ON THE DETERMINATION OF THE SOLAR INTERNAL ROTATION

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ABSTRACT The determination of the solar internal rotation remains one of the major goal of helioseismology. Indeed, the inversion of the solar oscillation rotational frequency splittings should provide us with an estimate of the solar rotation as a function of depth and latitude. After describing the mathematical formalism associated with solar rotation inversion, we discuss some key concepts of inverse theory. We then present an inverted solution based on the most recent frequency splitting determinations from the Mt. Wilson 60-Ft Solar Tower. Finally, we discuss some of the problems associated with inverting rotational frequency splittings.

INTRODUCTION

Solar rotation is the dominant process that breaks the Sun’s spherical symmetry and hence lifts the degeneracy of the oscillation eigenfrequencies with respect to the azimuthal order, $m$. The potential of inferring the solar internal rotation from these frequency splittings has been recognized early on (Rhodes et al., 1979), and has remained one of the major goals of helioseismology.

Moreover, our current understanding of the Sun’s internal rotation rate remains puzzling. Indeed, attempts to measure directly the Sun’s surface rotation rate (i.e., by contrast to inferring it from frequency splittings) have led to a plethora of apparently inconsistent values. While spectroscopic measurements have led to a “surface” rate of $452 \pm 2$ nHz; rotation rates determined from magnetic tracers depend on which class of tracers are used (i.e., on the size and age of sunspots) and while a value of 462 nHz is usually quoted, magnetic determinations span the range of 459 to 468 nHz; finally, determinations of the “surface” rate from supergranulation and Doppler correlation have led to an estimate of $473 \pm 0.1$ nHz (see review by Schröter, 1985, and references
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therein). These discrepancies most likely result from the different effective height associated with each tracer, but deriving precisely how the rotation rate varies with depth from such observations is somewhat contrived and ambiguous.

A second motivation to determine the solar internal rotation rate comes from constraints imposed by circulation models (models that attempt to explain the observed differential rotation) and kinematic models (models that are able to represent properties like the magnetic field reversal, the sunspot migration toward the equator, and the butterfly diagram). The first set of models presupposes a rotation rate that decreases with depth and is constant on cylinders (Gilman & Miller, 1986) while the second requires, on the contrary, a rotation rate that increases with depth (Gilman, 1986).

Moreover, stellar rotation evolution models predict a flat rotation rate in the outer layers with a fast rotating core (4 to 15 times the surface; Pinsonneault et al., 1989), while observed rotation rates of subgiants suggest that stellar rotation rate should increase with depth (Endal, 1987).

Finally, a fast rotating core could lead to a high value of the Sun's gravitational quadrupole moment, that would result in a conflictual value with respect to the general relativistic interpretation of Mercury's perihelion motion. While the upper limit associated to the current inferences of the solar rotation is consistent with the relativistic interpretation (Brown et al., 1989), the implication for the resulting gravitational quadrupole moment of any inferred rotational profile having a fast rotating core should be kept in mind.

For all these reasons, the inference of the sun's rotation rate as a function of depth and latitude has mobilized a significant fraction of the efforts in helioseismology, whether in the determination of accurate frequency splittings, or the derivation of the rotation rate from such splittings using forward or inverse approaches.

ROTATIONAL FREQUENCY SPLITTINGS

Formalism
The frequency splitting of a nonradial mode, \( \nu_{n,\ell} \), induced by slow differential rotation, \( \Omega(r, x) \ll 2\pi \nu_{n,\ell} \), is given by (Hansen et al., 1977):

\[
\Delta \nu_{n,\ell,m} = -\frac{m}{2\pi} \int \int K_{n,\ell,m}(r, x) \Omega(r, x) \, dr \, dx
\]

(1)

where the rotational kernels, \( K_{n,\ell,m} \), are given in term of the modal eigenfunctions, which are themselves derived from a resonant acoustic analysis of a given solar model. These kernels are:

\[
K_{n,\ell,m} = \frac{1}{K_{n,\ell}} \left[ \left( -\xi_{r,n,\ell}^2(r) + 2\xi_{r,n,\ell}(r)\xi_{h,n,\ell}(r) \right) P_{\ell,m}^2(x) \right.
\]

\[
\left. + \xi_{h,n,\ell}^2(r) Q_{\ell,m}^2(x) \right] \rho_o(r) \, r^2
\]

(2)

where \( \xi_{r,n,\ell} \) and \( \xi_{h,n,\ell} \) are respectively the radial and horizontal displacement eigenfunctions, \( \rho_o \) is the unperturbed density, the \( P_{\ell,m} \) are associated Legendre
polynomials, \( \theta \) is the co-latitude, \( z = \cos(\theta) \), \( K_{n,\ell} \) is a normalization coefficient, and \( Q_{\ell,m} \) is a function of the co-latitude, defined as:

\[
Q_{\ell,m}^2 = 2P_{\ell,m} \frac{dP_{\ell,m}}{d\theta} \frac{\cos \theta}{\sin \theta} - \left( \frac{dP_{\ell,m}}{d\theta} \right)^2 - \frac{m^2}{\sin^2 \theta} P_{\ell,m}^2
\]  

(3)

To illustrate the way different frequency splittings sample the solar interior, a few rotational kernels are shown in Fig. 1. Notice that as the order of the mode, \( n \), and hence its frequency, \( \nu \), increases or whether the degree of the mode, \( \ell \), decreases, the radial extent of the region sampled by that mode increases. Note also that it is only when the mode is nearly zonal \((m \ll \ell)\) that the mode samples the polar regions. Therefore, the information on the rotation rate in


Fig. 1. Rotational kernels, \( K_{n,\ell,m}(r, x) \), for a set of \((n, \ell, m)\). Note how the region sampled by the modes varies, confining the information on the deeper or higher latitude regions to low-\( \ell \) or low-\( m \) modes.

the deeper regions is confined to splittings of low-degree modes, while the information on the rotation rate at high latitude is confined to splittings of low-\( m \) modes. Since the observables, i.e. the frequency splittings, are all proportional to \( m \), which is itself never greater in magnitude than \( \ell \), they remain small for both cases (low-\( m \), or low-\( \ell \)), and hence have larger relative uncertainties.
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Semi-Parameterized Formulation
Since the rotational kernels are the sum of products of functions that each depend only on the depth or the latitude, we can reduce the two-dimensional inversion problem to a set of one-dimensional problems. Indeed, if we expand the latitudinal dependence of the rotation rate as a superposition of analytical functions, i.e.:

$$\Omega(r, z) = \sum_{s=0}^{s_{\text{max}}} R_s(r) T_s(z)$$  \hspace{1cm} (4)

the angular integral in Eq. 1 can be computed analytically.

Historically, observers have elected to parameterized the frequency splittings as a polynomial expansion in $m/L$, i.e.:

$$\Delta \nu_{n, \ell, m} = L \sum_{i=1}^{i_{\text{max}}} a_{i}^{n,\ell} P_i\left(-\frac{m}{L}\right)$$  \hspace{1cm} (5)

where $L^2 = \ell(\ell + 1)$, and $P_i$ are Legendre polynomials.

It can be shown (Brown et al., 1989) that, in the high-$\ell$ limit and if the angular functions $T_s$ are even polynomials in $x$ of degree $s$, the truncated expansions in Eqs. 4 and 5, with $s_{\text{max}} = i_{\text{max}} - 1$, lead to a triangular set of one-dimensional problems, i.e.:

$$a_i^{n,\ell} = \sum_{p=i}^{i_{\text{max}}} \int K^{(r,p)}_{n,\ell}(r) R_{p-1}(r) dr \quad \text{for odd } i = 1, \ldots, i_{\text{max}}$$  \hspace{1cm} (6)

where $K^{(r,p)}_{n,\ell}(r)$ are the resulting one-dimensional kernels. This triangular problem can be solved by elimination starting with $i = i_{\text{max}}$.

On the other hand, Ritzwoller and Lavelle (1991) have shown that if we select to expand the latitudinal dependence in terms of derivative of Legendre polynomial, i.e.:

$$T_s = \frac{dP_{s,0}}{dx}$$  \hspace{1cm} (7)

and if we expand the rotational splittings in terms of normalized Clebsch-Gordon coefficients, i.e.:

$$\Delta \nu_{n, \ell, m} = \sum_{s=1}^{s_{\text{max}}} b_s^{n,\ell} \beta_{s,m}^{\ell}$$  \hspace{1cm} (8)

the corresponding one-dimensional problems are decoupled, i.e.:

$$b_s^{n,\ell} = \int K^{(r,s)}_{n,\ell}(r) R_s(r) dr \quad \text{for odd } s = 1, \ldots, s_{\text{max}}$$  \hspace{1cm} (9)

where $\beta_{s,m}^{\ell}$ are normalized Clebsh-Gordon coefficients, while $K^{(r,s)}_{n,\ell}$ are the resulting one-dimensional kernels for this parameterization. In this case, each term of the expansion corresponds to an independent one-dimensional inverse problem.

Ritzwoller and Lavelle have shown that this expansion corresponds to identifying the differential rotation with the odd-degree, zonal part of the toroidal component of a general stationary laminar velocity flow field.

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Inverse theory in the helioseismic framework has been extensively discussed elsewhere (see for instance Christensen-Dalsgaard et al., 1990, and the references therein). In brief, the fundamental problems in any inverse problem are: that a) the set of observables is finite and the observables are known to a finite precision; and that b) the set of kernels associated with the available set of observables does not completely sample the domain of the unknown function.

It follows that the inverse problem is ill-posed, namely that a) the existence of the solution is not guaranteed, since there may not be a solution that satisfies exactly the set of integral equations; and that b) the uniqueness of the solution is not guaranteed, since there may be a family of functions whose contribution to the finite set of integral equations is null.

In a more abstract formulation, we can define the inverse problem as

$$y_k \pm \sigma(y_k) = \int \int K_k(r, z) x(r, z) \, dr \, dz \quad \text{for } k = 1, \ldots, N$$

where \(y_k\) represents the observables, and \(\sigma(y_k)\) their uncertainty, while \(K_k(r, z)\) represents the kernels and \(x(r, z)\) the unknown function.

To solve such a linear inverse problem, most methods construct an inverse operator \(H\), and obtain an estimate of the solution, \(\hat{x}\), at a target location \((r_0, z_0)\), from a linear combination of the observables, \(y_k\), i.e.:

$$\hat{x}(r_0, z_0) = \sum_{k=1}^{N} H_{k}^{(r_0, z_0)} y_k$$

From Eqs. 10 and 11, it follows that:

$$\hat{x}(r_0, z_0) = \int \left( \sum_{k} H_{k}^{(r_0, z_0)} K_k(r, z) \right) x(r, z) \, dr \, dz$$

$$= \int A(r, z; r_0, z_0) x(r, z) \, dr \, dz$$

where \(A(r, z; r_0, z_0)\) is the resolution kernel at the target location \((r_0, z_0)\).

The solution \(\hat{x}\) will be a good estimate of the solution at, or near \((r_0, z_0)\) if:

1. the solution represents the observables, i.e.:

$$\left| y_k - \int K_k(r, z) \hat{x}(r, z) \, dr \, dz \right| \lesssim \sigma(y_k) \quad \text{for all } k = 1, \ldots, N$$

2. the resolution kernel is a narrow function localized around \((r_0, z_0)\), i.e.:

$$A(r, z; r_0, z_0) \approx \delta(r - r_0) \delta(z - z_0)$$

3. the error magnification remains small, i.e.:

$$\sigma^2(\hat{x}) = \sum_{k} H_k^2 \sigma^2(y_k) \ll 1$$
While an estimate of the solution at any target location can always be computed, it is important to check whether the corresponding resolution kernel is localized near that target location.

In a semi-parametrized approach, since the estimate of the solution at any location is effectively a linear combination of the observables, a two-dimensional resolution kernel is defined. Such a two-dimensional resolution kernel is computed from a superposition of the the one-dimensional radial resolution kernels, whose explicit formulation is specific to the latitudinal parametrization that was chosen.

Fig. 2. Scaled frequency splittings, $\Delta \nu_{n,\ell,m}/\ell$, as a function of the ratio $m/\ell$ for a small subset of modes. The effect of a somewhat arbitrary solid body rotation has been subtracted to illustrate the effect of differential rotation.

MWO OBSERVATIONS

The inversion results presented here are based on frequency splittings derived from time series of full-disk dopplergrams from the 60-Ft Solar Tower of the Mt. Wilson Observatory. The dopplergrams were computed from a pair of images formed in the red and blue wing of the Na D-lines with a Cacciani two-cell magneto-optical filter. The images were recorded at a cadence of one a minute, by a JPL-built, fast readout, 1024 by 1024 pixel CCD camera (see
Rhodes et al., 1988 for a more detailed description). Daily observing sequences of up to 11 hours were taken during the summer of 1990, leading to some 110 days of useful data covering a 140-day long stretch.

In order to ease the management of such a large data set (the raw data set amounts to some 200 GB), we have broken the 140-day-long stretch into seven 20-day-long subsets. Each subset was separately Fourier transformed, using a 64k-point-long FFT. Then, for each \((\ell, m)\), we summed the seven separate power spectra together, generating in this way a spectrum with the same intrinsic frequency resolution \((0.57 \mu\text{Hz})\) as each individual subset had, but with a considerably improved signal-to-noise ratio. From that set of spectra, we have computed frequency splittings for each even-\(m\), resulting in some 613,000 splittings which correspond to some 4,400 modes of degree ranging between 10 and 332. Figure 2 illustrates the resulting splittings for a small, arbitrarily chosen, subset of modes.

INVERSION RESULTS

Method
The individual frequency splittings were fitted to an expansion in terms of normalized Clebsch-Gordon coefficients. For each of the 4400 modes \((i.e., each n, \ell)\), we have fitted as many coefficients as were significant, increasing the number of coefficients until the variance of the problem was minimized. This procedure lead to an expansion truncated at \(s = 21\) for the high-\(\ell\) modes \((i.e., 11\) odd terms). For the modes for which the expansion was truncated at a lower index, the coefficients corresponding to the higher order terms were set to zero and were given an uncertainty corresponding to the mean uncertainty for the corresponding ratio of \(\nu/\ell\). (A power law was fit in the least-squares sense to extrapolate the uncertainties at some \(\nu/\ell\) ratios.)

The resulting 11 one-dimensional inverse problems were solved using a regularized least-squares method, using a smoothness of the second derivative as the regularization constraint (Korzennik, 1990). Since in our parameterization the set of one-dimensional inverse problems is decoupled, and because the inversion error magnification was too large for the higher terms of the expansion, only the first five terms of the expansion have been kept in the inverted two-dimensional rotation rate presented here.

Equatorial Rotation Rate

Figure 3 shows the resulting equatorial rotation rate as a function of depth. Note that with the current low-\(\ell\) accuracy, we cannot resolve the rotation rate deeper than 0.6 of the solar radius. This is illustrated by the one-dimensional resolution kernels plotted in Fig. 4 which correspond to the first term of the latitudinal expansion. This is mainly due to our deliberate choice of limiting the frequency resolution of the power spectra by breaking the four-month-long time series in seven twenty-day-long subsets. We expect shortly to combine the complete time baseline into one long time series to increase the frequency resolution of our data set by a factor of seven.

The resulting equatorial rotation rate presents a steep increase with radius near \(r/R_\odot = 0.7\), suggesting the possibility of a discontinuity near the base of the convection zone. (Since the resulting resolution of the inversion is finite, an
actual discontinuity cannot be resolved.) Notice also that the equatorial rotation rate peaks near \( r/R_\odot = 0.9 \), then decreases with radius in the outer 10%.

![Graph showing the equatorial rotation rate](image)

**Fig. 3.** Equatorial rotation rate inverted from the Mt. Wilson 1990 frequency splittings. The error bars are formal 3-\( \sigma \) error on the smooth solution. Horizontal dashed lines represent "surface" rotation rate determinations as described in the introduction.

![Graphs showing one-dimensional resolution kernels](image)

**Fig. 4.** One-dimensional resolution kernels corresponding to the first term of the latitudinal expansion. In each panel, the vertical dashed line represents the location of the target radius.
Fig. 5  Inverted rotation rate for four selected latitudes. The error bars are formal 3-σ errors on the smooth solution.

**Latitudinal Dependence**

Figures 5 and 6 show the latitudinal dependence of the inverted profile. In the outer 30% of the solar radius, for latitude < 30°, the rotation rate is nearly constant on cylinders, due to a rapidly rotating “belt” centered near \( r/R_\odot = 0.9 \). In this belt the rotational flow is some 35 km/s faster than the flow speed that would correspond to the surface rate. At higher latitudes, the rotation rate is constant on cones. The differential character of the rotation disappears below a depth that corresponds to the base of the convection zone.

Figure 7 presents full two-dimensional resolution kernels. These important diagnostic plots indicate that we fail to resolve the rotation rate at high latitude (\( i.e., > 60° \)). Indeed, the resolution kernels are not localized, nor can they be localized at these high latitudes by simply adjusting the trade-off between resolution and error magnification. The inverted results for these high latitudes must therefore be discarded (as indicated by the regions which have been cross-hatched in Fig. 6).
Fig. 6  Inverted rotation rate as a function of depth and latitude. The inversion failed to produce localized resolution kernels for regions deeper that 0.6R☉, and at latitudes above 60°.

Comparison with Previous Results

The inverted profile presented here agrees qualitatively with previously published inverted profiles (Goode et al., 1991; Korzennik et al., 1990; Thompson, 1990; Brown et al., 1989). Indeed, a steep increase with radius at a location near the base of the convection zone seems to be a consistent feature in all inverted profiles. Similarly, the rotation rate appears to be constant on cones, for intermediate latitudes (30 to 60°). Finally, these results are consistent with the disappearance of differential rotation below the convection zone.

On the other hand, the inclusion of high-degree modes in inversions based on Mt. Wilson splittings allows us to extend the the inverted profile closer to the surface (see Korzennik, 1990). With the inclusion of high-degree modes with their azimuthal information in the inversion presented here, we are now able to determine the latitudinal extent of the fast rotating “belt” located close to the surface. The presence of this “belt” suggests that the rotation is constant on
cylinders for latitudes below 30°. It is tempting to point out that the transition between rotation constant on cylinders to rotation constant on cones occurs at the “active” latitudes.

Fig. 7 Two-dimensional resolution kernels corresponding to a selected set of target depths and latitudes. The respective target values of \((r, \theta)\) are indicated by the crosses.

CONCLUSIONS

As ground-based observations produce larger and larger frequency splittings data sets, the semi-parametrized approach, using the Ritzwoller & Lavely formalism, has, over full two-dimensional inversion, the distinct advantage of reducing significantly the size of the inversion problem. (In the case presented here, a 613,000 by 50 by 100 problem was reduced to eleven independent problems, each 4,400 by 100.)

Rather than selecting a-priori the number of terms kept in the frequency splitting parametrization, we have fit as many significant terms as required. While any full two-dimensional inversion method is not hampered by the choice of a given latitudinal parametrization, it can be shown that such a method will not be able to extract more latitudinal information. Indeed, while the high latitude information is contained in the low-\(m\) splittings, the latitudinal resolution is contained in the higher terms of the frequency splittings expansion. (Consider the forward problem of expanding a rotation law with “sharp” features). Since the one-dimensional inverse problems associated with each expansion term are decoupled, latitudinal resolution is improved by adding more terms for which a good inversion can be performed. Thus, to improve our latitudinal coverage and resolution, we need to improve the accuracy of the frequency splittings, hence the accuracy of the higher expansion terms.

As mentioned above, we plan to combine the complete four-months-long time baseline into a single time series in the near future. We should then be able
to improve the frequency resolution, and hence the accuracy of the determination of our low-\(l\) and low-\(m\) frequency splittings.

ACKNOWLEDGMENTS

The work described above is supported by NASA Grants NAGW-2893 and by Stanford University Subcontract \#6333 to SAO; by NASA grant NAGW-13, by Stanford University Subcontract \#6914, and NSF grant ATM-9119617 to USC, and by GSFC Award \#418-00-00-06-33 to JPL, and by GSFC HPCC Program GCI Award to SAO and USC.

Part of the computation carried out for this work was performed on JPL's Cray YMP supercomputer and on the Intel Touchstone Delta System operated by Caltech on behalf of the Concurrent Supercomputing Consortium. Access to the Delta was provided by NASA.

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