all lines gives the same result as assuming radiative equilibrium for every line individually.

In general, \( S_{\kappa}/\omega_\kappa \) approaches different values in the optically thin and thick limits, and so our assumption that it is independent of \( \kappa \) is not strictly valid. However, at the isoptropic expansion point (\( \sigma = 0 \)), \( S_{\kappa}/\omega_\kappa \) does become completely independent of \( \kappa \), while at other points equation (60) provides a reasonable average of the thick and thin forms. Furthermore, the associated differences in the line drag of thick and thin lines affect only the subtracted term in equation (25), which in practice is small compared to the first term. Thus this type of weak breakdown should have little effect on our estimates of wind instability.

The stronger type of breakdown of radiative equilibrium occurs when energy is exchanged between lines and energy reservoirs we do not include, such as bound-free continua. This could in principle have an important effect on the instability because photons introduced into a resonance zone by thermal processes will produce line drag and damp the instability. Likewise, when photons are removed from a resonance zone and not reintroduced into some other line, line drag is reduced and instability enhanced. However, we argue that for such transfer to represent an important fraction of the photon stream, appreciable thermalization must occur, which will tend to push all source functions toward LTE. That in turn restores detailed balance in all lines and continua and derails any net exchange of energy between them. Thus we argue that net energy transfer out of or into lines should not be prevalent enough to substantially alter the instability. However, this argument must be tested by more accurate detailed models outside the scope of this paper. Future efforts by the Munich group (e.g., J. Puls & U. Springmann 1994, private communication) will address such complex issues of excitation balance.

7. SUMMARY

We have investigated the radiative instability in W-R stars and found an effect that is fully analogous to OB wind instability, albeit somewhat reduced in strength by multiple-scattering effects. The essential quality yielding this reduction is the near isotropization of the radiation field by multiple scattering. We have quantified the angular character of the radiation field using a diffusion approximation with non-isotropic effectively gray opacity, and although the accuracy of these assumptions may be debated, we expect the following two salient features to be robust:

1. The inherent sphericity of the wind expansion tends always to lead to a net residual instability in optically thick W-R winds. Relative to OB winds, the growth rate magnitude \( \Omega \) is reduced by the ratio of the photon mean free path to the radius, \( \lambda/r \), which is characteristically of order the inverse wind-momentum number, \( P_{\text{wind}}^{-1} = L_*/c/Mv_w \sim 0.1 \).

2. Even with this reduction, the instability is still strong enough for base wind perturbations to be amplified a very large number of \( e \)-folds (typically at least 40) by the time they reach the outer wind. This can be expected to lead to extensive wind structure, including strong clumping, throughout the observable part of the wind.

Such extensively clumped wind structure could provide a natural explanation for the "moving bumps" commonly observed (e.g., Robert 1992; Moffat & Robert 1992) in optical line spectra formed in W-R winds. Indeed, the amplification may be even be strong enough to lead to significant structure at the electron scattering photosphere of the wind, and thus even provide a possible mechanism for observed continuum polarization variations (Robert et al. 1989). In any event, the significant level of clumping expected has important implications for inferring mass loss rates, which are usually based on diagnostics that are sensitive to the square of the density.

In the future we plan to apply the methods developed here...
toward extending our nonlinear instability simulations of OB winds to the optically thick W-R case. Because W-R winds can be diagnosed with comparatively high signal-to-noise optical line spectra, they can potentially provide much stronger tests of theoretical mechanisms for generating wind structure. Of course, meaningful comparisons with observed spectra will require moving beyond the one-dimensional spherically symmetric models computed so far, to include azimuthal as well as radial variations. Developing tractable methods for such multidimensional simulations of instability-generated wind structure represents a central challenge for future work. A major goal will be to test if synthetic line spectra derived from this diffusive instability can explain the observed characteristics of the moving bumps in W-R emission lines.

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APPENDIX A

THE EFFECTIVELY GRAY SOURCE FUNCTION

In the text, we used the effectively gray source function of FC, and asserted that it could also be obtained through a different set of approximations more consistent with our derivation. We show this now, and give a generalized form for the effectively gray source function $S(\mu)$, elucidating the applicability of the CAR approximation of FC.

The effectively gray source function $S(\mu)$ can be written as the emissivity per unit mass $\eta(\mu)$ divided by the cross section per unit mass $\kappa(\mu)$,

$$S(\mu) = \frac{\eta(\mu)}{\kappa(\mu)}.$$  \hspace{1cm} \text{(A1)}

As in the text,

$$\kappa(\mu) = \kappa_{\text{es}}[1 + \bar{Q}p_*(\mu)]$$  \hspace{1cm} \text{(A2)}

accounts for both free electron and resonance zone scattering. Thus we need merely to determine the emissivity, being careful to satisfy radiative equilibrium $\langle \eta \rangle = \langle \kappa I \rangle$.

To assist what follows, we define $Q_* = \kappa_{\text{es}}/\kappa_{\text{es,c}}$, which is essentially the ratio of the line strength to the free electron continuum. This is related to the flux-weighted quantity $Q$ by $Q = \sum w_k Q_k$. We then define the redistribution function $g(\kappa, \kappa')$ to be the probability that a parcel of continuum energy which was first absorbed by line $\kappa$ will ultimately emerge from the local resonance zone after reemission in line $\kappa'$. Note that $g(\kappa, \kappa')$ obeys the radiative-equilibrium condition

$$\sum_{\kappa'} g(\kappa, \kappa') = 1,$$  \hspace{1cm} \text{(A3)}

and also the reciprocity relation

$$g(\kappa, \kappa')Q_*\langle p_\kappa \rangle = g(\kappa', \kappa)Q_*\langle p_{\kappa'} \rangle.$$  \hspace{1cm} \text{(A4)}

This latter condition assures the time-reversal symmetry of the joint absorption and emission rate connecting $\kappa$ and $\kappa'$. Using the general function $g(\kappa, \kappa')$, we can write the effectively gray emissivity per unit mass as the sum of free electron scattering and line emission, such that

$$\eta(\mu) = \kappa_{\text{es}}\langle I \rangle + \kappa_{\text{es}} \sum_k \sum_{\kappa'} g(\kappa, \kappa') w_k Q_k \langle p_k I \rangle \frac{P_k(\mu)}{\langle p_k \rangle},$$  \hspace{1cm} \text{(A5)}

where the second term multiplies $g(\kappa, \kappa')$ by the absorption rate in line $\kappa$ and equips it with the normalized emission profile $P_k(\mu)/\langle p_k \rangle$.

In the CAR approximation, angular correlations in equation (7) are ignored by replacing $p_k(\mu)/\langle p_k \rangle$ by $p_*(\mu)/\langle p_\ast \rangle$, which sums over the emitting lines $\kappa'$ separately from the absorbing lines. In light of equation (7), this leads to

$$\eta(\mu) = \kappa_{\text{es}}\langle I \rangle + \kappa_{\text{es}} \sum_k w_k Q_k \langle p_k I \rangle \frac{P_k(\mu)}{\langle p_\ast \rangle},$$  \hspace{1cm} \text{(A6)}

which leads directly to the FC source function and equation (32). However, we obtain the same result by replacing the CAR approximation by assumptions already made in the derivation of the NID result for $\Omega$. In our expression for $S_k$ (eq. [7]), we effectively assumed that $g(\kappa, \kappa')$ was the Kronecker $\delta$-function $\delta_{\kappa\kappa'}$, as assumed by LA. This gives

$$\eta(\mu) = \kappa_{\text{es}}\langle I \rangle[1 + \bar{Q}p_*(\mu)] + \kappa_{\text{es}} \sum_k w_k Q_k \langle p_k I \rangle \frac{P_k(\mu)}{\langle p_\ast \rangle},$$  \hspace{1cm} \text{(A7)}

which is a complicated expression to work with, but note that the sum over $\kappa$ applies to the second-order term. However, to simplify our results, we have consistently replaced $p_k(\mu)/\langle p_k \rangle$ by $p_*(\mu)/\langle p_\ast \rangle$ in all second-order terms. Applying this approximation in

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equation (7) also gives rise to the FC source function. So when \( g(\kappa, \kappa') = \delta_{\kappa\kappa'} \), it is only necessary to apply CAR to second-order terms, which we have already done elsewhere to circumvent complicated sums over \( \kappa \). Thus the FC source function is fully consistent with our derivation.

In applications where a more accurate treatment of angular correlations is desired, we suggest that a natural way to generalize \( g(\kappa, \kappa') \) would be to define \( \epsilon_\kappa \) as the probability that a photon scattering in line \( \kappa \) will be thermalized or otherwise redistributed over the line list during the aggregate of all scatterings in the line, prior to escape from the resonance zone. Then the heuristic generalization

\[
g(\kappa, \kappa') = (1 - \epsilon_\kappa)\delta_{\kappa\kappa'} + \frac{\epsilon_\kappa Q_\kappa \langle p_\kappa \rangle}{\bar{\epsilon} \bar{Q} \langle p_\kappa \rangle},
\]

(A8)

where

\[
\bar{\epsilon} \bar{Q} \langle p_\kappa \rangle = \sum_\kappa \epsilon_\kappa Q_\kappa \langle p_\kappa \rangle,
\]

(A9)

asserts the assumption that incident photons will emerge from the same line with probability \( 1 - \epsilon_\kappa \), or else will be completely randomized over the line list prior to reemergence. This form for \( g(\kappa, \kappa') \) satisfies the requirements of radiative equilibrium and reciprocity. For our purposes, we simply follow LA and take \( \epsilon_\kappa \) to be negligibly small, which forces every line to be close to radiative detailed balance.

APPENDIX B

APPROXIMATING GRADIENTS OF THE ANGULAR OPACITY PROFILE

In a CAK wind (point star, no line overlap), the quantity \( w \) is constant. For a finite stellar disk or in the NID approximation, this no longer strictly holds, but assuming it nonetheless is useful because it allows us to evaluate \( \partial \ln f(\mu) / \partial \ln r \) in equation (42) without calculating the full steady-state solution. Neglecting the gradient of \( w \) allows us to write

\[
\frac{\partial \ln f(\mu)}{\partial \ln r} \cong \chi(\mu) \left( \frac{3 + 2\sigma \chi (1 - \mu^2)}{1 + \sigma \mu^2} \right),
\]

(B1)

where again \( \chi(\mu) \) is the fraction of effectively gray line opacity to total opacity. We also have in general

\[
\frac{\partial \ln f(\mu)}{\partial \ln \mu} = \chi(\mu) \frac{2\sigma \mu^2}{(1 + \sigma \mu^2)},
\]

(B2)

and combining these allows us to replace these terms in equation (42), allowing the subsequent development in the text. In principle, a more self-consistent determination of \( \partial \ln f(\mu) / \partial \ln r \) would be possible, but, since this term is never very significant, the improvement in accuracy in \( \Omega \) would be small, and so would not justify the increased complexity.

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