IV. Total Resonant Absorption and MHD Radiating Eigenmodes

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Abstract. Resonant absorption of MHD waves on a nonuniform flux tube is investigated as a driven problem for a 1D cylindrical equilibrium. The variation of the fractional absorption is studied as a function of the frequency and its relation to the eigenvalue problem of the MHD radiating eigenmodes of the nonuniform flux tube is established. The optimal frequencies producing maximal fractional absorption are determined and the condition for total absorption is obtained. This condition defines an impedance matching and is fulfilled for an equilibrium that is fine tuned with respect to the incoming wave. The variation of the spatial wave solutions with respect to the frequency is explained as due to the variation of the real and imaginary parts of the dispersion relation of the MHD radiating eigenmodes with respect to the real driving frequency.

1. Introduction

Resonant absorption is an efficient means for dissipating MHD waves in nonuniform plasmas. The physical basis of this dissipation mechanism can be understood in linear ideal MHD. The linear ideal MHD spectrum of a nonuniform plasma contains an Alfvén continuum and a slow continuum in addition to discrete eigenfrequencies. The spatial solutions that correspond to these Alfvén or slow continuum frequencies are non-square integrable (see, e.g., Goedbloed, 1983). An MHD wave with a frequency within the Alfvén continuum that impinges on the flux tube excites an Alfvén wave on the magnetic surface where the local Alfvén frequency matches the frequency of the incoming wave. In ideal MHD the amplitude of the Alfvén wave grows unbounded, but non-ideal effects cause dissipation of the Alfvén wave energy and local heating of the plasma.

Resonant absorption was first studied as a means for the supplementary heating of fusion plasmas (see, e.g., Tataronis and Grossmann, 1973; Grossmann and Tataronis, 1973; Hasegawa and Chen, 1974) and subsequently proposed as a mechanism for heating magnetic loops in the solar corona by Ionsom (1978). Since the original suggestion, resonant absorption has remained a popular mechanism for explaining the heating of the solar corona (see, e.g., Hollweg, 1990a). Recently, resonant absorption has been considered as a possible explanation of the observed loss of power of acoustic oscillations in sunspots (see, e.g., Bogdan, 1992, for a review of this subject).
Detailed results based on large-scale numerical simulations of the stationary state of resonant absorption in coronal loops were obtained by Grossmann and Smith (1988) in ideal MHD and by Poedts, Goossens, and Kerner (1989, 1990a, b) in resistive MHD. The aim of these simulations was to understand how the efficiency of resonant absorption depends on the equilibrium and on the incoming wave. Time-dependent computations of resonant absorption in magnetic loops were reported by Poedts, Goossens, and Kerner (1990c) and Poedts and Kerner (1992). Goossens and Poedts (1992) used their numerical code which initially was devised for computing the heating of magnetic loops in the solar corona to study resonant absorption of $p$-mode oscillations in sunspots. They verified the results that Lou (1990) had obtained by numerical integration of the viscous MHD equations and carried out an extensive survey of the relevant parameter space. They included twist in Lou’s equilibrium model and found very efficient absorption.

The numerical simulations by Poedts, Goossens, and Kerner (1989, 1990a, b) for magnetic loops in the solar corona show that the fractional absorption spectrum is often characterized by well-defined maxima and that for specific frequencies 100% absorption occurs. In addition there is a typical change in the spatial behaviour of the solutions for driving frequencies in an interval around the optimal frequency producing total absorption. The 100% absorption and the pronounced change in the spatial behaviour of the solutions were attributed to the presence of a so-called quasi-modus in the Alfvén continuum. The precise relation between this quasi-modus and total absorption was not explained and the surprising result of total absorption has not yet fully been understood. Goossens and Poedts (1992) found an absorption of up to 70% of the incident wave energy in their study of the interaction of $p$-mode oscillations with sunspots in contrast to 100% absorption for magnetic loops. The reason for this difference lies in the fact that Poedts, Goossens, and Kerner (1990b) carried out a systematic scan of the Alfvén continuum and in doing so found quasi-modes in the continuum while Goossens and Poedts (1992) considered observed values of frequencies and conservative values for radius and horizontal wave number. Recent computations by Stenuit, Goossens, and Poedts (1992) have shown that total absorption also occurs for $p$-mode oscillations in sunspots.

The aim of the present paper is to identify the conditions for total absorption and to explain the change in the spatial behaviour of the solutions for frequencies around the optimal frequency. The key tools are the procedure set up by Sakurai, Goossens, and Hollweg (1990a) for computing resonant absorption without solving the dissipative MHD equations and the correspondence between the eigenvalue-problem of MHD radiating eigenmodes in a nonuniform flux tube and resonant absorption.

The paper is organized as follows. In Section 2 we recall the basic procedure for computing resonant absorption with jump conditions and ideal MHD equations and also discuss the solutions in the uniform plasma surrounding the flux tube. In Section 3 we specialize on a flux tube with a ‘thin’ nonuniform layer in an attempt to make analytical progress. The optimal frequencies that produce maximal fractional absorption are found and the condition for total absorption is determined. In Section 4 we discuss
how the spatial solutions depend on the driving frequency and explain the change in the spatial solutions around the optimal frequency. In Section 5 we explain how the method of Section 3 can be generalized to a ‘thick’ nonuniform flux tube. Conclusions are summarized in Section 6.

2. Basic Equations and the SGH Procedure

A nonuniform flux tube is embedded in a uniform plasma which may be either magnetic or non-magnetic. In case of the absorption of \( p \)-mode oscillations in sunspots the uniform plasma is non-magnetic. In this uniform plasma there is a wave with given wave numbers and frequency that impinges on the nonuniform magnetic flux tube. The frequency of the wave lies within the Alfvén continuum of the nonuniform magnetic flux tube. Part of the energy of the wave will be absorbed in the flux tube and the amplitudes of the incoming and outgoing waves will differ.

In ideal MHD this driven problem is governed by a set of differential equations that are singular at the point where the local Alfvén frequency equals the frequency of the wave. The spatial solutions are non-square integrable, and ideal MHD cannot be used to describe the resonant behaviour of the wave. The singularity is removed by including non-ideal effects such as viscosity or resistivity but at the same time the order of the system of differential equations is raised. The non-ideal effects are only important in a narrow dissipative layer around the ideal MHD singular point. The thickness of this dissipative layer scales as \( \left( (v + \eta) |\omega'_A| \right)^{1/3} \), where \( v \) and \( \eta \) measure viscosity and electric resistivity, \( \omega_A(r) \) is the local Alfvén frequency and an accent denotes the derivative with respect to the radial coordinate. Outside this narrow dissipative layer, the behaviour of the wave is correctly described by the equations of ideal MHD. Non-ideal MHD is only necessary to treat the dissipative layer. From the viewpoint of the global behaviour of the solution, the solution in the dissipative layer only gives a means to connect the ideal MHD solutions across the dissipative layer. As a consequence, once these connection formulae are found, it is no longer necessary to solve the non-ideal MHD equations, unless of course the behaviour in the dissipative layer itself is the topic of interest. Sakurai, Goossens, and Hollweg (1991a) determined conserved quantities across the dissipative layer and connection formulae or jump conditions connecting the ideal MHD solutions to the left and the right of the dissipative layer. These connection formulae make it possible to determine the absorption of wave energy without solving the non-ideal MHD equations. This procedure was used by Hollweg (1988) as it were ‘avant la lettre’ in his study of resonant absorption of acoustic oscillations in sunspots. Hollweg considered a Cartesian slab geometry with a ‘thin’ nonuniform layer. The assumption that the Eulerian perturbation of the total pressure is a conserved quantity across the dissipative layer enabled Hollweg to obtain analytic expressions for the absorption coefficients. Sakurai, Goossens, and Hollweg (1991a) showed that for an equilibrium with a straight magnetic field the Eulerian perturbation of the total pressure is indeed the conserved quantity putting Hollweg’s assumption on a firm basis. Sakurai, Goossens, and Hollweg (1991b) used this procedure to study the
absorption of acoustic oscillations in sunspots for cylindrical flux tubes with straight magnetic fields. They considered a very simple equilibrium model and treated both ‘thin’ and ‘thick’ nonuniform layers. The conserved quantities and jump conditions are adequate means for studying resonant absorption. They enable us to understand the behaviour of the solutions across the dissipative layer found in the numerical solutions (see, e.g., Goossens, 1991; Goossens and Poedts, 1992). The strength of the method has been further underlined by the recent recovery of the results of Goossens and Poedts (1992) with the present jump conditions method (Stenuit, Debooscher, and Goossens, 1992). The Sakurai, Goossens, and Hollweg method (hereafter referred to as the SGH method) that uses jump conditions and integration of the ideal MHD equations to determine resonant absorption of MHD waves is illustrated in Figure 1.

There are four regions to consider. Region I is the uniform plasma that surrounds the magnetic flux tube. Region I may be non-magnetic as for example in the sunspot problem or magnetic as for example for a coronal loop. The nonuniform magnetic flux tube is divided in three regions. Region III is the thin dissipative layer around the ideal MHD singular point. Regions II and IV are, respectively, to the left and to the right of region III and contain the bulk of the magnetic flux tube. The MHD waves are correctly described by the equations of ideal MHD in regions I, II, and IV. In the uniform region I there is even an analytical solution for the MHD wave. Let us denote region III as \((r_A - \delta, r_A + \delta)\) where \(r_A\) is the position of the ideal MHD singularity and \(\delta/R \ll 1\) with \(R\) the radius of the flux tube. At \(r = 0\) and close to the origin the regular solution of the ideal MHD equations can be obtained in terms of a MacLaurin expansion. Numerical integration of the ideal MHD equations from \(r = r_1/r \ll 1\), where the solution is known as a power series, up to \(r = r_A - \delta\) gives the solution in

![Fig. 1](image)

Fig. 1. Schematic illustration of the SGH method. Ideal MHD is valid in regions I, II, and IV. Region III is the dissipative layer around the resonant point. Region I contains a uniform plasma with analytic solutions for the MHD waves.

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region IV. The connection formulae enable us to obtain the solution in \( r = r_A + \delta \) from its known value in \( r = r_A - \delta \). Numerical integration of the ideal MHD equations from \( r = r_A + \delta \) up to \( r = R \) completes the solution in the non-uniform flux tube. At \( r = R \) this solution has to be continuous with the analytic solution in region I.

The equations and conditions to use are the ideal MHD equations and the jump conditions and conserved quantities obtained by Sakurai, Goossens, and Hollweg (1991a). The ideal MHD equations for a wave in a cylindrical 1D magnetostatic equilibrium are (see, e.g., Sakurai, Goossens, and Hollweg, 1991a):

\[
D \frac{d}{dr} (r \xi_r) = C_1 r \xi_r - C_2 r P', \tag{1}
\]

\[
D \frac{dP'}{dr} = C_3 \xi_r - C_1 P'. \tag{2}
\]

In these equations \( \xi_r \) is the radial component of the Lagrangian displacement and \( P' \) is the Eulerian perturbation of total pressure \( (P' = p' + B \cdot B' / \mu, p \) is the plasma pressure, \( B \) is the magnetic field and an accent denotes the Eulerian perturbation). Equations (1)–(2) are obtained after the dependency on the ignorable spatial coordinates, \( \varphi, z \), and time, \( t \), has been factored out as

\[
\exp \{ i(m \varphi + kz) \} \exp(-i\omega t).
\]

In what follows we adopt the convention that \( \omega > 0 \) or at least \( \Re \omega > 0 \). The coefficient functions \( D, C_1, C_2, \) and \( C_3 \) are given by

\[
D = \rho (c^2 + \nu_A^2) (\omega^2 - \omega_A^2) (\omega^2 - \omega_C^2), \tag{3}
\]

\[
C_1 = 2 \omega^4 \frac{B_{\varphi}^2}{\mu r} - 2 m \frac{f_B B_{\varphi}}{\mu r^2} (c^2 + \nu_A^2) (\omega^2 - \omega_C^2), \tag{4}
\]

\[
C_2 = \omega^4 - (c^2 + \nu_A^2) \left( \frac{m^2}{r^2} + k^2 \right) (\omega^2 - \omega_C^2), \tag{5}
\]

\[
C_3 = D \left[ \rho (\omega^2 - \omega_A^2) + 2 \frac{B_{\varphi}}{\mu} \frac{d}{dr} \left( \frac{B_{\varphi}}{r} \right) \right] + 4 \left( \frac{B_{\varphi}^2}{\mu r} \right)^2 \omega^4 - 4 \rho (c^2 + \nu_A^2) (\omega^2 - \omega_C^2) \omega_A^2 \frac{B_{\varphi}^2}{\mu r^2}. \tag{6}
\]

Here \( c, \nu_A, \omega_A, \) and \( \omega_C \) denote the adiabatic speed of sound, the Alfvén speed, the Alfvén frequency, and the cusp frequency. The squares of these quantities are defined as

\[
c^2 = \gamma p / \rho, \quad \nu_A^2 = B^2 / \mu \rho, \quad \omega_A^2 = f_B^2 / \mu \rho, \quad \omega_C^2 = c^2 \omega_A^2 / (c^2 + \nu_A^2),
\]
with
\[ f_B = kB_z + mB_\phi / r. \] (7)

Of course \( \rho \) is density and \( P \) is the plasma pressure; \( k \) and \( m \) are the axial and the azimuthal wave numbers.

Equations (1)–(2) define an eigenvalue problem with \( \omega^2 \) as eigenvalue parameter when they are supplemented with proper boundary conditions. In a driven problem the frequency \( \omega \) is prescribed and real. Equations (1)–(2) have regular singular points at the zeros of the coefficient function \( D \). From Equation (3) there are mobile regular singularities at positions \( r \) where
\[ \omega^2 = \omega_\lambda^2(r), \] (8a)
or
\[ \omega^2 = \omega_\xi^2(r). \] (8b)

Equation (8a) defines the Alfvén resonance point, while Equation (8b) defines the cusp resonance point. Since \( \omega_\lambda^2(r) \) and \( \omega_\xi^2(r) \) are functions of position, Equations (8a) and (8b) define, from the viewpoint of the spectrum of ideal MHD, two continuous ranges in the spectrum which are classically referred to as the Alfvén continuum and the slow continuum. In the driven problem these two continua are ranges of resonant frequencies.

Equations (1)–(2) are not valid in the dissipative layer and the ideal solutions to the left and to the right of this dissipative layer are linked by the connection formulae (see Sakurai, Goossens, and Hollweg, 1991a)
\[ \left[ \xi_r \right] = -i\pi \frac{\text{sign } \omega}{|A|} \frac{g_B}{\rho B^2} C_A, \] (9)
\[ \left[ P' \right] = -i\pi \frac{\text{sign } \omega}{|A|} \frac{2B_\phi B_z f_B}{\rho B^2 \mu r} C_A, \] (10)

where \( C_A \) is the conserved quantity across the dissipative layer which is given by
\[ C_A = g_B P' - 2f_B B_\phi B_z \xi r / \mu r \] (11)

and
\[ g_B = mB_z / r - kB_\phi, \quad \Delta = -\frac{d\omega_\lambda^2}{dr}. \] (12)

In the present paper we consider a flux tube with a straight magnetic field, i.e., \( B_\phi = 0 \). The coefficient functions simplify and in particular
\[ C_1 = 0, \quad C_3 = D\rho (\omega^2 - \omega_\lambda^2). \] (13)

Equations (1)–(2) take the simple form
\[ D \frac{d}{dr} \left( r \xi_r \right) = -C_z r P', \quad (14) \]
\[ \frac{dP'}{dr} = \rho (\omega^2 - \omega^2_\lambda) \xi_r. \quad (15) \]

The connection formulae (9)–(11) take the simple form
\[ [\xi_r] = -i\pi \frac{\text{sign } \omega}{|A|} \frac{m^2}{\rho(r_A)r_A^2} P', \quad (16) \]
\[ [P'] = 0, \quad (17) \]
\[ C_A = \frac{m}{r_A} B_z P'. \quad (18) \]

In this case the Eulerian perturbation of total pressure is the conserved quantity across the dissipative layer.

Let us now rewrite the set of two first-order differential equations (14)–(15) as a second-order differential equation for \( P' \):
\[ \frac{d}{dr} \left\{ \frac{r}{\rho(\omega^2 - \omega^2_\lambda)} \frac{dP'}{dr} \right\} + \left\{ \frac{\omega^2 - k^2c^2}{\rho(c^2 + v^2_\lambda)(\omega^2 - \omega^2_\lambda)} - \frac{m^2/r^2}{\rho(\omega^2 - \omega^2_\lambda)} \right\} P' = 0. \quad (19) \]

For a uniform plasma this equation can be rewritten as a Bessel equation:
\[ \frac{d^2P'}{dr^2} + \frac{1}{r} \frac{dP'}{dr} + \left\{ \Gamma(\omega^2) - \frac{m^2}{r^2} \right\} P' = 0, \quad (20) \]
where
\[ \Gamma(\omega^2) = \frac{(\omega^2 - k^2c^2)(\omega^2 - \omega^2_\lambda)}{(c^2 + v^2_\lambda)(\omega^2 - \omega^2_\lambda)} \quad (21) \]
and
\[ \xi_r = \frac{1}{\rho(\omega^2 - \omega^2_\lambda)} \frac{dP'}{dr}. \quad (22) \]

Depending of the sign of \( \Gamma(\omega^2) \) the solutions to Equation (20) are \( J_{|m|} \) and \( Y_{|m|} \) (or \( H^{(1)}_{|m|} \) and \( H^{(2)}_{|m|} \) \( \Gamma(\omega^2) > 0 \)) or \( I_{|m|} \) and \( K_{|m|} \) \( \Gamma(\omega^2) < 0 \). Since we are concerned here with propagating waves in the uniform plasma we assume that \( \Gamma(\omega^2) > 0 \) there and we put
\[ \Gamma(\omega^2) = K^2_z. \quad (23) \]

\( K_z \) can be viewed as the horizontal wave number outside the nonuniform flux tube. In case the uniform plasma is non-magnetic the expression for \( \Gamma(\omega^2) \) reduces to
\begin{equation}
\Gamma(\omega^2) = (\omega^2 - k^2c^2)/c^2.
\end{equation}

The waves are then propagating if \( \omega^2 > k^2c^2 \) and are propagating acoustic waves.

Whether the uniform plasma is non-magnetic or magnetic the waves can be written in terms of the Bessel functions of the first kind:

\begin{equation}
P' = AJ_{|m|}(K_e r) + BY_{|m|}(K_e r),
\end{equation}

\begin{equation}
\xi_r = \frac{K_e}{\rho_e(\omega^2 - \omega_{he}^2)} \{ AJ_{|m|}(K_e r) + BY_{|m|}(K_e r) \},
\end{equation}

or in terms of the Hankel functions as

\begin{equation}
P' = QH_{|m|}^{(1)}(K_e r) + DH_{|m|}^{(2)}(K_e r),
\end{equation}

\begin{equation}
\xi_r = \frac{K_e}{\rho_e(\omega^2 - \omega_{he}^2)} \{ QH_{|m|}^{(1)}(K_e r) + DH_{|m|}^{(2)}(K_e r) \}.
\end{equation}

An accent on a Bessel function denotes the derivative with respect to the argument. In Equations (25)–(28) \( A, B, Q, \) and \( D \) are constants yet to be determined. They are not independent but related by

\begin{equation}
A = Q + D, \quad B = i(Q - D),
\end{equation}

\begin{equation}
Q = \frac{1}{2}(A - iB), \quad D = \frac{1}{2}(A + iB).
\end{equation}

The formulation with the Hankel functions is very convenient as it enables us to make an easy identification of incoming and outgoing waves. With the classic convention that Re \( \omega > 0 \), the time dependency exp \((-i\omega t)\) implies that \( H_{|m|} \) corresponds to an incoming wave and \( H_{|m|}^{(1)} \) to an outgoing wave, at least for waves that have phase and group velocities in the same direction (see Spruit, 1982, for possible complications with slow waves).

3. Driven Problem for a Thin Nonuniform Layer

In order to make further analytical progress we consider a thin nonuniform layer (of thickness \( 2a \)) which by assumption coincides with the dissipative layer. This is not a very realistic assumption. It is only made to avoid numerical integration of the differential equations (14)–(15) of ideal MHD and to illustrate the physical principles in the simplest possible mathematical way. The equilibrium is illustrated in Figure 2.

In the thin nonuniform layer the equilibrium quantities vary from their constant values inside the flux tube to their constant values outside the flux tube. The original region II ceases to exist and we use subscripts \( i \) and \( e \) to denote quantities in the flux tube and exterior to the flux tube. The aim is to study the resonant behaviour of a wave with a frequency in the Alfvén continuum and that is propagating in the exterior uniform plasma. We have different types of wave behaviour depending on the values of
the characteristic frequencies $\omega_{\text{Ai}}^2$, $\omega_{\text{Ci}}^2$, and $k^2c^2$ in the exterior and interior plasmas. We consider the case that $\omega_{\text{Ai}}^2 < \omega_{\text{Ai}}^2$, so that an Alfvén resonance occurs if

$$\omega_{\text{Ai}}^2 < \omega^2 < \omega_{\text{Ai}}^2.$$  \hspace{1cm} (30)

Since $\omega_{\text{Ci}}^2 < \omega_{\text{Ai}}^2$, the wave is propagating in the exterior plasma if

$$\omega^2 > k^2c_i^2.$$  \hspace{1cm} (31)

As we do not wish to treat cases where we have a slow resonance in addition to an Alfvén resonance we assume that

$$\omega^2 > \omega_{\text{Ci}}^2.$$  \hspace{1cm} (32)

In addition we assume that

$$\omega^2 > k^2c_i^2,$$

so that $\Gamma(\omega^2) < 0$ in the interior plasma and we can write

$$\Gamma(\omega^2) = -K_i^2.$$  \hspace{1cm} (34)

The solution to Equation (20) that is finite on the axis is then

$$P'(r) = CI_{m_1}(K_i r),$$  \hspace{1cm} (35)

$$\xi'(r) = C \frac{K_i}{\rho_i(\omega^2 - \omega_{\text{Ai}}^2)} I_{m_1}(K_i r).$$  \hspace{1cm} (36)

Solutions (35)–(36) in the flux tube and solutions (25)–(26) or (27) and (28) exterior to the flux tube have to satisfy the jump conditions (16)–(17), which now take the form

$$QH_{m}^{(1)}(K_e R) + DH_{m}^{(2)}(K_e R) = CI_{m_1}(K_i R),$$  \hspace{1cm} (37)
\[ QH_{m}^{(1)}(K_e R) + DH_{m}^{(2)}(K_e R) = C\{FI_{m}^{(1)}(K_i R) - iGI_{m}^{(1)}(K_i R)\}, \quad (38) \]

where
\[ F = \frac{K_\rho_e(\omega^2 - \omega_{Ae}^2)}{K_\rho_i(\omega^2 - \omega_{Ai}^2)}, \quad (39) \]
\[ G = \pi \frac{\text{sign } \omega}{|A|} \frac{m^2}{R^2} \frac{\rho_e(\omega^2 - \omega_{Ae}^2)}{\rho(r_A)K_e}. \quad (40) \]

The term \( G \) reflects the effect of the resonance. Equations (37)–(38) are two complex equations for three complex unknowns: namely, \( Q, D, \) and \( C \) or \( A, B, \) and \( C. \) For the time being we shall solve these equations for the ratios
\[ \alpha = \frac{A}{C}, \quad \beta = \frac{B}{C}, \quad (41) \]
\[ q = \frac{Q}{C}, \quad d = \frac{D}{C}. \quad (42) \]

Of course,
\[ d = \frac{1}{2}[\alpha_r - \beta_i + i(\beta_r + \alpha_i)], \quad (43) \]
\[ q = \frac{1}{2}[\alpha_r + \beta_i + i(\alpha_i - \beta_r)]. \quad (44) \]

The solutions for \( \alpha_r, \alpha_i, \beta_r, \) and \( \beta_i \) are straightforward to obtain
\[ \alpha_r = \frac{\pi K_e R}{2} \{I_{m}^{(1)}(K_i R)Y_{m}^{(1)}(K_e R) - FI_{m}^{(1)}(K_i R)Y_{m}^{(1)}(K_e R)\}, \quad (45) \]
\[ \alpha_i = \frac{\pi K_e R}{2} GI_{m}^{(1)}(K_i R)Y_{m}^{(1)}(K_e R), \quad (46) \]
\[ \beta_r = \frac{\pi K_e R}{2} \{FI_{m}^{(1)}(K_i R)J_{m}^{(1)}(K_e R) - I_{m}^{(1)}(K_i R)J_{m}^{(1)}(K_e R)\}, \quad (47) \]
\[ \beta_i = -\frac{\pi K_e R}{2} GI_{m}^{(1)}(K_i R)J_{m}^{(1)}(K_e R). \quad (48) \]

The effect of the resonance appears in \( \alpha_i \) and \( \beta_i. \) In the absence of a resonance these two coefficients vanish.

The fractional absorption of the wave in the flux tube is defined as
\[ \text{Abs} = \frac{|d|^2 - |q|^2}{|d|^2}. \quad (49) \]
The condition for total absorption is

\[ q = 0 \]  

(50)

and states that there is no reflected wave.

Before proceeding with the driven problem we have to turn to the corresponding eigenvalue problem. This is the eigenvalue problem of the MHD radiating eigenmodes of a nonuniform flux tube, where presently the nonuniformity is confined to a ‘thin’ layer. This eigenvalue problem has so far been considered only for a uniform flux tube by Spruit (1982) in the long wavelength limit or thin flux tube approximation \((kR \ll 1)\) and by Cally (1986) for a thick flux tube. For a uniform plasma there is no resonance and the eigenmodes are damped because of the outgoing radiation. The eigenvalue problem deals with perturbations of the flux tube in the absence of external driving, i.e., in the absence of an incoming wave. All we have to do is to put \(D = 0\) in Equations (37) and (38) to arrive at the conditions for the eigenvalue problem:

\[ QH^{(1)}_{|m|} (K, R) = CI_{|m|} (K, R) , \]  

(51)

\[ QH^{(1)'}_{|m|} (K, R) = C\{FI'_{|m|} (K, R) - iGI_{|m|} (K, R)\} . \]  

(52)

This is a set of two linear and homogeneous equations for two unknowns \(Q\) and \(C\). The condition for this set to have a non-trivial solution gives the dispersion relation which is

\[ d = 0 \]  

(53)

and just expresses the obvious fact that there is not any incoming wave. The eigenmodes are precisely those modes that produce perturbations in the flux tube in the absence of any external driving. The notation is very convenient; \(d\) represents the external driving (Equation (27)) and \(d = 0\) is the dispersion relation. The dispersion relation is complex because of the outgoing MHD radiation and the resonance. The eigenvalue has a real part \(\omega\), which is the oscillation frequency and an imaginary part \(\gamma\) and the time dependency of the mode is

\[ \exp(-i\omega t) \exp(i\gamma t) . \]  

(54)

The imaginary part \(\gamma\), which is negative here, is the damping rate \((\gamma < 0)\) of the oscillation. This damping is due to the leakage of wave energy by the outgoing MHD radiation and by the resonance in the dissipative layer. It is convenient to write explicitly the real and imaginary parts of the left-hand side of Equation (53) as

\[ d_r + id_i = 0 , \]  

(55)

where

\[ d_r = \frac{1}{2} (\alpha_r - \beta_r) , \quad d_i = \frac{1}{2} (\beta_r + \alpha_r) . \]  

(56)

Let us first consider the case of a true discontinuity so that there is no resonance. The dispersion relation reduces to
\[ \alpha_r + i \beta_r = 0, \]  

(57)

which can be shown to agree with Spruit’s equation (23) before Spruit used the thin flux tube approximation and with Cally’s equation (2.9).

Let us now make the reasonable assumption that the damping due to the outgoing MHD radiation is weak; more precisely if \( \gamma_{RAD} \) is the damping rate we then assume that \( |\gamma_{RAD}| \ll |\omega_r| \). A first approximation to the oscillation frequency of the eigenmode is then found by solving

\[ \alpha_r = 0. \]  

(58)

Let us denote this approximation of the oscillation frequency by \( \omega_0 \). The damping rate due to the outgoing MHD radiation, \( \gamma_{RAD} \), can then be found by using a Taylor expansion of the left-hand side of Equation (57) about \( \omega_0 \):

\[ \gamma_{RAD} = -\frac{\beta_r(\omega_0)}{\partial \alpha_r / \partial \omega_0}. \]  

(59)

Replacing the true discontinuity by a thin nonuniform layer enables us to include the effect of the resonance in the formulation. Both the real and the imaginary parts of the dispersion relation are altered in comparison to that of the true discontinuity. The change in the real part of the dispersion relation causes a change in the oscillation frequency while the change in the imaginary part introduces an additional damping. Let us denote the change in frequency by \( \omega_{RES} \) and the additional damping rate by \( \gamma_{RES} \). In the assumption that in addition to \( |\gamma_{RAD}| \ll \omega_r \) also \( |\omega_{RES}| \ll \omega_0 \) and \( |\gamma_{RES}| \ll \omega_r \), the oscillation frequency can be obtained by solving

\[ d_r = 0 \quad \text{or} \quad \alpha_r - \beta_r = 0. \]  

(60)

The change in frequency \( \omega_{RES} \) due to the resonance can then be obtained by a Taylor expansion of the left-hand side of Equation (60) about \( \omega_0 \) to yield

\[ \omega_{RES} = \frac{\beta_r(\omega_0)}{\partial \alpha_r / \partial \omega_0}. \]  

(61)

Similarly

\[ \gamma_{RES} = -\frac{\alpha_r(\omega_0)}{\partial \alpha_r / \partial \omega_0}. \]  

(62)

In what follows the damping rates \( \gamma_{RAD} \) and \( \gamma_{RES} \) will play a fundamental role. Spruit (1982) has concluded that damping due to outgoing MHD radiation is not efficient in thin flux tubes or for waves with long wavelengths. Resonant damping is another mechanism that can enhance wave damping and that can be a more efficient damping mechanism than outgoing MHD radiation.

To make further analytic progress we consider the thin flux tube or the long wavelength approximation \((K_i R \ll 1 \text{ and } K_e R \ll 1)\). We use the dominant terms in the expansions for the Bessel functions of small argument to obtain
\[ \alpha_r \approx \frac{1}{2} \left( \frac{K_i}{K_e} \right)^{|m|} \left( 1 + \frac{F}{K_e} \frac{K_e}{K_i} \right), \]

(63)

\[ \alpha_i \approx - \frac{K_e R G}{2 |m|} \left( \frac{K_i}{K_i} \right)^{|m|}, \]

(64)

\[ \beta_r \approx \frac{\pi |m|}{2(|m|!)^2 4^{|m|}} (K_i R)^{|m|} (K_e R)^{|m|} \left( \frac{F}{K_e} - 1 \right), \]

(65)

\[ \beta_i \approx \frac{-\pi}{2(|m|!)^2 4^{|m|}} K_e R G (K_i R)^{|m|} (K_e R)^{|m|}; \]

(66)

For a thin nonuniform layer we can replace \( |\Delta| \) by

\[ |\Delta| = (\omega^2_{\lambda i} - \omega^2_{\lambda e})/2a \]

(67)

and write

\[ K_e R G \approx 2\pi \frac{a}{R} m^2 \frac{\rho_e (\omega^2 - \omega^2_{\lambda e})}{\rho(r_\lambda) (\omega^2_{\lambda i} - \omega^2_{\lambda e})}, \]

(68)

so that

\[ \alpha_i \approx -\pi \frac{a}{R} |m| \left( \frac{K_i}{K_e} \right)^{|m|} \frac{\rho_e (\omega^2 - \omega^2_{\lambda e})}{\rho(r_\lambda) (\omega^2_{\lambda i} - \omega^2_{\lambda e})}. \]

(69)

The real part of the dispersion relation for the true discontinuity reduces to

\[ 1 + F \frac{K_e}{K_i} = 0 \]

(70)

and the solution to this dispersion relation gives a first approximation to the oscillation frequency of the MHD radiating eigenmode, i.e.,

\[ \omega^2_k \approx \omega^2_i \approx \frac{\rho_i \omega^2_{\lambda i} + \rho_e \omega^2_{\lambda e}}{\rho_i + \rho_e}. \]

(71)

The index 'k' indicates that it is traditional to refer to this frequency as the frequency of the kink mode. The damping rate due to the MHD radiation is

\[ \gamma_{RAD} = -\pi \frac{|m|}{(|m|!)^2} \left( \frac{K_e R^2}{4} \right)^{|m|} \frac{(\omega^2_{\lambda i} - \omega^2_k)(\omega^2_k - \omega^2_{\lambda e})}{(\omega^2_{\lambda i} - \omega^2_{\lambda e}) \omega_k |\omega_k|} < 0, \]

(72)

and the damping rate due to the resonance is

\[ \gamma_{RES} = -\pi \frac{a}{R} |m| \frac{(\omega^2_{\lambda i} - \omega^2_{\lambda e})(\omega^2_k - \omega^2_{\lambda e})^2 \rho_e}{(\omega^2_{\lambda i} - \omega^2_{\lambda e})^2 \omega_k \rho(r_\lambda)} < 0. \]

(73)

The oscillation frequency (71) agrees with the corresponding result by Spruit (1982). The expression for \( \gamma_{RAD} \) agrees with Equation (45) of Spruit apart from the factor \( 2\pi \).
which is missing in Spruit’s $\delta(\gamma_{\text{RAD}} = |\omega_k|/2)$. Equations (72) and (73) show that the damping rates $\gamma_{\text{RAD}}$ and $\gamma_{\text{RES}}$ are proportional to $(K_e R)^{2m}$ and $a/R$, respectively, so that depending on the characteristics of the equilibrium and the wave we can have the three possibilities $|\gamma_{\text{RAD}}| \gg |\gamma_{\text{RES}}|$, $|\gamma_{\text{RAD}}| \approx |\gamma_{\text{RES}}|$, $|\gamma_{\text{RES}}| \gg |\gamma_{\text{RAD}}|$. Straightforward algebra gives an expression for $\omega_{\text{RES}}$

$$
\omega_{\text{RES}} = \frac{\pi^2}{[(|m| - 1)!]^2} \frac{a}{R} \left(\frac{K_e^2 R^2}{4}\right)^{|m|} \frac{(\omega_{\Lambda_i}^2 - \omega_k^2)(\omega_{\Lambda_e}^2 - \omega_k^2)}{(\omega_{\Lambda_i}^2 - \omega_{\Lambda_e}^2) |\omega_k|} \times \frac{\rho_i \rho_e}{\rho(r_A)(\rho_i + \rho_e)} > 0.
$$

(74)

Since $\omega_{\text{RES}}$ is very small if $a/R \ll 1$ and $K_e R \ll 1$, it will usually turn out that $\omega_0$ differs from $\omega_k$ primarily because of higher-order terms in the expansion of Equation (45), which we have so far neglected. Thus $\omega_{\text{RES}}$ will not play an important role. This point will be illustrated by some specific examples below.

Let us now return to the driven problem. It is straightforward to rewrite Equation (49) as

$$
\text{Abs} = (\alpha_i \beta_r - \alpha_e \beta_i) |d|^2 = \frac{\pi}{2} K_e R G[I_m(K_i R)]^2 |d|^2.
$$

(75)

Equation (75) is general, and valid for any driving frequency $\omega$. Let us first consider the case that the driving frequency $\omega$ is far away from the oscillation frequency of an MHD radiating mode. In that case

$$
|1 + F K_e/K_i| \gg 0,
$$

(76)

and

$$
|d|^2 \simeq \frac{\alpha^2}{4},
$$

(77)

and Equation (75) can be rewritten in the thin flux tube approximation and the ‘thin’ nonuniform layer as

$$
\text{Abs} \simeq \frac{4\pi^2}{4^{|m| - 1}[(|m| - 1)!]^2} \frac{a}{R} (K_e R)^{2|m|} \frac{\rho_e (\omega^2 - \omega_{\Lambda_e}^2)}{\rho(r_A)(\omega_{\Lambda_i}^2 - \omega_{\Lambda_e}^2)} \left(1 + F \frac{K_e}{K_i}\right)^{-2}.
$$

(78)

This result confirms the intuitive expectation that for a ‘thin’ nonuniform layer the absorption coefficient is proportional to $a/R$ and as a consequence much smaller than 1. In addition, for waves with long horizontal wavelengths the absorption coefficient is also proportional to $(K_e R)^{2|m|}$; However, this intuitively plausible result holds for a driving frequency far away from the oscillation frequency of an MHD radiating mode.

Let us now consider driving frequencies around an oscillation frequency of an MHD
radiating eigenmode so that
\[ 1 + FK_e/K_i \approx 0, \quad \text{and also} \quad \alpha_r \approx 0. \]  
(79)

This means that we are considering frequencies for which \( \alpha_r \) is not any longer the dominant term in \( d \) and \( q \), but can be of the same size as or smaller than \( \alpha_r, \beta_r, \) and \( \beta_i \). Let us consider the possibility of total absorption. As noted earlier, total absorption requires that there is no reflected wave, i.e., \( q = 0 \) and this requires
\[ \alpha_r + \beta_i = 0 \]  
(80)
and
\[ \alpha_i - \beta_r = 0. \]  
(81)

Total absorption occurs if the two conditions (80) and (81) are fulfilled and it is not obvious that these two conditions can be fulfilled simultaneously since we have only the driving frequency to vary. Let us choose \( \omega \) so that condition (80) is fulfilled. This optimal driving frequency, \( \omega_{\text{opt}} \), is not exactly equal to the oscillation frequency of the MHD radiating eigenmode. The latter quantity is determined by Equation (60). For a thin flux tube \( KR \ll 1 \) and a ‘thin’ nonuniform layer \( \beta_i \) is proportional to \( (KR)^2 |^m| (a/R) \) and as a consequence is small. The optimal driving frequency slightly differs from the oscillation frequency of the MHD radiating eigenmode. The shift between \( \omega_{\text{opt}} \) and \( \omega_r \) is
\[ \omega_{\text{opt}} - \omega_r = -2\omega_{\text{RES}}. \]  
(82)

Whether this optimal driving frequency produces total absorption depends on the equilibrium. Total absorption occurs if
\[ \alpha_i = \beta_r, \]  
(83)

which, according to (59) and (62), means that for the optimal driving frequency, \( \omega_{\text{opt}} \), the damping due to resonant absorption and the damping due to the outgoing MHD radiation are equal. So we need impedance matching for total absorption and this depends on the equilibrium. For instance, if we use expressions (72) and (73) we get a simple relation between \( a/R \) and \( K_e R \):
\[ \frac{a}{R} = \frac{(K_e^2 R^2)^{|m|}}{|m|! 2^{|m|} \rho_r \rho_e} \left( \frac{\rho_e + \rho_i}{\rho_i \rho_e} \right). \]  
(84)

In deriving Equation (84) we have taken
\[ FK_e/K_i \approx -1. \]

For an incoming wave with a given external horizontal wave number, \( K_e \), and an optimal driving frequency, \( \omega_{\text{opt}} \), Equation (84) defines the nonuniform layer that totally absorbs the incoming wave. Usually resonant absorption for a ‘thin’ nonuniform layer is proportional to \( a/R \), but for driving frequencies around the oscillation frequency of an MHD radiating eigenmode, this simple intuitive statement is not true. Total ab-
sorption can occur even for a ‘thin’ nonuniform layer if the equilibrium is fine tuned to the incoming wave.

We illustrate these points for sound waves impinging on a thin magnetic flux tube from a field-free region; thus $\omega_{\alpha e}^2 = 0$. The conditions $\omega_{\alpha i} > \omega > k c_e$ require that $\omega / \omega_k$ be in the range

$$
\left(1 + \frac{\rho_e}{\rho_i}\right)^{1/2} \frac{\omega}{\omega_k} > \left(1 + \frac{\rho_e}{\rho_i}\right)^{1/2} \frac{c_e}{v_{Ai}},
$$

with $c_e / v_{Ai}$ determined by pressure balance:

$$
\frac{c_e}{v_{Ai}} = \left[\frac{2}{\gamma} \left(\frac{\rho_e}{\rho_i} - \frac{T_i}{T_e}\right)\right]^{-1/2}.
$$

Here $\gamma$ is the usual ratio of specific heats, and $T$ denotes temperature. As a numerical illustration we will take $\rho_e / \rho_i = 4$, $T_i / T_e = 0.8$, and $\gamma = 1.2$; then $2.24 > \omega / \omega_k > 0.968$. We further assume that $\omega_{\alpha e}^2 / r^{-1}$, and $c^2$ vary linearly with $r$ in the thin nonuniform layer; from Equation (22) of Sakurai, Goossens, and Hollweg (1991b) we find that

$$
\frac{\rho_e}{\rho(r_A)} = \frac{\rho_e}{\rho_i} + \left(1 - \frac{\rho_e}{\rho_i}\right) \left(1 - \frac{\omega^2 / \omega_k^2}{1 + \rho_e / \rho_i}\right).
$$

The value of $(a/R)$ for optimum absorption is given approximately by Equation (84) evaluated at $\omega / \omega_k = 1$. If we take $m = 1$ and $kR = 0.3$, we obtain $(a/R)_{\text{opt}} = 4.679 \times 10^{-3}$. In Figure 3 we plot the absorption coefficient as a function of $\omega / \omega_k$ (with $k$ held fixed) for the above parameters: the curve is generated without approximation using Equations (75), (45)–(48), and the full expression for $K_e$, $K_t$, $F$, $G$, and $\rho(r_A)$. The maximum absorption coefficient is very nearly unity. It is not exactly unity because we have calculated $(a/R)_{\text{opt}}$ by assuming that the maximum absorption occurs at $\omega / \omega_k = 1$, which is not precisely true. However, it should be noted that $\omega_{\text{RES}} / \omega_k = 1.77 \times 10^{-5}$ (Equation (74)). Thus $\omega_{\text{RES}}$ is too small to account for the fact that the maximum absorption does not occur at $\omega / \omega_k = 1$. Instead, the frequency of maximum absorption is determined primarily by the frequency at which $\alpha_r = 0$, but Equation (45) must be expanded to higher order than was done in obtaining Equation (63). For the present case ($m = 1$ and $\omega_{\alpha e}^2 = 0$) we find by expanding Equation (45) that the frequency of maximum absorption is given by $(\omega / \omega_k)_{\text{opt}} = 1 + \varepsilon$, where

$$
\varepsilon \approx \frac{\rho_e}{8(\rho_e + \rho_i)} \left[4E(K_e R)^2 - (K_r R)^2 + 4(K_e R)^2 \ln \left(\frac{K_e R}{2}\right)\right]
$$

and $E$ is Euler’s constant. In Figure 4 we plot the absorption coefficient both as a function of $\omega / \omega_k$ and $a/R$ for the same parameters used in Figure 3. The figure shows that away from the optimal driving frequency the absorption coefficient is very nearly a linear function of $a/R$. But Figure 4 also shows the somewhat surprising result that
the absorption coefficient can still be large and exhibit a distinct maximum even when $a/R$ is considerably larger than $(a/R)_{opt}$. Thus substantial absorption can occur even if $a/R$ is not perfectly tuned to the incoming wave, as long as $\omega$ is close to the optimal frequency. If the flux tube is driven at the optimal frequency, and if the usually small term $\beta_i$ is ignored, then the absorption coefficient varies with $a/R$ approximately as

$$\text{Abs} \approx \frac{4(a/R)(a/R)_{opt}}{[(a/R) + (a/R)_{opt}]^2},$$

which is not a sensitive function of $a/R$ for $a/R \geq (a/R)_{opt}$.

The application of these results to the interaction of solar magnetic flux tubes with sound waves is problematical. Total absorption occurs close to $\omega/\omega_k = 1$, and total absorption will therefore occur only if the minimum allowed value of $\omega/\omega_k$ is less than unity. This in turn requires a rather large value of $\rho_e/\rho_i$, i.e.,

$$\frac{\rho_e}{\rho_i} > \frac{\gamma/2 + T_i/T_e}{1 - \gamma/2}.$$  

(This is why we took $\rho_e/\rho_i = 4$ in the above numerical example.) It is not known whether solar flux tubes have such strong density contrast. On the other hand, strong absorption peaks will occur close to the eigenfrequencies of the magnetic structures. If the absorption peaks were observed, then the eigenfrequencies could be deduced, and we
would have a powerful diagnostic tool for probing the subsurface properties of magnetic structures on the Sun.

In this regard, we should note that Penn and LaBonte (1992) have recently reported that the absorption of sound waves in sunspots seems to be rather uniformly distributed throughout the interior of the sunspot. This result could possibly be explained by our model, if the subsurface sunspot structure consisted of numerous thin flux tubes, as in the 'spaghetti model' of Parker (1979). Braun et al. (1992) have also recently reported observations of a frequency dependence of the absorption coefficient. Further work will be needed to determine whether the observed frequency dependence is compatible with that predicted by our model.

Total absorption on account of a reflection coefficient that vanishes is also found by Okretič and Cadez (1991) for a 'thin' nonuniform layer in planar geometry. Okretič and Cadez find optimal frequencies producing maximal absorption in their numerical results and for one specific equilibrium with $a/D = 1/150$ (in their notation) this frequency produces total absorption. They note that the absorption coefficient may reach unity if the wave parameters are adequately chosen. The present analysis shows that this adequate choice means impedance matching. Hollweg (1990b) also found total absorption of fast waves propagating in a cold plasma with a linear density variation: he did not relate the strong absorption to impedance matching, however.
4. Spatial Behaviour of the Solutions as a Function of the Driving Frequency

Let us now consider the spatial behaviour of the solutions. For a flux tube with a straight magnetic field the solution $\xi_r(s)$ in the vicinity of the ideal singularity is (see Equation (33) of SGH)

$$
\xi_r(s) = \frac{m^2/r_\Delta^2}{\rho(r_\Delta)\Delta} P' \left\{ \ln |s| - i\pi \frac{\text{sign } \omega}{\text{sign } \Delta} H(s) \right\},
$$

where $r_\Delta$ is the position of the ideal singularity, $s = r - r_\Delta$, and $H(s)$ is the Heaviside function; $H(s) = 0$ for $s < 0$ and $H(s) = 1$ for $s > 0$. The Eulerian perturbation of total pressure is a conserved quantity for a straight magnetic field.

Let us now consider a ‘thin’ nonuniform layer. In the interior region the solution is given by Equations (35)–(36) and in the exterior by Equations (27)–(28). Close to the singularity we have

$$
P' = CI_{m1} (K_i R), \quad (85)
$$

$$
\xi_r(s) = C \frac{K_i}{\rho(\omega^2 - \omega^2_\Delta)} I_{m1} (K_i R) + \frac{m^2/r_\Delta^2}{\rho(r_\Delta)\Delta} CI_{m1} (K_i R) \left\{ \ln |s| - i\pi \frac{\text{sign } \omega}{\text{sign } \Delta} H(s) \right\}. \quad (86)
$$

In the previous section we have imposed the continuity of the solutions (85)–(86) and (27)–(28). We have neglected the $\ln |s|$ terms since these terms are only important for very small $|s|$. This yields the system of Equations (37)–(38) which we have solved for the ratios $Q/C$ and $D/C$. This means that we have scaled the solution by prescribing its value on the axis (e.g., $C = 1$) or in other words that we have prescribed the amplitude of the transmitted wave and subsequently have determined the amplitudes of the corresponding incoming and outgoing waves. From a mathematical point of view this is a perfectly sound procedure since we are dealing with a homogeneous problem and can choose the scale factor as we wish.

From a physical point of view this is not the logical procedure. In a driven problem we want to know how a system behaves under a given external driving. We want to determine the amplitudes of the transmitted and outgoing waves for a prescribed amplitude of the incoming wave. From that point of view the obvious choice is to take $D = 1$ and then to determine $Q$ and $C$. From (42) we get

$$
C = \frac{1}{d}, \quad Q = \frac{q}{d}, \quad (87)
$$

where $d = 0$ is the dispersion relation of the MHD radiating eigenmodes. Remember, however, that the driving frequency is real and cannot make $d$ vanish. The solution (86) can now be rewritten as
\[ \xi_r(s) = \frac{1}{|d|^2} \left\{ d_r (G_1 + G_2 \ln |s|) + d_i G_3 H(s) + i\left[ -d_r (G_1 + G_2 \ln |s|) + d_i G_3 H(s) \right] \right\}, \quad (88) \]

where

\[ G_1 = \frac{K_i}{\rho_i(\omega^2 - \omega_{\Lambda}^2)} I_{|m|} (K_i R), \quad (89) \]

\[ G_2 = \frac{m^2/r_{\Lambda}^2}{\rho(r_{\Lambda} \Delta)} I_{|m|} (K_i R), \quad (90) \]

\[ G_3 = -\pi G_2 \frac{\text{sign } \omega}{\text{sign } \Delta}. \quad (91) \]

The object is to see how \( \xi_r(s) \) varies with the driving frequency. The dependency of \( \xi_r(s) \) on \( \omega \) is hidden in \( d_r \) and \( d_i \) which are functions of \( \omega \). It is important to remember that \( d_r \) vanishes when \( \omega \) is equal to the real part \( \omega_r \) of the eigenvalue of an MHD radiating eigenmode; in addition there is not any real \( \omega \) for which \( d_r \) and \( d_i \) vanish at the same time. For a driving frequency \( \omega \) far enough away from the oscillation frequency \( \omega_r \) of an MHD radiating eigenmode both \( d_r \) and \( d_i \) are different from zero and from Equation (78) it follows that both \( \Re \xi_r(s) \) and \( \Im \xi_r(s) \) have \( \ln |s| \) and \( H(s) \) contributions. For \( \omega = \omega_r \), we have \( d_r = 0 \) and as a consequence \( \Re \xi_r(s) \) has only a \( H(s) \) contribution and \( \Im \xi_r(s) \) has only a \( \ln |s| \) contribution. Because of continuity there is an interval around \( \omega_r \) in which \( |d_r| \) is small and as explained in the previous section \( \omega_{\text{opt}} \) is in that interval. The importance of the \( \ln |s| \) and \( H(s) \) contributions to \( \Re \xi_r(s) \) and \( \Im \xi_r(s) \) for a frequency, \( \omega \), different from \( \omega_r \) depends on the relative sizes of \( |d_r| \) and \( |d_i| \). Let us assume that

\[ |d_i| \ll |d_r| \]

except in an interval around \( \omega_r \). This inequality can easily be proven in the long wavelength limit and is true in general unless the waves are severely damped by MHD radiation or by the resonance. We have then a situation as given in Table I. Outside the interval around \( \omega_r \), \( \Re \xi_r(s) \) is characterized by a \( \ln |s| \) contribution and \( \Im \xi_r(s) \) by a \( H(s) \) contribution. Inside the interval around \( \omega_r \), \( |d_i| \) first decreases with \( \omega \), vanishes at \( \omega = \omega_r \), and subsequently increases as \( \omega \) is further increased. This means that for \( \Re \xi_r \), the \( \ln |s| \) contribution decreases in size, vanishes at \( \omega = \omega_r \), and again increases in size when \( \omega \) is increased. At the same time the \( H(s) \) contribution increases in size becomes maximal at \( \omega = \omega_r \) and decreases. The pure \( H(s) \) behaviour of \( \Re \xi_r(s) \) is attained at \( \omega = \omega_r \) and not at \( \omega = \omega_{\text{opt}} \). Since \( \omega_{\text{opt}} \) only slightly differs from \( \omega_r \), the solution has primarily a \( H(s) \) behaviour, but there is also a small \( \ln |s| \) contribution. The behaviour of \( \Im \xi_r(s) \) follows the same pattern. The \( H(s) \) contribution which is dominant outside the interval around \( \omega_r \) decreases in size, vanishes at \( \omega = \omega_r \), and again


TABLE I
Variation of Re $\xi_r(s)$ and Im $\xi_r(s)$ as a function of the driving frequency. The interval in which $|d_r| \approx |d_0|$ is enlarged for the sake of clarity. Frequency $\omega$ increases from bottom to top. Also we have taken $\omega_{\text{opt}} < \omega_r$, which is not necessarily always the case. Brackets around $\ln |s|$ or $H(s)$ indicate that these contributions are non-important.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>Re $\xi_r(s)$</th>
<th>Im $\xi_r(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>d_r</td>
<td>\ll</td>
</tr>
<tr>
<td>$</td>
<td>d_r</td>
<td>\approx</td>
</tr>
<tr>
<td>$d_r = 0 \quad \omega_{\text{opt}}$</td>
<td>$-\omega_r$</td>
<td>$-H(s)$</td>
</tr>
<tr>
<td>$</td>
<td>d_r</td>
<td>\approx</td>
</tr>
<tr>
<td>$</td>
<td>d_r</td>
<td>\ll</td>
</tr>
</tbody>
</table>

Increases when $\omega$ is increased. At the same time the $\ln |s|$ contribution increases in size, becomes maximal at $\omega = \omega_r$ and decreases subsequently. The pure $\ln |s|$ behaviour of Im $\xi_r(s)$ is attained at $\omega = \omega_r$ and not at $\omega = \omega_{\text{opt}}$. At $\omega = \omega_{\text{opt}}$ the solution has a dominant $\ln |s|$ behaviour, but in addition there is a weak $H(s)$ contribution. The prominent change in the spatial behaviour of the solutions for frequencies around the optimal frequency is completely due to the dependency of $d_r$ and $d_0$ on $\omega$ and especially on the fact that $d_r$ vanishes for $\omega = \omega_r$. Note that the discussion applies to an incoming wave with coefficient $D = 1$. If the incoming wave is given a complex coefficient say for instance $(1 + i)/\sqrt{2} = \exp(i\pi/4)$ then the complete solution has to be multiplied by the phase factor $\exp(i\pi/4)$.

5. Driven Problem for a ‘Thick’ Nonuniform Layer

Let us now see how the results of Sections 3 and 4 have to be modified for a ‘thick’ nonuniform layer. In the external uniform plasma (region I) the solution is still given by Equations (25)–(26) or (27)–(28). In region IV we no longer have an analytical solution. Close to the origin we can use a series expansion or just replace the coefficients in the differential equations by their values at $r = 0$ and obtain the solution given by Equations (35)–(36). This solution is now only valid close to $r = 0$ and $K_r$ has to be considered as the local value of this quantity at $r = 0$. A multiplicative factor $C$ scales the solution and will emerge in the boundary conditions at $r = R$. Numerical integration of the ideal MHD equation from $r = r_1$ ($r_1/R \ll 1$) up to $r = r_A - \delta$ gives the solution in the interval $[r_1, r_A - \delta]$. The connection formulae (16)–(18) then enable us to find the solution in $r = r_A + \delta$. Here the solutions for $\xi_r$ and $P'$ become complex because of the phase shifts that are introduced by the resonance. Numerical integration of the ideal MHD equations from $r = r_A + \delta$ up to $r = R$ gives the solution in the interval $[r_A + \delta, R]$. Hence we have the complete solution in $[0, R]$ (except for the dissipative layer $[r_A - \delta, r_A + \delta]$). To make the multiplicative factor $C$ visible we write the
solutions as

$$P' (r) = C \varphi (r), \quad (92)$$

$$\xi_r (r) = \frac{C}{\rho (\omega^2 - \omega_n^2 (r))} \frac{d \varphi (r)}{dr}, \quad (93)$$

where $\varphi (r)$ is a known function obtained by series expansion and numerical integration and

$$\varphi (r) = I_m (K_e r) \quad \text{for} \quad 0 \leq r \leq r_1,$$

$$\varphi (r) \quad \text{is real for} \quad 0 \leq r \leq r_A - \delta,$$

$$\varphi (r) \quad \text{is complex for} \quad r_A + \delta \leq r \leq R.$$

At $r = R$ the solutions (92)–(93) and (27)–(28) have to be continuous. This continuity requirement leads to conditions equivalent to the conditions (37)–(38) for the ‘thin’ nonuniform layer:

$$QH^{(1)}_{\mid m} (K_e R) + DH^{(2)}_{\mid m} (K_e R) = C \{ \varphi_r (R) + i \varphi_i (R) \}, \quad (94)$$

$$QH^{(1)}_{\mid m} (K_e R) + DH^{(2)}_{\mid m} (K_e R) = C \{ \varphi'_r (R) + i \varphi'_i (R) \}. \quad (95)$$

An accent now denotes a derivative with respect to $z = K_e r$. This derivative is introduced for convenience because otherwise a factor $K_e$ would float around in (95). In comparison with Equations (37)–(38) for the ‘thin’ layer not only $\xi_r$ (or $dP'/dr$) but also $P'$ has an imaginary part. The absence of an imaginary part in $P'$ in case of the ‘thin’ layer is due to the fact that to the right of the resonance there is not any nonuniform layer in which $\operatorname{Im} P'$ can become different from zero. In the ‘thick’ nonuniform layer the imaginary part of $P'$ is zero for $r = r_A + \delta$ but has a non-zero derivative there. Numerical integration over the interval $[r_A + \delta, R]$ gives a non-zero imaginary part of $P'$. In obtaining conditions (94)–(95) we have implicitly assumed that the equilibrium quantities are continuous at $r = R$ (e.g., $\omega^2_n (R) = \omega^2_n (R)$) so that $F = 1$. The correspondence between conditions (94)–(95) and conditions (37)–(38) is straightforward if we note that

$$\varphi_r (R) \rightarrow I_{\mid m} (K_e R), \quad \varphi_i (R) \rightarrow 0,$$

$$\varphi'_r (R) \rightarrow I_{\mid m} (K_e R), \quad \varphi'_i (R) \rightarrow G I_{\mid m} (K_e R).$$

Equations (94)–(95) are solved for $q = Q/C$ and $d = D/C$ in the same way as we have solved Equations (37)–(38). The solutions are given by Equations (43)–(44). The expressions (45)–(48) for $\alpha_r, \alpha_i, \beta_r$, and $\beta_i$ have to be modified but this can easily be done by using the correspondences indicated above. We obtain

$$\alpha_r = \frac{\pi K_e r}{2} \left[ \varphi_r (R) Y_{\mid m} (K_e R) - \varphi'_r (R) Y_{\mid m} (K_e R) \right], \quad (96)$$
\[ \alpha_i = \frac{\pi K_e R}{2} \left[ \varphi_i(R) Y_{m+1}(K_e R) - \varphi_i'(R) Y_{m+1}(K_e R) \right], \quad (97) \]

\[ \beta_r = \frac{\pi K_e R}{2} \left[ \varphi_r'(R) J_{m+1}(K_e R) - \varphi_r(R) J_{m+1}(K_e R) \right], \quad (98) \]

\[ \beta_i = \frac{\pi K_e R}{2} \left[ \varphi_i'(R) J_{m+1}(K_e R) - \varphi_i(R) J_{m+1}(K_e R) \right], \quad (99) \]

The effect of the resonance appears in \( \alpha_i \) and \( \beta_i \), since in the absence of a resonance \( \varphi(r) \) and its derivative are real functions. The fractional absorption of the wave is given by Equation (49) and the condition for total absorption is still Equation (50) which can be rewritten as Equations (80)–(81).

For a ‘thin’ nonuniform layer and waves with long wavelengths we could deduce from (80) that the optimal driving frequency slightly differs from the oscillation frequency of the MHD radiating eigenmode (the difference being twice the correction of the oscillation frequency due to the resonance). Equation (81) then told us that total absorption occurs if the equilibrium is fine tuned to the wave so that there is impedance matching. Let us now see how this is modified for the ‘thick’ nonuniform layer. The eigenvalue problem of the MHD radiating eigenmodes of ‘thick’ nonuniform flux tubes has so far not been studied in the literature. The dispersion relation can be obtained following the line of reasoning of Section 3. It is again Equation (53) and the time dependency of the eigenmode is given by (54). Equations (55)–(56) remain valid if the modified expressions for \( \alpha_r, \alpha_i, \beta_r, \) and \( \beta_i \) are used (Equations (96)–(99)). The eigenvalues of the MHD radiating eigenmodes of ‘thick’ nonuniform flux tubes have to be obtained numerically. Numerical integration of the ideal MHD differential equations and numerical solution of the dispersion relation are required. At present further analytic progress can only be made if we make additional assumptions. Let us recall that for a ‘thin’ nonuniform layer and waves with long wavelengths

\[ |\alpha_i/\alpha_r| \sim a/R, \quad |\beta_i/\alpha_r| \sim (KR)^2|m|, \quad |\beta_i/\alpha_r| \sim (a/R)(KR)^2|m| \quad (100) \]

for frequencies away from an oscillation frequency of an MHD radiating eigenmode. This enabled us to obtain a first approximation to the oscillation frequency by solving \( \alpha_r = 0 \). The correction to the frequency due to the resonance, and the damping due to the resonance and due to the MHD radiation could then be obtained by a Taylor expansion of the dispersion relation.

Consider now waves in an equilibrium state that satisfy

\[ |\alpha_i/\alpha_r| \ll 1, \quad |\beta_i/\alpha_r| \ll 1 \quad (101) \]

for frequencies away from the oscillation frequency of an MHD radiating eigenmode; that is to say waves that are not drastically affected by the resonance nor by the MHD radiation. For such waves we can essentially repeat the arguments of Section 3 both for the eigenvalue problem and the driven problem. Optimal driving frequencies are close to the oscillation frequencies of MHD radiating eigenmodes and total absorption occurs when there is impedance matching.
If the resonance and the MHD radiation are very strong (i.e., the ratios \( \alpha_i/\alpha_r \), \( \beta_i/\alpha_r \), and \( \beta_i/\alpha_r \) are of order unity for frequencies away from eigenmodes) so that the oscillation frequencies of the eigenmodes are substantially altered and the eigenmodes are strongly damped, the above interpretation of the optimal driving frequencies and total absorption does not hold. Such cases cannot be excluded although apparently they have not been encountered in numerical simulations. Nevertheless it is absolute necessary to carry out a systematic numerical study of the MHD radiating eigenmodes of 'thick' nonuniform flux tubes when resonant absorption is studied. These two problems are so tightly related that they should not be considered separately.

6. Conclusions

In this paper we have shown how the efficiency of resonant absorption and the spatial wave solutions depend on the frequency of the incoming wave. The tool to be used for carrying out this investigation is the SGH method. This method avoids integration of the non-ideal MHD equations and combines jump conditions and conserved quantities with the integration of the ideal MHD equations. The overall result of the investigation is that in order to understand resonant absorption, a simultaneous study of resonant absorption and the eigenvalue problem of MHD radiating eigenmodes on nonuniform plasmas is required. The MHD radiating eigenmodes of a nonuniform cylindrical plasma have complex eigenvalues. The real part of these eigenvalues are the oscillation frequencies of the eigenmodes. The MHD radiating eigenmodes are damped by radiation of MHD waves into the surrounding plasma and also by the ideal resonance, if the oscillation frequency of the eigenmode lies within the ideal Alfvén continuum of the nonuniform plasma. Unless the nature of the MHD eigenmode is totally altered by the MHD radiation and/or the ideal resonance, the oscillation frequency of the MHD radiating eigenmode only slightly differs from the oscillation frequency of the classic non-radiating MHD eigenmode. The correction to the oscillation frequency and the damping due to the MHD radiation and the ideal resonance can be obtained by a Taylor expansion of the dispersion relation. This dispersion relation was obtained from the driven problem by imposing that there was no incoming wave.

Total absorption, or no reflected wave, occurs if two conditions are satisfied. These two conditions both involve the frequency of the incoming wave. In contrast to the eigenvalue of an MHD radiating eigenmode, this frequency is real and does not satisfy the complex dispersion relation. The first condition for total absorption only slightly differs from the real part of the dispersion relation and involves quantities that for an arbitrary frequency are an order of magnitude larger than the quantities in the second condition. The frequency can always be chosen so that this first condition is satisfied and optimal absorption is attained. These optimal driving frequencies are not exactly equal to the oscillation frequencies of the MHD radiating eigenmodes but are slightly different. Whether these optimal frequencies produce total absorption depends on how well the equilibrium is tuned to the incoming wave. If the equilibrium has the property that the damping due to the MHD radiation is equal to the damping due to the ideal
resonance then the incoming wave is totally absorbed. This impedance matching guarantees total absorption even for a 'thin' nonuniform layer.

The variation of the spatial wave solutions with respect to the driving frequency is determined by the dependence of the real and imaginary parts of the complex dispersion relation on frequency. Since the driving frequency is real, the real and imaginary parts of the dispersion relation cannot vanish simultaneously. For an arbitrary frequency, the real part is in absolute value much larger than the imaginary part of the dispersion relation and \( \text{Re} \xi \), has a dominant \( \ln |s| \) behaviour and a negligible \( H(s) \) contribution while \( \text{Im} \xi \), has a dominant \( H(s) \) behaviour and a negligible \( \ln |s| \) behaviour. When the driving frequency equals the oscillation frequency of an MHD radiating eigenmode, the real part of the dispersion relation vanishes. For such a frequency \( \text{Re} \xi \), has only a \( H(s) \) contribution and \( \text{Im} \xi \), only a \( \ln |s| \) contribution. In a small interval around the oscillation frequency of this MHD radiating eigenmode, the spatial wave solutions undergo a rapid change.

The final conclusion is that resonant absorption and the eigenvalue problem of MHD radiating eigenmodes are so tightly related that they should be studied simultaneously in future. This, in turn, means that resonant absorption may, under some circumstances, be used to infer properties of the eigenmodes of solar magnetic structures. Thus, for example, resonant absorption of solar \( p \)-modes might be used as a diagnostic tool to probe the subsurface magnetic structure of the Sun. The recent observation (Braun et al., 1992) of a frequency-dependent \( p \)-mode absorption coefficient at sunspots suggests that this is a viable possibility.

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