ASYMPTOTIC MODAL INERTIA

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ABSTRACT An asymptotic formalism is summarized for estimating the inertia of a stellar p mode and, under somewhat more restricting circumstances, a g mode of moderate degree. The importance of the inertia lies principally in its connecting of the energy in the mode with the amplitude of the associated disturbance in the surface layers of the star. Thus, it occurs in the relation between observed oscillatory power and the energetics of excitation and decay. It also arises in perturbation analysis when the perturbation takes place in the surface layers near where the eigenfunction is normalized.

INTRODUCTION
The inertia $I_{nml}$ of a linear adiabatic mode of stellar oscillation of principal order $n$, degree $l$ and azimuthal order $m$ is defined by

$$I_{nml} = \int_V \rho \xi \cdot \xi^* dV,$$

where $\xi$ is the appropriately normalized spatial factor in the displacement eigenfunction, $\rho$ is the density of the unperturbed star and the integral is taken over the volume $V$ of the star. The eigenfunction is normalized in the unperturbed surface of the star which, referred to spherical polar coordinates $(r, \theta, \phi)$, is given by $r = R$. I confine explicit results to spherically symmetrical equilibrium configurations, so that $R$ is independent of $\theta$ and $\phi$ and $I_{nml} = I_{nl}$ is independent of $m$; the analysis can be generalized. Normalization of the eigenfunctions is not standard; the most natural, perhaps, is

$$\frac{1}{4\pi} \int_{r=R} \xi \cdot \xi^* \sin \theta d\theta d\phi = 1.$$ 

Except where it is explicitly stated to the contrary, however, I adopt for analytical simplicity a commonly used alternative obtained by replacing $\xi \cdot \xi^*$ in equation (2) by the modulus of the square of the radial component $\xi_r$ of $\xi$.

It is commonly the case that the surface of the star lies in a region where the mode is evanescent. In that case, $I_{nl}$ is not a direct measure of what one would naturally regard as the physical inertia, which should relate, one might
think, the energy in the mode to the motion in the region of propagation, where most of the energy resides. Instead, \( I_{nl} \) is more usefully regarded as a measure of evanescence, and thus, at least if there is just a single region of propagation, it is basically a property of the eigenfunction above the upper turning point. Thus \( I_{nl} \) can be regarded as a diagnostic of the structure of the outer layers of the star.

It is this view that suggests the procedure for estimating the inertia. The star is considered to be divided into two overlapping regions. Region 1 is near the surface of the star; it includes the entire evanescent zone and extends into the region of propagation. Region 2 is basically the region of propagation, but it includes implicitly neighbouring portions of the evanescent zones. Different simple representations of the eigenfunctions are obtained in the two regions, and then they are matched in the common region of validity. The eigenfunction in region 2 reflects the dominant dynamics of the mode, and can be characterized by the dependence of frequency \( \omega \) on degree \( l \), which is observable. Dynamically, the role of region 1 is essentially to present region 2 with an upper boundary condition, and thereby influences somewhat the value of \( \omega \). It also connects the dynamically important region 2 with the surface \( r = R \) where the eigenfunction is normalized.

In the case of \( p \) modes, region 1 is usually thin. Thus, except for modes of very high degree, the horizontal structure of the mode is unimportant; it is merely the frequency provided by the oscillator below that controls the evanescence. The eigenfunction in region 1 can therefore be approximated by that of a radial mode, which provides substantial analytical simplification. When it is adequate to represent the outer layers of the star by a plane-parallel polytrope, of index \( \mu \), yet further simplification ensues. The illustrations below both adopt that representation.

**P-MODE INERTIA**

In region 1, density \( \rho \), pressure \( p \) and adiabatic sound speed \( c \) satisfy \( \rho = \rho_0 z^\mu \), \( p = (\mu + 1)^{-1} \rho_0 z^{\mu+1} \) and \( c^2 = c_0^2 \), where \( z \) is depth below the fiducial point at which \( p, \rho \) and \( c \) formally vanish, and \( c_0^2 = (\mu + 1)^{-1} \gamma g \), in which \( g \) is the gravitational acceleration and \( \gamma \) is the first adiabatic exponent. The vertical component of the displacement eigenfunction is given by

\[
\xi_r = \xi_0 z^{-\mu/2} J_\mu \left( \frac{2\omega \sqrt{z}}{c_0} \right),
\]

where \( J_\mu \) is the Bessel function of the first kind and \( \xi_0 \) is a constant. As \( z \to 0 \),

\[
\xi_r \sim \frac{1}{\Gamma(\mu + 1)} \left( \frac{\omega}{c_0} \right)^\mu \xi_0,
\]

a result that is required for the normalization of \( \xi \). Matching with the eigenfunction in region 2 requires also the asymptotic representation of (3) as \( z \to \infty \), namely

\[
\xi_r \sim \left( \frac{c_0}{\pi \omega} \right)^{\frac{1}{2}} \xi_0 z^{-(\mu+\frac{1}{2})/2} \cos \left[ \frac{2\omega \sqrt{z}}{c_0} - \frac{1}{2} (\mu + \frac{1}{2}) \pi \right] \\
\sim \pi^{-\frac{1}{2}} \omega^{-1} c_0 \rho_0^{\frac{1}{2}} \xi_0 (K/\rho)^{\frac{1}{2}} \cos \left( \int K(r) dr' + \pi/4 \right),
\]

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where \( K(r) \) is the radial component of local wave number, defined by equation (6) below for \( l \neq 0 \) but in which here the \( l \) dependence is neglected, and the integral is evaluated from \( r \) to \( R \). In relating the two expressions on the right-hand side of equations (5) I have equated \( \sqrt{\mu(\mu + 2)} \) with \( \mu + 1 \).

The second of expressions (5) is of the same functional form as the usual high-frequency asymptotic representation in region 2 provided that now the \( l \) dependence of \( K \) is included:

\[
K^2 \sim \frac{\omega^2 - \omega_c^2}{c^2} - \frac{L^2}{r^2},
\]

where \( L = l + \frac{1}{2} \) and \( \omega_c = (c/2H)\sqrt{1 - 2dH/dr} \) is the acoustic cutoff frequency, where \( H \) is the density scale height. The corresponding horizontal displacement eigenfunction is asymptotically

\[
\xi_h \sim \left( \frac{L}{rK} \right) \xi_t \tan \left( \int K(r')dr' + \pi/4 \right).
\]

Moreover, the oscillation eigenfrequency satisfies Duvall’s law:

\[
\frac{\pi(n + \alpha_p)}{L} \sim \mathcal{F}(w) := L^{-1} \int_{r_t}^{R} K(r)dr,
\]

where \( w = \omega/L \) and where \( \omega_c \) has been set to zero in \( K \), its small contribution having been absorbed into the phase \( \alpha_p \), which is a slowly varying function of \( \omega \) and whose value is approximately \( \mu/2 \). The lower limit of integration is given by \( K(r_t) = 0 \). After substituting expressions (5) - (8) into equation (1), and applying the normalization condition at the surface, the constant \( \xi_0 \) cancels and one obtains

\[
I_{nl} \sim \frac{1}{2\pi} \left[ \Gamma(\mu + 1) \right]^2 \rho_0 R^{\mu+3} \left( \frac{\gamma}{\mu + 1} \right)^{\mu+1} \left( \frac{\omega_e}{\omega} \right)^{2(\mu+1)} \frac{d\mathcal{F}}{dw},
\]

where \( \omega_e^2 = GM/R^3 \) in which \( G \) is the gravitational constant and \( M \) is the mass of the star. Thus, the inertia can be represented in terms of the parameters \( \rho_0 \), \( \mu \) and \( \gamma \), which characterize the structure of the upper evanescent layers of the star, and the measurable function \( \mathcal{F}(w) \), which embodies the salient properties of the region of propagation. Note that no polytropic assumption was made for region 2. Note also that because the inertia is a connexion between the two distinct regions, its form is the product of a function of \( \omega \), which describes the degree of evanescece, and a function of \( w \), which measures the energy.

It is of interest to record the effect of using the normalization given by equation (2) rather than requiring that the rms vertical component of displacement be zero. Then, the formula for \( I_{nl} \) given by equation (9) must be divided by \( 1 + u^2 \), where \( u \) is the ratio of the horizontal to the vertical component of the displacement at \( r = R \). If the surface \( r = R \) is regarded to lie in the star’s atmosphere, which is considered both to be isothermal with a constant value \( \gamma_a \) of \( \gamma \) and to be well above the upper turning point of the mode, for high-frequency p modes \( (\omega/\omega_c \gg LH_aR^{-1}, \) where \( H_a \) is the atmospheric scale height) \( u \) is given by

\[
u \sim (\omega_e/\omega_c)^2 \sigma^{-2} \left( 1 - \frac{1}{2} \gamma_a [1 - (1 - \sigma^2)^{\frac{1}{2}}] \right) I,
\]

where \( \sigma = \omega/\omega_c \).
G-MODE INERTIA

Gravity modes depend more sensitively on the details of the stellar stratification, and the polytropic representation of the outer layers of the star has more restricted applicability than it has for p modes. Therefore I shall not present the details of the calculation. Since the dynamics of g modes in region 1 depends directly on \( l \) as well as on \( \omega \), it is necessary to take horizontal variation into account everywhere. The displacement eigenfunction is determined in terms of confluent hypergeometric functions, and needs to be expanded for large and small \( z \) and matched with appropriate asymptotic interior eigenfunctions as in the previous section. For p modes, the outcome is identical to equation (9), as it must be. For g modes, the result is

\[
I_{nl} \sim -\frac{1}{\pi} \left[ \Gamma(\mu + 1) \right]^2 \rho_0 R^{\mu + 3} \beta^{-(\mu + 1)} \left( \frac{w}{\omega_a} \right)^{2\mu - 3} \left( \frac{\omega_a}{\omega} \right)^3 \frac{dG}{d\ln \omega},
\]

where \( \beta = \mu - (\mu + 1)/\gamma \) and where \( G(\omega) \) is the g-mode analogue of \( F(\omega) \), given in terms of the buoyancy frequency \( N \) and satisfying a similar asymptotic equation:

\[
\frac{\pi(n + \alpha_g)}{L} \sim G(\omega) := \int_{r_t}^{R} \left( \frac{N^2}{\omega^2} - 1 \right)^{\frac{1}{2}} d\ln r,
\]

in which \( G \) is related to \( n, l \) and the slowly varying function \( \alpha_g \) (whose value is also approximately \( \mu/2 \)), and which, in principle, is therefore observable. The lower turning point is given by \( N(r_t) = \omega \). Note that although formula (11) is expressed as the product of a function of \( w \) and a function of \( \omega \), physically the separation is not as straightforward as it is in the case of p modes, because evanescence in region 1 depends on both \( w \) and \( \omega \).

Finally, I record that for high-degree f modes,

\[
I_{0l} \sim 8\pi \Gamma(\mu + 1) \rho_0 R^{\mu + 3}(2L)^{-(\mu + 1)}.
\]

COMPLETE POLYTROPIC STELLAR MODELS

A polytrope satisfies \( p = K \rho^{\Gamma} \) with \( \Gamma = 1 + \mu^{-1} \). If \( \rho = \rho_c \theta^\mu \) and \( r = \alpha x \), where \( \rho_c \) is the central density and \( \alpha^2 = (\mu + 1)K\rho_c^{\Gamma - 2}/4\pi G \), then \( \theta(x) \) satisfies the Lane-Emden equation subject to \( \theta = 1 \) at \( x = 0 \); the surface \( x = x_1 \) of a complete polytrope is given by the position of the first zero of \( \theta \). After expanding the solution about \( x = x_1 \) in a power series in \( x_1 - x \), one finds

\[
\rho_0 R^{\mu + 3} = \frac{x_1}{4\pi} \left( -x_1 \frac{d\theta}{dx} \bigg|_{x=x_1} \right)^{\mu-1} M.
\]

The value of this quantity, which occurs in all the expressions for \( I_{nl} \), can be obtained in terms of \( M \) from standard tables of polytropic stellar models. For the complete polytrope \( u = 0 \) for acoustic and internal gravity modes, and therefore formulae (9) and (11) apply also for the normalization given by equation (2); formula (13), however, must be divided by 2 for that normalization.