COOLING OF SOLAR FLARE PLASMAS. I. THEORETICAL CONSIDERATIONS

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ABSTRACT

Theoretical models of the cooling of flare plasma are reexamined. By assuming that the cooling occurs in two separate phases where conduction and radiation, respectively, dominate, a simple analytic formula for the cooling time of a flare plasma is derived. Unlike earlier order-of-magnitude scalings, this result accounts for the effect of the evolution of the loop plasma parameters on the cooling time. When the conductive cooling leads to an "evaporation" of chromospheric material, the cooling time scales as $L^{1/4}/n^{1/4}$, where the coronal radiative loss function is assumed to vary as $T^{-3/2}$, and quantities are evaluated at the start of the decay phase (defined as the time of maximum temperature). When the conductive cooling is static, the cooling time scales as $L^{3/4}/n^{1/4}$. In deriving these results, use was made of an important scaling law ($T \propto n^2$) during the radiative cooling phase that was first noted in one-dimensional hydrodynamic numerical simulations (Serio et al. 1991; Jakimiec et al. 1992). Our own simulations show that this result is restricted to approximately the radiative loss function of Rosner, Tucker, & Vaiana (1978). For different radiative loss functions, other scalings result, with $T$ and $n$ scaling almost linearly when the radiative loss falls off as $T^{-2}$. It is shown that these scaling laws are part of a class of analytic solutions developed by Antiochos (1980b).

Subject headings: MHD — plasmas — Sun: corona — Sun: flares

1. INTRODUCTION

It is well known that one result of a solar flare is the existence of a hot, dense plasma confined within a coronal loop (or series of loops). Observations from Skylab, the Solar Maximum Mission (SMM), and Yohkoh indicated that temperatures of order $(1-2) \times 10^7$ K and densities as high as $10^{11}$ cm$^{-3}$ existed after the conclusion of the impulsive phase (Moore et al. 1980; Wu et al. 1986). Typically, the peak temperature and emission measure (defined as $\int n_e^2 dV$, where $n_e$ is the electron density and $V$ the volume of hot plasma) peaked at different times, with the temperature reaching a maximum a few minutes before the emission measure.

In the absence of energy input, such a plasma will cool mainly by conduction and radiation. Given the initial conditions of the flare plasma, cooling times can be predicted from theoretical considerations. An important issue is how this predicted cooling time compares with the cooling time deduced from observations of the temperature evolution (Moore et al. 1980; Wu et al. 1986). If the two times are roughly equal, there is no need for significant energy deposition during the decay phase, while if the theoretical cooling time is shorter than the observed one, then additional energy must be deposited to account for the high observed temperatures. Clearly such an analysis is an extremely important way of determining the temporal profile of energy input throughout a flare.

An application of such an analysis is to the decay phase of large so-called two-ribbon flares (Priest 1981). For example, in the Skylab flare of 1973 July 29, bright X-ray loops were seen for over 12 h after the flare onset. Since the predicted cooling times were of order a few tens of minutes, it was clear that continual heating was occurring in this event. A widely accepted picture of this flare is that a filament eruption distorts the coronal magnetic field which then relaxes by reconnecting and forming a series of loops at different heights in the corona (Kopp & Pneuman 1976; Pneuman 1982; Cargill & Priest 1982, 1983; Svestka 1989). Each newly formed loop cools quite rapidly, but the continual formation of new loops gives the appearance of a very long lived X-ray source. Other two-ribbon flares from the SMM database were examined (Cargill & Priest 1983; Wu et al. 1986), with similar conclusions being drawn. For smaller events (often referred to as compact flares; Priest 1981), the situation is less clear since the theoretical cooling times can be closer to the observed ones, making the issue of whether extra energy deposition is required less cut (e.g., Wu et al. 1986). Also, the physical picture in these smaller events is probably different from two-ribbon flares, with perhaps only one loop being heated.

Flare cooling has been studied extensively by means of one-dimensional numerical simulations (e.g., Antiochos 1980a; Doschek et al. 1982; Serio et al. 1991; Jakimiec et al. 1992). In addition there have been many analytical estimates of flare cooling times (see Wu et al. 1986 and §2 for a discussion). This paper extends this earlier body of work in two ways. In §§ 2 and 3 we present a simple method that can be used to calculate quickly and easily a theoretical prediction of the total cooling time of a flare plasma from the upper end of the soft X-ray emitting temperature range to a few times $10^7$ K. During the course of the development of this analysis, we noted that numerical simulations led Serio et al. (1991) and Jakimiec et al. (1992) to propose a very simple scaling ($T \propto n^2$) during the radiative cooling phase of a flare. The second goal of this paper is to investigate this important scaling law. In §4 we show that the precise scaling between $T$ and $n$ depends on the radiative loss function and the Appendix compares these results to the analytic solutions for radiative cooling developed by Antiochos (1980b).

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2. MODELS OF FLARE COOLING

The evolution of the temperature during the decay phase of a flaring loop is governed by

\[
\frac{\partial p}{\partial t} + V_s \frac{\partial p}{\partial s} = -\gamma p \frac{\partial V_s}{\partial s} + (\gamma - 1) \left( \frac{\partial}{\partial s} \left( \kappa_0 \frac{T^{5/2}}{T} \frac{\partial T}{\partial s} \right) - P_{rad} \right),
\]

where \(p\) and \(T\) are the loop pressure and temperature, respectively, \(s\) is the coordinate along a magnetic field line, and \(V_s\) is the field-aligned velocity. \(P_{rad}\) is the optically thin radiative loss function in the region \(10^5 < T < 2 \times 10^7\), generally parameterized as a piecewise continuous function, \(P_{rad} = n^2 \chi T^3\) (e.g., Rosner, Tucker & Vaiana 1978; Priest 1982) with \(\chi < 0\) for \(T > 10^5\). \(\kappa_0\) is the coefficient of thermal conductivity, taken as \(10^{-6}\) in cgs units. Equation (1) assumes that all motions are along the magnetic field, which is treated as rigid. This implies that the plasma beta (\(\beta = 8\pi p / B^2\)) is small, a condition readily satisfied in most postflare situations. It is also assumed that the heat conduction perpendicular to the magnetic field is small (e.g., Priest 1982).

The simplest way to estimate conductive and radiative cooling times is to set \(V_s = 0\), assume that the density is constant and then equate the first term on the left-hand side with the second and third on the right-hand side, respectively. This gives cooling times due to conduction (\(\tau_c\)) and radiation (\(\tau_r\)):

\[
\tau_c = 4 \times 10^{-10} \frac{\eta L^2}{T^{5/2}}, \quad \tau_r = \frac{3k}{n_\lambda T^{\alpha-1}},
\]

where \(L\) is the half-length of the loop. It is important to realize that \(T\) and \(n\) are average quantities in the coronal part of the loop and that \(\tau_c\) and \(\tau_r\) do not account for the evolution of \(n\) and \(T\) as the loop cools. Hence they cannot be used to estimate the time a loop takes to cool from one temperature to another, where the temperatures differ significantly. These simple formulae are also widely used in the stellar flare community (e.g., Stern, Underwood, & Antiochos 1983). They do, however, indicate which loss process is the more important at a given time. More accurate estimates of conductive and radiative cooling times can be obtained by neglecting various terms in equation (1) and solving the resultant differential equation. We discuss these below, but are unaware of any analytic solutions of equation (1) where both conductive and radiative losses can be accounted for properly.

2.1. Conductive Cooling

When radiation is neglected, analytic solutions for conductive cooling exist either when \(V_s = 0\) (Antiochos & Sturrock 1976) or when \(V_s\) is subsonic (Antiochos & Sturrock 1978). In the former (latter) case, the loop density (pressure) is constant and the downward heat flux must be entirely radiated away by the lower atmosphere (drives an upflow). The latter case is often referred to as evaporative cooling. The assumption of constant loop density is unlikely to be valid, since the emission measure first increases and then decreases during the temperature decay (see Svestka et al. 1982 for two excellent examples): if the flare volume is constant, then the density must vary. The assumption of constant pressure is primarily made for analytic convenience, but has some basis in the observation that the loop pressure first increases, and then decreases near the time of maximum temperature (e.g., Underwood et al. 1978; Svestka et al. 1982). The temporal evolution of the temperature is given by

\[
T(t) = T_0 \left( 1 + \frac{t}{\tau_{0r}} \right)^{-2/5},
\]

\[
T(t) = T_0 \left( 1 + \frac{t}{\tau_{0c}} \right)^{-2/7},
\]

for static and evaporative cooling, respectively (Antiochos & Sturrock 1976, 1978). Here \(\tau_{0c}\) is the conductive cooling time at the start of the cooling, as defined in equation (2). The evaporative cooling rate (eq. [3b]) is slower than the static rate (eq. [3a]), since the energy is no longer lost from the system, and heats up cool material.

2.2. Radiative Cooling

Solutions of equation (1) where conduction is neglected can also be found (Antiochos 1980b). For \(V_s = 0\) the density is constant and the temperature satisfies

\[
T(s, t) = T_0 \left[ 1 - (1 - \alpha) \frac{t}{\tau_{0r}} \right]^{1/(1-\alpha)},
\]

where \(\tau_{0r}\) is the radiative cooling time at the start of the radiative phase.

As noted above, the density is not constant during the decay phase, and one way to incorporate density changes is to assume that the flare decay phase proceeds through a series of steady state solutions. Then a solution of equation (1) gives \(T \propto n^{1/2}\) (Rosner et al. 1978). Note that this scaling assumes that the loop is being heated throughout the decay phase, so that an extra heating term is required in equation (1) (see Rosner et al. 1978). This approach has been adopted by some workers in the stellar flare community (Van den Oord & Mewe 1989) to analyze very hot (\(T > 4 \times 10^7\) K) flares. However, the observations suggest this scaling holds in about 20% of solar flares (Sylwester et al. 1993), with most other flares satisfying a different proportionality relation between \(T\) and \(n\). This indicates that most solar flares do not cool through a series of steady state solutions. In order to ascertain whether a flare needs any additional heating, we need to know the fastest cooling time, i.e., the rate of cooling in the absence of any heating, so that a different approach is required.

Serio et al. (1991) and Jakimiec et al. (1992) have presented empirical evidence based on one-dimensional hydrodynamic numerical simulations that

\[
T \propto n^2
\]

during the decay phase of flares when long-term heating is absent. This scaling was found to be approximately independent of the loop length, indicating that thermal conduction is not playing a role in this part of the decay. Equation (5) is thus relevant only in the radiative cooling phase and can also be related to analytic solutions for radiative cooling developed by Antiochos (1980b; see the Appendix of this paper). In particular, equation (5) is not valid in the conductive cooling regime discussed above. Jakimiec et al. (1992) have also shown that the scaling between \(T\) and \(n\) is a function of the amount of heating in the decay phase, and they do in fact show that when extensive long-term heating is present, \(T \propto n^{1/2}\), as discussed above.
Before making use of equation (5), some remarks are appropriate. In § 4 we will show that this scaling is less general than claimed by Serio et al. (1991) and Jakimiec et al. (1992) since the exact power of proportionality in equation (5) is a function of the coefficient (α) of the radiative loss function. However, equation (5) does correspond to the popular radiative loss function of Rosner et al. (1978), with α ≈ − 1 2, and we use equation (5) to develop our analytic analysis of cooling times. Note also that for a constant flare volume, equation (5) implies that \( T \propto EM \), where \( EM = \int n^2 dV \) is the emission measure. This scaling was found in a number of observed flares by Syntetos et al. (1993).

On use of equation (5), and setting \( \alpha = -\frac{1}{2} \), the solutions of Antiochos (1980b; see the Appendix for details) imply a time dependence of the temperature:

\[
T(t) = T_0 \left( 1 - \frac{3}{2} \frac{t}{\tau_{r0}} \right),
\]

Equation (6) can be generalized to different values of both \( \alpha \) and the scaling between \( T \) and \( n \). This is discussed further in the Appendix.

3. A SIMPLE COOLING MODEL

The results of this section are based on the assumption that during the evolution of a flare loop, conductive cooling dominates initially and at later times radiative cooling takes over. If radiative cooling dominates initially, then it will always dominate, since as the temperature falls, \( \tau_r \) decreases and \( \tau_c \) increases. In this case, equations (4) or (6) can be used to compute the cooling time. However, if conductive cooling dominates initially, then as the temperature falls (and the density rises if the cooling is evaporative), \( \tau_c \) and \( \tau_r \) will change in such a way that they will become equal at some time, referred to as \( t_* \). For \( t_* \), radiative cooling takes over. It is thus assumed that the loop cools in three distinct stages, with conductive cooling being important at the start, a transition to radiative cooling occurring at a temperature \( T_* = T(t = t_*) \), and finally a rapid draining of the loop when its temperature falls below \( 10^8 \) K. We do not address this final stage, which arises because thermal pressure gradients cannot support a significant amount of cool material against gravity.

To demonstrate this approach, we consider a case where \( P_{rad} = 1.2 \times 10^{-19} n^2 T^{-3/2} \), which gives a reasonable fit between \( 10^8 \) and \( 10^7 \) K, and with draining radiative cooling described by equation (6). Initially we will demonstrate the method for evaporative (E) conductive cooling (eq. (3b)): the corresponding results for static (S) conductive cooling (eq. [3a]) will be given later in this section. Conductive and radiative cooling become approximately equal when \( \tau_c \approx \tau_r \) at time \( t_* \), and up to this time, the loop pressure is assumed constant, as required by equation (3b). If the loop initially has a temperature and a density of \( T_0 \) and \( n_0 \), respectively, substituting equation (3b) into equation (2) gives

\[
t_* = \tau_{r0} \left[ \frac{T_{r0}}{T_{r0}} \right]^{7/12} - 1 \]

with \( T_* = T(t = t_*) \) given by

\[
\frac{T_*}{T_0} = \left( \frac{T_{r0}}{T_{r0}} \right)^{-1/6}
\]

where we have used the result of constant loop pressure. In equations (7E) and (8E),

\[
\frac{T_0}{\tau_{r0}} = 8.67 \times 10^{12} \frac{T_{r0}^4}{n_0^2 L^2},
\]

where equation (2) has been used. To give the reader a feel for the parameters involved in flare decay, for a typical flare loop at the temperature peak, one might have \( T = 1.5 \times 10^7 \) K, \( n = 10^{10} \text{ cm}^{-3} \), \( L = 3 \times 10^9 \) cm, so that \( \tau_{r0}/\tau_{r0} \approx 490 \).

Below \( T = T_* \), radiative cooling takes over, and it is assumed that \( T \propto n^2 \), with the pressure now allowed to vary with time. The time to cool from \( T_* \) to \( T_L \) (where \( T_L \) is the temperature at which the single power law of our radiative loss function becomes inaccurate) is given by

\[
t_{*+} = \frac{2 \tau_{r*}}{3} \left[ 1 - \left( \frac{T_{L}}{T_*} \right) \right],
\]

where \( \tau_{r*} \) is the radiative cooling time at \( t = t_* \) defined as

\[
\tau_{r*} = \tau_{r0} \left( \frac{T_{r0}}{T_{r0}} \right)^{5/12}.
\]

The total loop cooling time (\( \tau_{cool} \)) is then \( t_* + t_{*+} \):

\[
\tau_{cool} = \tau_{c0} \left[ \left( \frac{T_{r0}}{T_{r0}} \right)^{7/12} - 1 \right] + \frac{2}{3} \tau_{r*} \left( \frac{T_{r0}}{T_{r0}} \right)^{5/12} \left[ 1 - \left( \frac{T_{L}}{T_*} \right) \right].
\]

For given values of the loop length, \( T_0 \) and \( n_{0} \), all of which can be obtained experimentally, equation (12E) gives the cooling time of the flare plasma. This result can be further simplified by making two approximations. If \( \tau_{r0} \ll \tau_{r0} \) and \( T_* \gg T_L \), then equation (12E) becomes

\[
\tau_{cool} \approx \left( \frac{2}{3} \right)^{7/12} \tau_{r0}^{5/12}.
\]

In this limit, \( \tau_{cool} \) scales approximately as \( 1/p_{1/6}^4 \), the precise result being

\[
\tau_{cool} \approx 2.35 \times 10^{-2} \frac{L^{5/6}}{T_0^{1/6} n_{0}^{1/6}} = 6.06 \times 10^{-2} \frac{L^{5/6}}{p_0^{1/6} s}.
\]

Subscripts "0" correspond to plasma properties at the start of the cooling.

For static conductive cooling, a similar analysis can be carried out using equation (3a). Here the density is constant in the conductive phase and a function of time in the radiative phase, with \( T \propto n^2 \). The results are

\[
t_* = \tau_{r0} \left( \frac{T_{r0}}{T_{r0}} \right)^{5/18} - 1
\]

\[
\frac{T_*}{T_0} = \left( \frac{T_{r0}}{T_{r0}} \right)^{-1/6}
\]

\[
\tau_{r*} = \tau_{r0} \left( \frac{T_{r0}}{T_{r0}} \right)^{3/8}
\]

\[
\tau_{cool} = \tau_{c0} \left[ \left( \frac{T_{r0}}{T_{r0}} \right)^{5/18} - 1 \right] + \frac{2}{3} \tau_{r*} \left( \frac{T_{r0}}{T_{r0}} \right)^{3/8} \left[ 1 - \left( \frac{T_{L}}{T_*} \right) \right],
\]
\[ R = \frac{\tau_{\text{cool (evaporative)}}}{\tau_{\text{cool (static)}}} = \left( \frac{\tau_0}{\tau_0} \right)^{1/24}, \]

so that the cooling rates are similar. In particular, for the flare parameters discussed above, \( R = 0.75 \). In § 2.1 we noted that evaporative cooling was a factor \( T/T_0 \) slower than static cooling. However, since evaporative cooling increases the loop density, the radiative cooling, when it begins, will result in a faster temperature decrease than will happen when the conductive cooling is static. The two offsetting effects lead to cooling times that are similar.

3.1. Comparison of Equation (14) with Existing Numerical Solutions

Equation (14E) can be used to determine whether long-term heating is required in the decay phase of flares. If the loop cools according to equation (14E), then long-term heating is not required. If the cooling is significantly slower than equation (14E) predicts, then one can infer continual heating. However, comparison of equations (14E) and (14S) with observations can tell us nothing about the accuracy of the assumptions that went into obtaining it. To do this, a comparison with more complete theoretical cooling models (those using hydrodynamic simulations) is required.

Such a comparison is difficult for two reasons. One is that some workers who have examined the decay phase of a flare have chosen initial conditions that are only appropriate for examining the radiative cooling phase. We present such simulations in § 4 (see also Doschek et al. 1982). They are initialized at a time corresponding to the maximum of the emission measure rather than the maximum of the temperature, with the initial state satisfying \( \tau_e \approx \tau_r \). The conduction-dominated phase, which occurs between the temperature and emission measures peaks (Svestka et al. 1982), is thus missing. Antiochos & Krall (1979) carried out simulations of the decay of a flaring loop where the initial conditions had \( \tau_e \ll \tau_r \) and showed that a conductive phase with an increasing emission measure was followed by a radiative cooling phase. However, comparison with equations (14E) and (14S) is not possible, since Antiochos & Krall assumed a rather large magnetic flux tube divergence in the corona, so that the conductive cooling cannot be described analytically.

A more realistic comparison would be one with simulations of the energization and decay of the flare plasma (e.g., Nagai 1980; Pallavicini et al. 1983; MacNeice et al. 1984; Nagai & Emslie 1984; Fisher, Canfield, & McClymont 1985; Mariska, Emslie, & Li 1989; and references therein). We compare our analytic results to a number of these cases, bearing in mind that there are a considerable number of free parameters that govern how the flare energy is deposited.

Pallavicini et al. (1983) present results where a loop is heated by a thermal pulse at its apex for \( 10^2 \) s and then allowed to evolve. At the end of this heating, \( T_0 \approx 2 \times 10^7 \) K, \( n_0 \approx 6 \times 10^{10} \text{ cm}^{-3} \) and \( L = 1.8 \times 10^9 \) cm (Fig. 9 of Pallavicini et al. 1983). Using equation (14E), we find \( \tau_{\text{cool}} \approx 1150 \) s. Pallavicini et al. show that the loop cools to a few times \( 10^6 \) K in \( \sim 1050 \) s: unfortunately the later part of their decay phase will be influenced by the presence of an ambient coronal heating term that will modify the temperature evolution below a few million degrees. It is worth comparing the good agreement of our \( \tau_{\text{cool}} \) with the simulations with the "simple" cooling times given by equation (2). Using the initial values of \( T \) and \( n \), equation (2) gives \( \tau_e = 51 \) s and \( \tau_r = 4470 \) s. However, it should be noted that the pressure in the simulations of Pallavicini et al. (1983) decays during the conductive phase rather markedly, in contrast with the assumption of the evaporative model of constant pressure.

Nagai (1980) carried out a similar numerical experiment to that of Pallavicini et al. (1983). At the end of energy injection, he has \( T_0 \approx 2 \times 10^7 \) K, \( n_0 \approx 4 \times 10^9 \text{ cm}^{-3} \), and \( L \approx 3 \times 10^9 \) cm. Equation (14E) gives \( \tau_{\text{cool}} \approx 2840 \) s, whereas Nagai finds a cooling time of (very) approximately 50 minutes, or 3000 s. Again the agreement is encouraging.

Perhaps the most systematic set of numerical simulations of one-dimensional flare plasma evolution have been carried out by Jakimiec et al. (1992). Figure 1 of their paper shows results where the flare temperature maximum is \( \approx 2.2 \times 10^7 \) K with a density of \( 6 \times 10^{10} \text{ cm}^{-3} \) in a loop of half-length \( 2 \times 10^9 \) cm. We find \( \tau_{\text{cool}} \approx 850 \) s, which compares with their cooling time of 770 s. Unfortunately, in this case the flare heating is allowed to persist until close to the emission measure maximum, so that a direct comparison is difficult. (Both Pallavicini et al. and Nagai used very impulsive bursts of heating.) While this comparison with selected simulations is by no means systematic, it does suggest that equation (14E) may be an adequate description of flare plasma cooling.

4. ANALYSIS OF THE \( T \propto n^2 \) SCALING

The principal results of the preceding section depended on an empirical scaling during the radiative decay phase of the flare (Serio et al. 1991; Jakimiec et al. 1992). We were sufficiently curious about this result, as well as its relationship to the analytic solutions of Antiochos (1980b), which make assumptions appropriate to the regime of validity of the scaling, that we felt a detailed investigation of the \( T \propto n^2 \) scaling was warranted.

It is important to note that Jakimiec et al. found that the precise form of the scaling law differed slightly between the various simulations that they carried out: in particular, the duration of the heating function played an important role (Figs. 7a and 7b of their paper). However, the overall impression that one forms from the work of Jakimiec et al. is that \( T \propto n^2 \) is quite a robust scaling for the radiative loss function used.

In order to investigate the temperature-density scalings in the radiative phase, we have performed our own one-dimensional hydrodynamic simulation of a cooling flare loop. We solve the one-dimensional equations of mass, momentum, and energy conservation along a single magnetic field line. At the base of the loop, there is a model chromosphere with \( T = 10^4 \) K which can respond to incoming coronal pertur-
bations by either radiating away an incoming heat flux or by " evaporating" material into the corona. The mesh is nonuniform, with high resolution (5 km) in the transition region and a coarser mesh in the corona. Further details can be found in Mariska (1987).

In the first example we ran, the radiative loss function is that of Raymond (1978) below $T = 10^{4.75}$ and is a power law, $8.23 \times 10^{-20} T^{-1.2}$, above that point. The initial state is a loop with half-length $2 \times 10^9$ cm and an initial atmosphere in hydrostatic equilibrium with a maximum temperature of $10^7$ K and density of $8.5 \times 10^{13}$ cm$^{-3}$. These conditions satisfy a loop scaling law that is equivalent to that of Craig et al. (1978) and Rosner et al. (1978), with a small difference due to the slightly different radiative loss function used in each case.

The upper curve in Figure 1 shows a log-log plot of $T_e$ against $n_e^2$, where subscript "e" refers to quantities plotted at the loop summit. The lower curve shows the average coronal temperature plotted against the average coronal density. The average is over temperatures between $10^6$ K and the maximum temperature of the loop. The curve consists of three distinct parts. At high temperatures, the density changes relatively slowly as a function of temperature. This is a regime where radiative cooling is still important. Since the conductive cooling time is a strong function of temperature, large changes in $T$ are to be expected for small changes in $n$. In the intermediate part of the curve, temperature and density do appear to be related by a power law (discussed in more detail below). At lower temperatures, there is a tendency for the curve to deviate from this power-law behavior. It is worth noting that in all the cases discussed here, the downflows remained subsonic.

In the central part of the curve (from $T = 10^6.8$ to $T = 10^6.4$), we find that $T_e \propto n_e^{1.94 \pm 0.17}$. The power for the average temperature is $1.92 \pm 0.02$, in good agreement with the value discussed by Jakimiec et al. (1992), who found $T \propto n_0^{1.94 \pm 0.17}$. Note that their large values of $\sigma$ appear to be due to the very noisy (and nonmonotonic) behavior of their apex temperature as a function of time (see Fig. 3 of Jakimiec et al.). Another useful diagnostic of flare cooling is to examine the relative contributions of the various loss terms in equation (1) throughout the decay phase. This is best achieved by integrating over the loop length to get

$$\frac{3}{2} \frac{d\Phi}{dt} = -\frac{5}{2} \frac{pV_b}{L} - \left( \frac{\kappa_0 T^{5/2}}{L} \right) \frac{dT}{ds} \bigg|_{s=s_b} - \dot{P}_{\text{rad}}, \quad (16)$$

where $V_b$ is the (downflowing) velocity through the base, $s_b$ is the base position, and we have assumed symmetry (no flow or heat flux at the loop summit). The averaged quantities in equation 16 are defined as

$$\dot{\Phi} = \frac{1}{L} \int_{s_b}^{s_L} \Phi \, ds.$$

Figure 2 shows $d\Phi/dt$ (squares) and the three terms on the right-hand side of equation (17) as a function of time, where radiative, enthalpy, and heat flux losses are denoted by triangles, stars, and diamonds, respectively, and the base temperature is taken as $10^6$ K. It is clear that radiation dominates the decay of the plasma, with enthalpy losses becoming important toward the end and conductive losses being important at the very start. This is to be expected given the fact that the initial loop has $\tau_r \approx \tau_c^*$. 

![Fig. 1.—Scaling of temperature (horizontal axis) with density (vertical axis) for the first simulation described in § 4. The upper and lower curves show, respectively, the quantities at the loop apex and those averaged over the coronal part of the loop. The loop length is $2 \times 10^9$ cm, and the initial temperatures and densities are $10^7$ K and $3 \times 10^{13}$ cm$^{-3}$, respectively. The radiative loss function is that of Raymond (1978) below $T = 10^4.75$ and scales as $T^{-1.2}$ above that temperature.](image1)

![Fig. 2.—Relative magnitudes of the four terms in eq. (16) for the example shown in Fig. 1. $-1.5 \frac{d\Phi}{dt}$, radiative, enthalpy, and heat flux losses are denoted by squares, triangles, stars, and diamonds, respectively. All physical quantities are averaged over the loop above $10^6$ K.](image2)
To test the sensitivity of the results to the radiative loss function, we also ran a number of cases using the complete Raymond (1978) radiative loss function for different values of \( L \). The results are summarized in Table 1. Assuming \( T \propto n^\alpha \), the proportionality constant \( l \) shows weak variation with a loop length, as well as a small dependence on the radiative loss function (contrast the case of \( L = 2 \times 10^9 \) cm in the table with the results in Fig. 1).

<table>
<thead>
<tr>
<th>( l ) (cm)</th>
<th>( 10^9 )</th>
<th>( 2 \times 10^9 )</th>
<th>( 4 \times 10^9 )</th>
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<td>1.87</td>
<td>1.75</td>
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<tr>
<td>( l ) (average)</td>
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<td>1.74</td>
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</tbody>
</table>

To investigate further the importance of the radiative loss on the \( T-n \) scaling, the radiative loss function was modified above \( T = 10^{5.75} \) K to be of the form \( 5.83 \times 10^{-11} T^{-2} \). This case was run to investigate any strong dependence of the scaling on the radiative loss parameter \( \alpha \). The loop still had a half-length of \( 2 \times 10^9 \) cm, but the maximum temperature was now \( 7 \times 10^6 \) K and the density \( 10^{11} \) cm\(^{-3} \). The different values were those required to satisfy the initial condition of a hydrostatic loop. The generation of the initial loop state in this case presented some interesting features. In general, a loop in hydrostatic thermal equilibrium has \( \tau_e \approx \tau_r \), where quantities are averaged over the loop. This equality gives (approximately) the Rosner et al. (1978) scaling law. However, if \( \alpha \) becomes too negative above some temperature, the arguments of Rosner et al. (1978) need to be revised somewhat. Specifically, for \( \alpha < -3/2 \), the initial loop condition no longer satisfies \( \tau_e \approx \tau_r \), but has \( \tau_e < \tau_r \). The problem this poses in the present context is that the loop is in the conductive regime for some time during the decay and only moves into the radiative regime later on.

Figure 3 shows the log-log plot of \( T \) and \( n \) in the same format as Figure 1, and Figure 4 shows the various losses as a function of time. Looking at Figure 4 first, it can be seen that conductive losses are equal to, or exceed, the radiative losses during the first half of the cooling. Therefore, it is only the lower parts of the curves in Figure 3 that are of interest for radiative cooling models. From \( T = 10^6 \), we find that the apex (average) temperature scales as \( n^{1.27 \pm 0.01} (n^{1.21 \pm 0.01}) \). Clearly these differ significantly from the results in Figure 1, indicating that the scaling discussed by Jakimiec et al. (1992) is dependent on the chosen radiative loss function.

This dependence on the radiative loss function would appear to be rather a moot point at first sight, since the loss function used by Jakimiec et al. and ourselves (in Fig. 1) has been generally accepted as being reasonably accurate for over 15 years. However, recent work on coronal element abundances (Cook et al. 1989) indicate that this viewpoint may be due for revision. In particular, Cook et al. showed that the radiative loss function may fall off rather more rapidly in the temperature range above \( 10^6 \) K than has been assumed. Indeed Cargill (1994) made a very crude estimate that in this regime \( \alpha \approx -2.5 \). The importance of the result in Figure 3 then becomes apparent.

4.1. Relation of Simulation Results to Solutions of Antiochos

Antiochos (1980b) examined the radiative decay phase of a flare by assuming that thermal conduction was negligible and that any flows were subsonic. The Appendix outlines the math-
emathematical development of Antioclos (1980b) and corrects a number of typographical errors in his equations. He obtained an infinite number of solutions, and in the Appendix, we show that both of the empirical scalings derived above correspond to different solutions.

It should be noted that each of the solutions permitted by the formalism of Antioclos (1980b) in fact correspond to different pressure gradients between loop apex and base. This weak pressure gradient can be readily determined a posteriori by computing \((1/2)nmV^2\) at the base of the loop and then making the assumption that \(p + (1/2)nmV^2\) is constant along a field line. Thus \(p = p_0\) to lowest order and \(p = p(s, t)\) to first order, the deviation from isobaricity being \(\propto V^2/C_s^2\). Since the flows are subsonic, the deviation is small. However, the Antioclos model has nothing in it to distinguish between this infinite number of valid solutions and so cannot be used to determine why the scalings derived in the previous section arise. The relevant solution is determined by the value of the weak pressure gradient between loop top and base.

It is a straightforward exercise to compare the simulation results with these analytic solutions. Figure 5 shows the value of the loop apex temperature (solid line and squares) and the average loop temperature (solid line and triangles) for the case with \(\alpha = -\frac{1}{2}\); the symbols that begin at 200 s (stars and diamonds, respectively) are the analytic solutions of Antioclos (1980b) described in the Appendix, with \(T \propto n^2\) assumed. We must be careful to begin the analytic solution at a temperature where conductive cooling is unimportant, so that the analytic solution cannot be used at \(t = 0\). As can be seen, the two cases are not identical: the analytic solution decays linearly with temperature, while the numerical solution is slightly faster. A good way to compare the two cases is by the time the loop takes to cool from one to another temperature in the radiative phase. We choose the upper and lower temperatures as \(6 \times 10^6\) and \(2 \times 10^6\), respectively. The numerical (analytic) solution takes \(\approx 400\) (500) s to cool, an error in the analytic solution of 25%. All of the physical assumptions for the validity of the Antioclos (1980b) model are satisfied, such as subsonic flows, lack of conduction, etc. We have found that the key mathematical assumption, namely, separability of temporal and spatial variations is not satisfied. Specifically, the two time-dependent coefficients multiplying the first and last terms in equation (A3) are not constant. Separability requires that these terms be constant. However, in view of the impossibility of obtaining analytic solutions to the dynamic radiative cooling equations without the separability approximation, an error of 25% is quite tolerable as a price for making progress.

Figure 6 shows the corresponding results for \(\alpha = -2\), with an identical notation. In plotting the analytic solutions, we have assumed that \(T \propto n^{1.25}\), as discussed above. Note also that we have to start the analytic radiative solution at \(\approx 4 \times 10^6\) K, since conduction is more important at lower temperatures than in the case shown in Figure 5. In this case, the errors in the cooling time are of order 50%, with the numerical and analytic times being \(\approx 600\) and 900 s, respectively. The separability assumption is again violated, but much more severely than in the \(\alpha = -\frac{1}{2}\) case. One must conclude that, while the assumptions of Antioclos (1980b) are reasonable for loss functions that are a weak function of temperature, care needs to be taken with their use when the loss function varies strongly with temperature.

\[\text{Fig. 5.—Time dependence of temperature for the radiative loss function}\]

\[\text{Fig. 6.—Mean time dependence of the loop temperature for the radiative loss function}\]

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4.2. The Origin of the $T$-$n$ scalings

It can be readily seen on simple physical grounds that the temperature should vary more slowly as a function of density when the coronal radiative loss curve has a steeper slope. For cases when $\alpha$ is more negative, the loop top will cool more slowly by radiation than it will when $\alpha$ is larger. Now the pressure gradient that drives material out of the loop must be set up by the difference in the radiative losses in the base of the corona, $10^6$ K say, and the loop top. If the radiative losses are fixed at $10^9$, then more negative values of $\alpha$ will lead to stronger pressure gradients. Note that these pressure gradients are rapidly equalized by flows, and the pressure remains constant in space to lowest order. However, the point is that, in order to maintain a constant pressure, a larger outflow of material from the loop is expected for more negative $\alpha$. The larger the outflow, the more rapid is the change in coronal density, so that temperature will scale more weakly with the density as $\alpha$ becomes more negative.

From examination of the equations of mass and energy conservation, we can show that this relationship between density and temperature is expected to be an approximate power law. During the cooling phase when conduction is negligible and the motions are highly subsonic, the energy equation can be written as

$$\frac{5}{2} \frac{\partial}{\partial s} (pV_s) = - \frac{3}{2} \frac{dp}{dt} - P_{\text{rad}}.$$  \hspace{1cm} (18)

The key point is that, at any instant, this equation has the identical physical form as the energy equation of the well-known static loop models (e.g., Rosner et al. 1978; Vesecky, Antiochos, & Underwood 1979). The radiative loss term $P_{\text{rad}}$ is, of course, the same in both cases. The pressure decrease term $-1.5 \frac{dp}{dt}$ is spatially constant and is positive; hence, it acts as a uniform heating rate. The enthalpy term $2.5 \frac{\partial (pV_s)}{\partial s}$ is negative at the location of the temperature maximum (the loop midpoint for a symmetric model), since the plasma accelerates away from this point; however, the term becomes positive at slightly lower temperatures where the plasma begins to decelerate. This behavior is identical to that of the conductive term in the static models (e.g., Fig. 3 of Vesecky et al. 1979). At the temperature maximum, the energy balance is between the pressure decrease and the sum of radiative losses and convective cooling. For uniform pressure, however, the radiative losses have a very strong temperature dependence, so that at lower temperatures the balance is between radiative losses and convective heating.

Note, also, that the cooling loop will have a very thin transition region as in the static loop models. We expect that since the velocities are highly subsonic in the transition region, an approximately steady sate is set up there with a spatially constant mass flux. It is shown in the Appendix that, in fact, all the analytic solutions of Antiochos (1980b) exhibit this property. Using the results that the mass flux is constant in the transition region and that the energy balance there is between radiation and convection, the temperature variation of the temperature scale height can be obtained from equation (18) as

$$\left( \frac{1}{T} \frac{\partial T}{\partial s} \right)^{-1} = T^{3-\alpha}.$$ \hspace{1cm} (19)

This dependence on temperature is actually stronger than in the static models, so we expect that an even thinner transition region in the cooling loop.

We reach the interesting conclusion that the instantaneous spatial structure of a cooling loop is the same as that of a static loop model, with convection playing the role of conduction and pressure decrease playing the role of uniform coronal heating. Consequently, the analogous scaling laws hold; at coronal temperatures all three terms in the energy equation are of similar order of magnitude,

$$\frac{dp}{dt} \sim n^2 T^3 \sim \frac{nV_s}{L}.$$ \hspace{1cm} (20)

and in the transition region only the second part of this relation holds. The density decrease, on the other hand, can be estimated from the continuity equation,

$$\frac{dn}{dt} \sim \frac{nV_s}{L}.$$ \hspace{1cm} (21)

Using the relations above we find that $(dp/dt)/(dn/dt) \sim p/n$; hence the temporal variation of the coronal pressure and/or temperature with density is a power law. Unfortunately, the exact value of the power law depends on the magnitudes of the coefficients in these relations which will change with different models, as shown by the results of our simulations.

5. DISCUSSION AND CONCLUSIONS

In this paper, we have examined two aspects of the flare cooling problem. First, we have presented a simple analytic formula for computing the cooling time of a flare plasma. By breaking up the cooling into two separate phases (conductive at high temperatures and low densities and radiative in the opposite limit), we are able to draw on existing knowledge of how solar plasmas cool and present a result that we believe can be of great use to the observational community. It is our intention to apply these results to well-observed solar flares from the Yohkoh satellite. This will be described in a subsequent paper.

These simple results relied on the use of a simple scaling law connecting temperature and density, originally discussed by Serio et al. (1991) and Jakimiec et al. (1992). For the standard coronal radiative loss functions, this scaling has $T \propto n^{1/2}$, with an error of perhaps 20% in the power. We have shown that this scaling is, in fact, a function of the chosen radiative loss function, with $T \propto n^{1/2.5}$ when the loss function is proportional to $T^{-2}$. The true value of such scaling laws is that they can be used to make significant analytic progress in understanding flare cooling, and so it is important to understand the precise role that the scaling plays.

If we assume that $T \propto n^3$, then the radiative cooling time can be expressed as

$$\tau_c \propto \frac{1}{T^{1.5}}.$$ \hspace{1cm} (22)

Then, if $\alpha = -\frac{1}{2}$, $\tau_c \propto T^{-1} (T^{-3+1/1.5})$, a difference of roughly $T^{-1}$. If the density is constant in the cooling, then the ratio of the two times is $T^{-1.5}$. So, while the incorporation of the density changes plays some role, the major change in the cooling rates is due the change in $\alpha$ itself. The incorporation of the density variation just decreases the difference in the cooling rates slightly.

This dependence of the scaling between $T$ and $n$ on the radiative loss function also has consequences for the use of $T$-$n$
diagrams to infer the heating rate in flares (Sylwester et al. 1993). These authors argued that flares in which no long-term heating was taking place would show $T \propto n^2$, and those with heating would have $T$ proportional to a lower power of $n$, with steady heating having $T \propto n^{1/2}$. In view of the importance of the loss function in determining the $T\sim n$ scaling, serious doubt must be cast on their claim that 80% of flares have heating operating well into the decay phase. This also serves to highlight the crucial importance that should be placed on the need for accurate coronal abundance measurements, which play a key role in determining the loss function.

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APPENDIX

THE ANALYTIC SOLUTIONS OF ANTIOCHOS

Antiochos (1980b) examined the radiative decay phase of a flare by assuming that thermal conduction was negligible and that any flows were subsonic. The latter constraint immediately led to

$$p(s, t) = p(t), \quad \text{(A1)}$$

and an equation describing the evolution of the loop temperature was derived (Antiochos 1980b, eq. [29]; there is a $\partial/\partial s$ missing in the second-to-last term of this equation). It was assumed that separable solutions existed, so that

$$y = T(s, t)^{2-\alpha} = T_0^{2-\alpha} \exp(\theta(t)\xi(s)), \quad \text{(A2)}$$

where Antiochos's original notation is used. The separability assumption then gives a basic equation:

$$\frac{5}{3} \frac{\tau_{ao}}{\phi} \frac{d\phi}{dt} \left[ \frac{\xi \frac{d^2 \xi}{ds^2} - \left( \frac{d \xi}{ds} \right)^2}{\xi^2 - (2 - \alpha) \frac{d^2 \xi}{ds^2} + \frac{1}{\xi} \left( \frac{d \xi}{ds} \right)^2 + \frac{\tau_{ao}}{3 \phi^2} \frac{d\phi}{dt} \left( 7 - 2 \alpha \right) \left( \frac{d \xi}{ds} \right)^2 - \frac{2(2 - \alpha) \xi}{ds^2} \frac{d^2 \xi}{ds^2} \right] = 0, \quad \text{(A3)}$$

where a number of typos in the original paper are corrected. Note also that in distinction from Antiochos (1980b), the loop cross-sectional area is assumed constant. By requiring that the temporal and spatial evolution be separable, it is found that

$$p(t) = p_0 \phi(t) = p_0 (1 + \eta t)^{-\nu - 1}, \quad \text{(A4)}$$

$$\theta(t) = (1 + \eta t)^{-\nu}, \quad \text{(A5)}$$

$$\frac{d \xi}{ds} = B \xi^{-1/(2 - \alpha)}(\xi + \mu)^{\nu}, \quad \text{(A6)}$$

where $B$ is a constant of integration and $\mu$ and $\nu$ are two arbitrary parameters, related to the separation constants, while $g$ and $\eta$ are given by

$$g(\nu) = \frac{3(1 + \nu)}{4 - 2\alpha - \nu(2\alpha + 1)} + \frac{3 - \alpha}{2 - \alpha}, \quad \text{(A7)}$$

$$\eta(\nu) = \frac{3(2 - \alpha)}{[4 - 2\alpha - \nu(2\alpha + 1)] \mu \tau_{ao}}, \quad \text{(A8)}$$

To specify completely the loop temperature structure, we need to proceed beyond Antiochos (1980b) and determine $B$, $\mu$, and $\nu$. Two boundary conditions immediately suggest themselves. At the loop summit, it is reasonable to assume a symmetry condition, so that $d\xi/ds = 0$ there ($\xi = 1$). Then, equation (A6) immediately gives $\mu = -1$. At the loop base, we can assume $\xi \rightarrow \xi_b$, so that equation (A6) becomes

$$\frac{d \xi}{ds} = \left( \frac{d \xi}{ds} \right)_b \left( \frac{\xi}{\xi_b} \right)^{-1/(2 - \alpha)} \left( 1 - \xi \right)^\nu, \quad \text{(A9)}$$

where it is assumed that $\xi_b \ll 1$. At this point $\nu$ is still undetermined, and it appears as if different values of $\nu$ correspond to an infinite set of possible solutions. Antiochos (1980b) outlines various general constraints on $\nu$, but it is unclear what these really correspond to.

If we require that $T$ and $n$ are related by $T(t) \propto n(t)^l$, then equations (A3) and (A4) show that

$$\nu = \frac{(\alpha - 2)l}{(2 - \alpha)l - (l + 1)}, \quad \text{(A10)}$$
For $\alpha = -1/2$, we find $v = -5/2$ and $g(v) = 1/2$. Thus, the temperature cools linearly with time, and it can also be readily seen that $\eta = -3/(2\tau_0)$. This is where equation (9) in §2.2 comes from. Hence, the empirical scaling of Serio et al. (1991) and Jakimiec et al. (1992) is one of a family of analytic solutions for radiative cooling with subsonic flows.

However, it should be noted that below a critical value of $l$, $g(v) < 0$. In this regime equation (A9) indicates that our solution is invalid, since the temperature diverges at the loop summit. Equations (A7) and (A10) can be combined to show that

$$l > \frac{9 - 2\alpha}{6 - 3\alpha}$$

is required for solutions. For $\alpha = -1/2(-2)$, the critical value of $l$ is $4/3 (13/12)$, so that both of our empirical scalings can also be described by the solutions of Antiochos (1980b).

An important feature of all the analytic solutions is that the mass flux $\eta(s, t)(s, t)$ has almost no spatial variation except near the temperature maximum. From the expressions above for $p(s, t)$ and $T(s, t)$, the density is obtained immediately:

$$\eta(s, t) = \eta_0(1 + \eta)^{1-\alpha(2-\alpha)}\xi^{1/2-\alpha}$$

The velocity $\xi(s, t)$ can now be calculated from the continuity equation,

$$\xi(s, t) = C(1 + \eta)^{-1/2-\alpha}(1 - C)^{-1-\alpha}$$

where $C$ is a constant of integration that can be determined from the constants in equation (A9). Multiplying equations (A12) and (A13) we find that the spatial variation of the mass flux $\eta v$ is given by $(1 - \xi)^{-1-\alpha}$. Since $\xi$ is a strong function of temperature $\xi \sim T^{2-\alpha}$, $\xi < 1$ for $T < T_{\text{max}}$; hence the mass flux is approximately constant over most of the temperature range of the loop. Note that this result does not depend on the particular values of $\alpha$ and $v$; it holds for any acceptable solution.

REFERENCES

— 1982, Solar Magnetohydrodynamics (Dordrecht: Kluwer)