ARE THE 1986–1988 CHANGES IN SOLAR FREE-OSCILLATION FREQUENCY SPLITTING SIGNIFICANT?

DOUGLAS GOUGH
Institute of Astronomy and Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Madingley Road, Cambridge CB3 0HA, UK

AND

PHILIP B. STARK
Department of Statistics, University of California, Berkeley, CA 94720
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ABSTRACT

The solar normal-mode splitting coefficients deduced from Big Bear Solar Observatory (BBSO) data differ between 1986 and 1988; inversions for equatorial rotation are slower at depth and faster near the surface in 1988 than in 1986. The significance of the change has been disputed. The data sets overlap for five splitting coefficients \( a_j \) \( j = 1 \) to \( 5 \), associated with 710 multiplets. On the assumption that rotation rate varies smoothly with radius, both data sets are satisfied (within the published uncertainties) by the same rotation model at all colatitudes except near 30°–40° and near 70° (and at their southern hemisphere reflections 140°–150° and 110°). The evidence for equatorial change is weak. Nonparametric tests show a significant offset in the magnitudes of \( a_1 \), \( a_2 \), and \( a_4 \), and of linear combinations sensitive to rotation at colatitudes of 60°–80° (and 100°–120°). Nonparametric tests show significant radial trends in the changes to \( a_2 \), \( a_4 \), and (less significantly) \( a_5 \). The evidence for radial trends in the changes to rotationally sensitive combinations of \( a_1 \), \( a_3 \), and \( a_5 \) is weak. There is strong anticorrelation between \( a_1 \) and \( a_4 \), \( a_2 \) and \( a_3 \), and \( a_1 \) and \( a_5 \), suggesting that the estimates are not independent. Individual coefficients \( a_j \) show more evidence for change than do “physical” linear combinations, adding weight to this hypothesis. Some of the changes in splitting might be related to solar activity, which changed most near colatitude 70° from 1986 to 1988.

Subject headings: Sun: rotation — Sun: oscillations — Sun: interior

1. INTRODUCTION

Libbrecht (1989) and Libbrecht & Woodard (1990) provide splitting coefficients \( a_j \) \( j = 1 \) to \( s \), \( s = 5 \) or 6, characterizing solar free-oscillation spectra for the years 1986 and 1988 obtained from observations at the Big Bear Solar Observatory (BBSO). The splitting coefficients are the coefficients of Legendre polynomial expansions of the shifts of individual singlet frequencies \( v_{nm} \) (as a function of \( m/l \)) from the central frequency \( v_{nl0} \) of the multiplet. Here \( n \) is the order of the mode, \( l \) is the degree, and \( m \) is the azimuthal order. Splitting of a multiplet into singlets with different frequencies that depend on \( m \) occurs in theory when the spherical symmetry of the system is broken, for example by rotation or asphericity in the structure of the Sun. Coefficients \( a_1 \), \( a_3 \), and \( a_5 \) depend on rotation: appropriate averages with respect to the radial coordinate \( r \) of the angular velocity \( \Omega(r, \theta) \) at colatitude \( \theta \) are approximately linearly related to

\[
S_\theta \equiv a_1 + (1 - 5 \cos^2 \theta) a_3 + (1 - 14 \cos^2 \theta + 21 \cos^4 \theta) a_5
\]

(1)

(see Brown et al. 1989). Nonlinear corrections to the relation are extremely small. Since, in addition, the combination \( S_\theta \) is an even function of \( \cos \theta \), north–south asymmetry cannot at present be detected using frequency information alone. Coefficients \( a_2 \) and \( a_4 \) depend on asphericity of the Sun and are relatively insensitive to angular velocity.

Goode & Dziembowski (1991) and Schou (1991) inverted these two sets of data and found changes from 1986 to 1988 in the Sun’s internal equatorial rotation rate. Schou claims the differences are not significant, whereas Goode & Dziembowski suggest that they are, and present evidence that the changes are related to the solar cycle. This disagreement has prompted us to take a closer look at the data.

The angular velocity is related to the splitting (approximately) by an integral of the form

\[
S_\theta = \int K_\theta \Omega dr ,
\]

(2)

where \( K_\theta \) is a nonnegative function of \( r \). Within this approximation, \( S_\theta \) changes only if the angular velocity distribution changes, although changes in the angular velocity need not change the rotational splitting. Thus we may reduce the question of the necessity of change in angular velocity to the question of change in the splitting coefficients. In particular, we try to answer the questions.

Q1. Can the 1986 and 1988 splitting data be fitted adequately by a single model of the angular velocity, or are differences in the angular velocity required to account for the splitting?

Q2. Is there a significant change to the “typical” amount of splitting?

Q3. Is there a significant radial trend in the changes?

Q4. Are the odd coefficients determined better or worse than linear combinations of them that are more directly sensitive to the angular velocity?

2. DATA, ASSUMPTIONS, AND STATISTICAL CAVEATS

As always, assumptions about the data play a heavy role in the conclusions, and the use of statistical tests invites the abuse
of statistical tests. We have tried to minimize the assumptions and abuses.

2.1. Data

The "data" in this study are the splitting coefficients obtained by Libbrecht (1989) and Libbrecht & Woodard (1990) for the BBSO observations. The intersection of the two data sets comprises the first five coefficients of 710 multiplets with frequencies ranging from 1516.1 \( \mu \text{Hz} \) to 3965.5 \( \mu \text{Hz} \), and values of \( l \) between 5 and 60. For each \( (l, n) \) pair, the five coefficients \( a^n\ell_l \) are found by a nonlinear least-squares procedure that fits the splitting coefficients, the background power near the multiplet frequency, and the amplitudes of the individual singlets to an estimate of the spatio-temporal power spectrum of the solar surface motions derived from transformed Doppler observations (see Libbrecht 1989 and Libbrecht & Woodard 1990 for details).

In the linearization of the problem about a least-squares solution, the covariance matrix of the fitted coefficients is diagonally dominant, since the Legendre polynomials are nearly orthogonal with respect to sums at the sample points (K. Libbrecht 1992, private communication). The diagonal dominance increases as \( l \) (and hence the number of sample points in the power spectrum) increases. The errors in the data to which the expansion is fitted are usually taken to be Gaussian by invoking the central limit theorem: the spectra involve linear combinations of vast numbers of the original Doppler observations, which are not totally dependent. The errors in the splitting coefficients are then approximately Gaussian and independent, with standard deviations given by Libbrecht (1989) and Libbrecht & Woodard (1990). (To find a formal covariance matrix would require repeating their analysis, and would give information only locally, in the neighborhood of the least-squares solution. Below we compute the Spearman rank correlation of changes in the estimated coefficients.)

We hope to get an idea of the variability and covariance of splitting coefficients empirically. The distributional details of the errors and uncertainty estimates are not important for the tests we perform, except in \$3.1\$ where we ask whether the rotation rate must have changed to account for the observed changes in splitting. We try to state clearly the assumptions used in each section.

2.2. Caveats

When performing multiple tests on the same data there is a risk of "data mining": if we examine enough projections of the data there is a good chance that we can find some projections that give apparently but erroneously highly "significant" results. Significance levels should be adjusted to take account of the fact that more than one test is being made.

The "significance level" of an hypothesis test is the chance that it erroneously rejects the null hypothesis when the null hypothesis is in fact true. This is called a "type I" error. The "power" of an hypothesis test against a specific alternative hypothesis is the chance that the hypothesis test correctly rejects the null hypothesis when the alternative is in fact true. Failure to reject the null hypothesis when it is false is a "type II" error; the "power" of a test is \( 1 - \alpha \) when a specific alternative is true, where \( \alpha \) is the chance of a type II error.

Suppose we had 1000 independent sets of data and performed the same 0.01 significance-level hypothesis test on each. If the null hypotheses were true, we should expect about 10 false alarms (i.e., type I errors). The chance of at least one type I error is

\[
1 - (\text{chance of no type I errors}) = 1 - (0.99)^{1000} \approx 0.99996 .
\]

Another way to express this result is that the significance level of the composite test that rejects the null hypothesis if any one of the 1000 tests indicates rejection is 0.99996.

In the problem at hand, we have only one set of data, but we are performing many tests against different alternatives. The outcomes of the tests are not independent since the data are the same; however, we can still get an upper bound on the chance that we make any type I errors in terms of the individual significance levels of the tests: Let \( E \) be the event that one or more of the \( N \) null hypotheses is incorrectly rejected when all the null hypotheses are true (we assume it is possible for all to be true simultaneously), and let \( E_i \) be the event that the \( i \)th null hypothesis is incorrectly rejected. Let \( P(E_i) = p_i \), where \( P(\cdot) \) denotes the probability of the event in parentheses. Then

\[
P(E) = P(\bigcup_{i=1}^{N} E_i) \leq \sum_{i=1}^{N} p_i ;
\]

that is, the chance of at least one type I error is at most the sum of the chances of type I errors in the original tests. Because we perform dozens of tests, we would like to keep the significance level of each especially low (<0.0001 in most of the paper). This keeps small the chance that we incorrectly reject one or more of the null hypotheses.

We must point out that our test for radial trends in the splitting is not strictly legitimate, since the models of Good & Dziembowski (1991) and Schou (1991), based on the same data, suggested to us that such trends might exist. Hypotheses should really be formulated before examining the data used to test them; for example, they might be suggested by a physical theory of how the distribution of angular momentum in the Sun changes over time, or by different data. We must await the publication of splitting coefficients for 1990 and 1991 BBSO data to make a better test.

3. The Tests and their Outcomes

We have performed a number of significance tests on various combinations of the two sets of splitting coefficients. The maximum depth to which a mode is sensitive is monotone in \( w \equiv v/(l + 1/2) \), where \( v \) is the central frequency of the multiplet. We computed \( w \) for each mode and sorted the modes on their \( w \) values in order of increasing maximum depth of sensitivity. (For some tests, such as the \( \chi^2 \) and Wilcoxon signed-rank test, sorting is irrelevant—only the pairing of observations for the same \( l \) and \( n \) matters.)

We have tested the hypothesis of no change between 1986 and 1988 against several alternatives: any change whatsoever (the omnibus alternative), a shift in the "typical" splitting, and a radial trend in the changes. The omnibus alternative was tested parametrically, relying on the assumed normality of the estimates, their independence, and their published uncertainties. To test for changes in the typical value and for trends we used nonparametric tests based on ranks, without reference to the published uncertainties. Tests against specific alternatives are usually more powerful than tests against the omnibus alternative (they have smaller probability of a type II error, namely, failing to reject the null hypothesis when the alternative is true), and parametric tests tend to be more powerful.
than nonparametric tests when the parametric assumptions hold.

3.1. Omnibus Test for Change

Here we examine the evidence for change of any kind in the splitting, and therefore, in particular, we determine whether the splitting changes require different rotation models. We assume in this section that the published uncertainty estimates are reasonable and that the errors are independent and not too far from normal. We also presume that the Sun does not look like an onion (i.e., rotation rate does not vary wildly with radius) to justify taking averages of linear combinations of the splitting coefficients for modes with turning points at nearly the same depth (small nonoverlapping ranges of \( \omega \)); although, because we do find evidence for change, the validity of our basic conclusion does not depend on that presumption.

In order to determine the change in angular velocity required to account for the observed change in splitting coefficients, we must specify how well (and in what sense) the true splitting should agree with the estimated splitting. Our model for the observational errors is

\[
a_i = a_i^0 + \epsilon_i ,
\]

where \( a_i \) is the estimate of the coefficient, \( a_i^0 \) is its true value and \( \epsilon_i \) is random error. According to K. G. Libbrecht (1992, personal communication), the errors \( \epsilon_i \) may be taken to be independent, zero-mean random variables with standard deviations \( \{ \sigma_i \} \) given by Libbrecht (1989) and Libbrecht & Woodard (1990). Then the error in a linear combination

\[
\lambda = \sum_{i=1}^{l} a_i a_i
\]

is

\[
\epsilon = \sum_{i=1}^{l} a_i \epsilon_i ,
\]

which has variance

\[
\sigma^2 = \sum_{i=1}^{l} a_i^2 \sigma_i^2 .
\]

Using rules (5)–(7) we construct vectors of differences between linear combinations of the 1986 coefficients and the corresponding 1988 coefficients. That is, we construct a \( J \)-vector \( \delta \) whose components \( \delta_j \) are averages (over multiplets with similar turning depths) of differences between values of \( \lambda \) for 1986 and 1988. To test whether the differences are significant, we evaluate a weighted two-norm of the vector of differences \( \delta \):

\[
\| \delta \|_2 = \sqrt{\sum_{j=1}^{J} \left( \frac{\delta_j}{\tau_j} \right)^2} ,
\]

where \( \tau_j \) is the standard deviation of \( \delta_j \). If \( \{ \delta_j \} \) are independent zero-mean random variables, the distribution of \( \| \delta \|_2^2 \) is approximately the \( \chi^2 \) distribution with \( J \) degrees of freedom. For \( J > 30 \) the distribution of \( Z = (2)^{1/2} \| \delta \|_2 - (2J - 1)^{1/2} \) is approximately the standard normal; by convention, the value \( Z \) takes is called the "z-score." The probability that a standard normal random variable exceeds the z-score is thus a good approximation to the probability that the splitting differences would be as large as observed or larger (the p-value), under the null hypothesis that the values \( a_i^0 \) are the same in both years and that the errors \( \epsilon_i \) are independent zero-mean random variables with standard deviation \( \sigma \), and with nearly normal distributions. Table 1 gives z-scores and approximate p-values for this hypothesis when different numbers of observations are averaged together in bins in \( w \) (the number of degrees of freedom ranges from 710, when there is no averaging, to 71, when sets of 10 observations are averaged; in every case the approximation of the \( \chi^2 \) distribution by the normal should be excellent). In the bin averaging, we weighted observations by the inverse of their standard deviations to obtain minimum-variance averages.

The changes in \( a_1, a_2, a_4, S_{50}, S_{80}, \) and \( S_{70} \) appear to be significant, and \( S_{20} \) marginally so. The z-scores for the even coefficients are quite high, both with and without averaging, while those of the odd coefficients and linear combinations of them are typically substantially lower when the data are averaged with depth. The decreased significance with small amounts of averaging leads us to doubt the reality of the changes in \( a_1, a_2, S_{50}, S_{10}, S_{20}, S_{50}, S_{60}, S_{80}, \) and \( S_{90} \), since the significance depends on rapid fluctuations over small depth ranges, which a priori we deem unlikely. \( S_{90} \) stands out particularly with a strong local maximum of the z-score.

If the average is legitimate, the data are evidence for changes in angular velocity at \( 30° \)–\( 40° \) and at about \( 70° \); elsewhere no change in the rotation rate is required to fit the data at about the 95% confidence level. The empirical covariance of \( \{ a_i \} \) (§ 3.4) casts some doubt on the significance of the changes at small colatitude. The evidence for change in the equatorial rotation rate is extremely weak, even without averaging. Interestingly, the colatitude of maximum change in sunspot activity from 1986 to 1988 was near \( 70° \).

3.2. Test for Shift in the Typical Value

A more specific alternative hypothesis is that the splitting coefficients increased or decreased on the whole over 1986–1988; this alternative can be tested nonparametrically using Wilcoxon’s signed rank test (Lehmann 1975). The null hypothesis for this test is that the true coefficients (or linear combinations thereof) are equal in 1986 and 1988, but have independent errors whose distribution is the same for both members of each 1986–1988 pair. The results of the test are in Table 2.

Wilcoxon’s signed rank test statistic \( V \) is based on the rank of the differences between the 1986 and 1988 values of the splitting coefficients (or linear combinations thereof). Suppose that the 1986 values of the variables in question are \( x_{ij} \), and that the 1988 values of the variables are \( y_{ij} \). Let \( t_j = x_j - y_j \), and suppose for simplicity that all \( n \) of the \( t_j \) are distinct (the treatment of ties is discussed by Lehmann 1975). Suppose \( n_+ \) of the \( t_j \) are positive and \( n_- \) of them are negative (\( n_+ + n_- = n \)). Now sort the \( t_j \) by increasing absolute value. Let \( S_k \) be the position in this sorted list of the \( k \)-th positive \( t_j \), \( k = 1, \ldots, n_+ \), and let \( V = \sum_{k=1}^{n_+} S_k \). The smallest \( V \) can be is \( n(n+1)/2 \), which in our analysis takes the value \( 2.52 \times 10^3 \), and which occurs if all the splitting coefficients (or combinations) in 1986 are in 1988. The statistic \( V \) tends to be large if many of the differences are positive, and if the largest differences are positive, i.e., when there is a tendency for the splitting to be larger in 1986 than in 1988. Similarly, the sum \( V \) will tend to be smallest when most of the differences are negative, and when the positive differences are slight, i.e., when there is a tendency for the splitting to be larger in 1988 than in
1986. When, on the whole, the splitting is the same in both years, $V_S$ will tend to have a moderate value (about $n(n + 1)/4 = 1.26 \times 10^5$, half its greatest possible value). Note that although the sum $V_S$ counts explicitly the positions $S_i$ of only the positive values of $t_j$, the values of those positions are themselves influenced by the values of the negative $t_j$. Indeed, despite its superficial asymmetry, the probability distribution of $V_S$ is actually symmetric with respect to change of sign of $t_j$

when the null hypothesis is true. The probability that a very large or small value of $V_S$ is observed when in fact there is no difference in the distribution of splitting coefficients can be calculated using a counting argument. For large $n$ (as here), there are excellent approximations using the normal distribution (see Lehmann 1975). If the null hypothesis is true, the random variable

$$Z = \frac{V_S - E(V_S)}{\sigma(V_S)} = \frac{V_S - n(n + 1)/4}{\sqrt{n(n + 1)(2n + 1)/24}}$$

has approximately the standard normal distribution when $n$ is large; here $E(V_S)$ and $\sigma(V_S)$ are the mean and standard deviation of $V_S$ corresponding to a random distribution of $S_i$. The value that $Z$ takes is called the z-score, and the probability that the absolute value of a standard normal random variable exceeds the absolute value of the z-score is a good approximation to the (two-sided) p-value for large $n$.

Table 2 shows significant shifts in $a_1$, $a_2$, and $a_4$, and to linear combinations corresponding to rotation changes in the range 60°–80° (most near 70°), and their southern-hemispherical reflections. The evidence for shifts in the amount of splitting at other colatitudes is weak, especially near the poles.

### 3.3. Tests for Radial Trends

A different alternative hypothesis is that there is a radial trend in the changes to the splitting coefficients, which we tested with Spearman’s rank correlation coefficient $p_S$ (Lehmann 1975): see Table 3. The null hypothesis for this test is that the differences between 1986 and 1988 are independent of radius $r$; the alternative hypothesis is that there is an association with $r$.

The Spearman rank correlation coefficient is defined as follows. As before, let $t_j$ be the difference between the 1986 and

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**Note.**—Values of the Wilcoxon signed-rank test statistic $V_S$, corresponding z-scores, and approximate p-values for the null hypothesis of no change in the magnitude of the splitting coefficients (or combinations of them) between 1986 and 1988, against the alternative hypothesis of a shift in the typical splitting. The apparent lack of uniqueness in the relation between the values of $V_S$ and $t_j$ is due to the next significant digit. The statistic was computed for 1986 minus 1988: negative values $z$ of the test statistic $Z$ indicate the typical value of the coefficient $a_i$ or the combination $S_i$ of coefficients was larger in 1988 than in 1986.

### Table 1

<table>
<thead>
<tr>
<th>Coefficient (1)</th>
<th>$z$: Average 1 (2)</th>
<th>$p$: Average 1 (3)</th>
<th>$z$: Average 2 (4)</th>
<th>$p$: Average 2 (5)</th>
<th>$z$: Average 3 (6)</th>
<th>$p$: Average 3 (7)</th>
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</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>5.67</td>
<td>&lt;0.0001</td>
<td>5.35</td>
<td>&lt;0.0001</td>
<td>6.59</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>$a_2$</td>
<td>9.27</td>
<td>&lt;0.0001</td>
<td>11.64</td>
<td>&lt;0.0001</td>
<td>12.79</td>
<td>&lt;0.0001</td>
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<tr>
<td>$a_3$</td>
<td>3.48</td>
<td>0.0003</td>
<td>1.81</td>
<td>0.0351</td>
<td>2.73</td>
<td>0.0032</td>
</tr>
<tr>
<td>$a_4$</td>
<td>7.15</td>
<td>0.0001</td>
<td>7.16</td>
<td>&lt;0.0001</td>
<td>8.93</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>$S_0$</td>
<td>1.22</td>
<td>0.0009</td>
<td>−1.01</td>
<td>0.8438</td>
<td>−0.40</td>
<td>0.6554</td>
</tr>
<tr>
<td>$S_{10}$</td>
<td>2.74</td>
<td>&lt;0.0001</td>
<td>1.48</td>
<td>0.0694</td>
<td>2.16</td>
<td>0.0154</td>
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<tr>
<td>$S_{20}$</td>
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<td>&lt;0.0001</td>
<td>1.63</td>
<td>0.0526</td>
<td>2.33</td>
<td>0.0099</td>
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<td>$S_{30}$</td>
<td>8.85</td>
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<td>2.16</td>
<td>0.0154</td>
<td>2.91</td>
<td>0.0018</td>
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<td>10.47</td>
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<td>3.34</td>
<td>0.0004</td>
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<td>$S_{50}$</td>
<td>6.62</td>
<td>&lt;0.0001</td>
<td>3.12</td>
<td>0.0009</td>
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<td>&lt;0.0001</td>
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<tr>
<td>$S_{60}$</td>
<td>−1.20</td>
<td>0.8849</td>
<td>−1.63</td>
<td>0.9484</td>
<td>−1.01</td>
<td>0.8438</td>
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<td>$S_{70}$</td>
<td>1.54</td>
<td>0.0618</td>
<td>−0.78</td>
<td>0.7823</td>
<td>−0.21</td>
<td>0.5832</td>
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<td>$S_{80}$</td>
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<td>&lt;0.0001</td>
<td>3.20</td>
<td>0.0006</td>
<td>4.32</td>
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<td>$S_{90}$</td>
<td>−4.72</td>
<td>&gt;0.9999</td>
<td>−1.23</td>
<td>0.8907</td>
<td>0.25</td>
<td>0.4013</td>
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<td>$S_{100}$</td>
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<td>−2.25</td>
<td>0.9878</td>
<td>−0.94</td>
<td>0.8264</td>
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### Table 2

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>$V_S$</th>
<th>$z$</th>
<th>$p$</th>
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</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>9.71 $\times 10^4$</td>
<td>−4.840</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>$a_2$</td>
<td>6.65 $\times 10^5$</td>
<td>−10.50</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>$a_3$</td>
<td>1.17 $\times 10^5$</td>
<td>−1.185</td>
<td>0.2360</td>
</tr>
<tr>
<td>$a_4$</td>
<td>1.70 $\times 10^3$</td>
<td>−8.823</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>$a_5$</td>
<td>1.33 $\times 10^5$</td>
<td>−1.844</td>
<td>0.0651</td>
</tr>
<tr>
<td>$S_0$</td>
<td>1.34 $\times 10^3$</td>
<td>−1.371</td>
<td>0.1704</td>
</tr>
<tr>
<td>$S_{10}$</td>
<td>1.33 $\times 10^3$</td>
<td>−1.359</td>
<td>0.1838</td>
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<td>0.2464</td>
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<td>1.31 $\times 10^3$</td>
<td>−0.833</td>
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<td>1.24 $\times 10^3$</td>
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<td>0.6621</td>
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<tr>
<td>$S_{50}$</td>
<td>1.16 $\times 10^3$</td>
<td>−0.835</td>
<td>0.0666</td>
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<td>$S_{60}$</td>
<td>1.07 $\times 10^3$</td>
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<td>9.76 $\times 10^3$</td>
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<td>$S_{90}$</td>
<td>1.17 $\times 10^3$</td>
<td>−1.730</td>
<td>0.0837</td>
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TABLE 3
SPEARMAN’S TEST FOR RADIAL TRENDS IN THE ANGULAR VELOCITY CHANGES

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>ρₙ</th>
<th>z</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁</td>
<td>0.064</td>
<td>1.69</td>
<td>0.0909</td>
</tr>
<tr>
<td>a₂</td>
<td>-0.259</td>
<td>-6.897</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>a₃</td>
<td>-0.051</td>
<td>-1.365</td>
<td>0.1722</td>
</tr>
<tr>
<td>a₄</td>
<td>0.233</td>
<td>6.207</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>a₅</td>
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<td>2.811</td>
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<tr>
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<tr>
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<tr>
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<td>0.0140</td>
</tr>
<tr>
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<td>1.741</td>
<td>0.0817</td>
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<tr>
<td>S₅₀</td>
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<td>0.7011</td>
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<td>S₉₀</td>
<td>0.080</td>
<td>2.140</td>
<td>0.0324</td>
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</table>

NOTE.—Values of Spearman’s rank correlation coefficient ρₙ, corresponding z-scores and approximate p-values for the null hypothesis of no radial trend in the change in coefficients (or combinations of them) between 1986 and 1988.

1988 values of a splitting coefficient or a linear combination of splitting coefficients of the multiplet j, whose reduced frequency w takes the value wᵢ. Let Rᵢ be the rank of tᵢ, i.e., its position in a list of tᵢ sorted from smallest to largest tᵢ. Let Tᵢ be the rank of the turning depth of the modes associated with the multiplet j; this is the same as the rank of wᵢ. The Spearman rank correlation coefficient ρₙ is the ordinary correlation coefficient of the vectors of ranks (Rᵢ) and (Tᵢ):

$$\rho_n = \frac{\sum (R_i - \bar{R})(T_i - \bar{T})}{\sqrt{\sum (R_i - \bar{R})^2 \sum (T_i - \bar{T})^2}}$$  \hspace{1cm} (9)

where $$\bar{R} = \frac{1}{n} \sum_{i=1}^{n} R_i$$ and $$\bar{T} = \frac{1}{n} \sum_{i=1}^{n} T_i$$ are the mean ranks. The rank correlation coefficient satisfies $$-1 \leq \rho_n \leq 1$$ (as does the ordinary linear correlation coefficient). Note that $$\bar{R} = \bar{T} = (n+1)/2$$, since both (Rᵢ) and (Tᵢ) are permutations of {1, 2, ..., n}. If (Rᵢ) and (Tᵢ) are similarly ordered, so that the largest splitting change is for the mode with the greatest turning depth, the next largest splitting change is for the next largest depth, and so on, then ρₙ = 1 (perfect rank correlation). If they are oppositely ordered, so that the largest splitting change is for the mode with the shallowest turning depth, etc., then ρₙ = -1 (perfect rank anticorrelation). In between, ρₙ measures the degree to which large changes of the splitting are associated with more deeply penetrating modes. Again there is a transformation of ρₙ that is approximately normally distributed when the null hypothesis is true and n is large, and ties can be accounted for (Lehmann 1975). Table 3 gives the z-scores using that transformation, and the corresponding approximate p-values.

None of the odd coefficients nor linear combinations of them shows a trend at significance level 0.0001. At significance level 0.01, only a₃ and S₀ show significant (positive) trends (S₀ just misses), and S₀ (the poles) only barely so. A 1% probability of random occurrence is low, but not overwhelmingly so; it is especially suspect since we are performing so many tests, and because the angular velocity in the polar regions contributes only weakly to the frequency-splitting coefficients.

3.4. Correlation among the Coefficients

We also used Spearman’s rank correlation coefficient ρₙ to see whether the coefficients {aᵢ}_{j=1} are correlated with each other: see Table 4. In this case, the ranks of one coefficient for each mode were correlated with the ranks of a different coefficient for the same mode, to see if the magnitudes of changes in the coefficients were associated.

The high negative correlation (significance p < 0.0001) among even and among odd coefficients supports the hypothesis that sums of the odd (or even) coefficients are determined better than the odd (or even) coefficients individually, and that differences are probably determined worse. The correlation is expected since the least-squares procedure uses only a finite number of points in the power spectrum: the Legendre polynomials are not really orthogonal with respect to the data inner product, leading to tradeoff among even and among odd coefficients. The combinations S₀ involve differences for small colatitudes; all three terms are positive for $$\theta > 70^\circ$$, so we expect angular velocity to be determined better near the equator than near the poles. This is also expected from the observation geometry—polar regions are smaller, less well observed and noisier than equatorial regions. We note, in addition, that asphericity giving rise to the even coefficients is likely to be given at all latitudes by a difference between a₃ and 2a₁, e.g., Gough & Thompson (1988), with greater cancellation, and, therefore, lesser reliability, near the poles.

4. DISCUSSION

4.1. Correlation of Equatorial Rotation with Sunspot Number

Figure 2 of Goode & Dziembowski (1991) shows inversions for the equatorial rotation rate at 0.4R and the inverse of the mean sunspot number, at five epochs from 1983 to late 1988. Both plots exhibit maxima at the third point (in late 1986), suggesting a relation between rotation and sunspot number. One might like to know the chance of this occurring randomly. The mean sunspot number is a fixed curve with a minimum at the third point, and increasing monotonically on both sides. Since one might equally well have plotted the mean sunspot number as its inverse, the appropriate question is how likely it is that five random points (representing the rotation rate

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inversions) have an extremum at the third point, and monotonic behavior either side of the extremum. Inspection of the error bars on the third and fourth points in Figure 2 of Goode & Dziembowski (1991) shows the maximum of the rotation rate may have occurred at the fourth rather than the third point; since the sign of the time axis is not important we are led to seek the probability that five random points have an extremum at the second, third, or fourth point, and monotonic behavior about the extremum.

Were the data independent and identically distributed (iid), all 5! = 120 possible orderings of their ranks would have equal probability. The probability of an extremum at the second, third, or fourth point is the number of orderings giving such an extremum, divided by 120. Enumeration gives 28 such orderings, so the probability of a coincidental maximum is 7/30 = 0.233 if the rotational inversions were iid noise. As a result, Goode & Dziembowski’s (1991) Figure 2 does not persuade us that the correlation of the inferred rotation near the equatorial plane with the solar cycle is significant.

4.2. Splitting and Active Regions

The strongest evidence for changes in the splitting we found was at colatitudes near 30°–40° and near 70°. In particular, the Wilcoxon signed-rank test (Table 2) shows much more significant change near 70° than elsewhere. While the observed changes may well reflect changes in solar angular velocity, there are other possibilities. For example, we may be committing a type II error—incorrectly rejecting the null hypothesis of no change—simply because we observed data that are unlikely, but possible, under the null hypothesis. Other possibilities include the violation of assumptions we made in the analysis, such as independence, validity of the uncertainties assigned by the observers, and perhaps most importantly, the assumption that the splitting coefficients are related to the angular velocity by equations (1) and (2). The increased solar activity near colatitude 70° in 1988 suggests a different alternative from change in the angular velocity: the estimation of splitting coefficients might be affected by solar activity.

It has been observed that the power in acoustic waves with frequencies in the range of those analyzed here is lower in regions of high magnetic field (e.g., Woods & Cram 1981; Brown et al. 1992), presumably brought about by the modification to the structure of the atmosphere. The spatial distribution of sunspots has power at azimuthal wave numbers comparable to those of the modes we study here. Sunspots and other large-scale magnetic features superrotate relative to the photosphere, and inversions for rotation in the interior for 1988, using the same data that we analyze here, show rotation declining slightly with depth near colatitude 70° (Gough et al. 1992). Therefore, it is plausible that sunspots superrotate relative to the material in the acoustic cavity of colatitude 70°. The reduced amplitudes of waves near active regions superposes a set of “troughs” that might be precessing faster than the acoustic modes. One would expect this to bias estimates of the splitting upward. This is borne out by the negative sign of the Wilcoxon statistic for S70 in Table 2—the splitting was typically larger in 1988 than in 1986.

We have estimated the magnitude of the effect using information provided by Brown et al. (1992) and find it probably to be rather smaller than the differences reported in the BBSO splitting data. Moreover, this potential explanation cannot account for the observed changes in linear combinations sensitive to angular velocity near 30°–40°. Thus the data do appear to provide evidence for a real change in the angular velocity.

5. Conclusions

The evidence for changes in the Sun’s internal rotation rate between 1986 and 1988 from the BBSO splitting data given by Libbrecht (1989) and Libbrecht & Woodard (1990) is weak except at colatitudes between 30° and 40° and near 70°. The strongest evidence for change in the typical site size of the coefficients is at colatitude 70°, where changes in sunspot activity were greater. The evidence for changes in the equatorial rotation rate is weak.

The coefficients pairs (a1, a2), (a3, a4), and (a3, a5) are significantly rank correlated. We do not know if this is reflected in the uncertainty estimates given by Libbrecht (1989) and Libbrecht & Woodard (1990). Rank correlation is a different measure of association than linear correlation, and it is possible for Spearman’s rank correlation coefficient to be larger in absolute value than the ordinary correlation coefficient.

The answers to questions Q1–Q4 are as follows.

A1. A χ² test (Table 1) suggests that the 1986 and 1988 data require different angular velocity models near colatitudes 30°–40° and 70°. Elsewhere, the rotation need not vary with time. Changes in individual splitting coefficients a1, a2, and a3 (which are essentially unassociated with rotation), are significant.

A2. From the Wilcoxon signed-rank test (Table 2), there appears to have been an increase from 1986 to 1988 both in the “typical” value of coefficient a1 and in the rotation-sensitive linear combination for colatitude 70°. Coefficients a2 and a4, which are only weakly associated with rotation, also changed significantly.

A3. The Spearman rank correlation test (Table 3) finds strongly significant radial correlations for coefficients a1 and a3. Some rotation-sensitive linear combinations (S9 and S10) of odd coefficients show evidence of change, the sense being that near the poles the angular velocity at the greater depths tend to have increased from 1986 to 1988 relative to the values nearer the surface.

A4. On the χ² test, the evidence for change in a1 is stronger than the evidence for change in coefficients a3 and a5, and stronger than the evidence for change in rotationally sensitive linear combinations. On the other hand, the evidence for change in typical values given by the Wilcoxon signed-rank test is stronger for S70 than for the odd coefficients a1, a3, and a5. The differences in the smallest significances (most significant changes) are small in both cases. The strong anti-correlation among the odd coefficients found by the Spearman rank correlation test (Table 4) suggests that the reliability of rotation-sensitive linear combinations depends on the latitude (through the signs and magnitudes of the terms). For some latitudes the linear combinations are determined better than the individual coefficients; for other latitudes we believe they are determined worse. A definitive answer to Q4 would require reanalysis of the original spectral data, which is beyond the scope of this work.

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