THE DYNAMICS OF MAGNETIC FLUX RINGS
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ABSTRACT

Understanding the evolution of magnetic fields in the presence of turbulent convection is essential for understanding the solar cycle dynamo. In this paper, we present results from numerical simulations of closed magnetic fibrils moving in a steady "ABC" flow, which we believe approximates some of the important characteristics of a turbulent, convecting flow field. Three different evolutionary scenarios are found: expansion to a steady, deformed ring; collapse to a compact fat flux ring; and occasionally, evolution toward an advecting, oscillatory state. There is a "critical length scale" for a closed magnetic fibril which divides the collapsing and expanding solutions.

A simple scaling analysis predicts the existence of the expanding and collapsing solutions, as well as the amplitude of the asymptotic field strength for the expanding solutions. The form of the asymptotic field strength, $B_{\infty}$, is well approximated by $B_{\infty} \approx 3.7 \rho^{2/3} l^{1/3} \Phi^{4/3} \Phi^{-1/3}$, where $\rho$ is the mass density, $l$ is the size scale of the most vigorous motions, $V_\infty$ is the velocity amplitude associated with the size scale, and $\Phi$ is the magnetic flux per fibril. The scaling analysis further suggests that small-scale turbulent velocities are unimportant for amplification of strong magnetic fields.

Subject headings: convection — MHD — Sun: magnetic fields — turbulence

1. INTRODUCTION

The dynamics of magnetic flux tubes has a long history in solar physics from the early discussions of magnetic buoyancy (Parker 1955), through the derivation of the thin flux tube equations (Parker 1979; Spruit 1981), to the buoyant rise of flux tubes in the solar atmosphere (Moreno-Insertis 1986; Chou & Gilman 1987; Chou & Fisher 1989; Choudhuri 1989; Choudhuri & D'Silva 1990; Fan, Fisher, & DeLuca 1993, hereafter FFD). An important related problem, namely understanding how "fibril magnetic fields" evolve in the presence of vigorous motions in the convection zone (see, e.g., Parker 1982a, b, c, d), will be the main topic of this paper. We consider the evolution of isolated closed flux rings in a steady flow with chaotic streamlines. We find that for the flow field considered in this paper, thin magnetic fibrils with $10^{18}$ Mx of flux can be amplified to a field strength of $10^6$ G—more than 50 times larger than the equipartition field strength of the flow ($1.3 \times 10^4$ G).

A critical component of any solar or stellar dynamo model is the amplification of magnetic fields through the stretching of magnetic field lines. In kinematic dynamo models, a flow field is specified and the amplification of the magnetic field subject to (turbulent) diffusion is determined. However, as the field strength increases, the Lorentz force feedback of the magnetic field on the flow can become important; this fact is ignored in kinematic models. For diffuse (i.e., nonfibril) fields, subjected to turbulent motions, the feedback can be important even for field strengths well below the equipartition strength (Vainshtein & Rosner 1991; Cattaneo & Vainshtein 1991; Vainshtein & Cattaneo 1992). When the magnetic field is in fibril form, the feedback of the magnetic field on the fluid takes a slightly different form. The fluid is separated into two regions: magnetized (i.e., the magnetic flux tubes) and field-free gas. The time evolution of the magnetized regions is governed by a balance between the aerodynamic drag on the magnetized fluid by the unmagnetized fluid, the magnetic tension force, and other body forces that may be present, typically magnetic buoyancy and the Coriolis force (see discussions in Parker 1998a, b, c, d). The importance of the magnetic feedback is strongly dependent on the length scales of the flow field: even flux tubes with modest magnetic field strengths are effective at resisting deformation from small-scale motions, while field strengths much greater than the equipartition strength are needed to resist deformation by large-scale flows. In a similar way, simulations of diffuse fields show that the Lorentz force greatly alters the flow at small scales even when the magnetic energy is small compared with the kinetic energy (Cattaneo & Vainshtein 1991 and Vainshtein & Cattaneo 1992).

Note that the simple distinction between kinematic and dynamic models in MHD (kinematic models solve the induction equation with the velocity field given, while dynamic models solve both the momentum and induction equations) becomes less clear when the fibril field approximation is made. Our model is kinematic in the sense that the velocity field of the unmagnetized fluid is specified. However, for the flux tubes both the momentum and induction equations are solved, so the model is dynamical for the magnetized fluid. In fact, because the magnetic fields in our fibrils are strong, the tension force (part of the Lorentz force in full MHD) plays an important role in controlling the dynamical evolution of the fibril field. As a result, important nonlinear effects (i.e., those that limit the field amplitude) are included in the equations. The assumption that the field-free fluid is unaffected by the presence of the magnetic fibrils should be valid if the magnetic filling factor (mass of magnetized/mass of unmagnetized) is sufficiently small. While the assumption of a small magnetic filling factor has strong support from photospheric observations, it cannot be simply understood by theories of MHD turbulence. (However, Parker 1984 has argued that the fibril

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state can be understood as a reduced energy state for the total solar atmosphere.)

An important assumption made in this paper is that the resistivity of the plasma is small enough that we can treat the dynamics of magnetic fibrils in the ideal MHD limit. While this should generally be true, the operation of the true dynamo requires changes in the magnetic topology which in turn demands a finite resistivity. We believe that changes in the topology will occur when two fibrils collide, in which case “fast” reconnection will change the connections between different parts of the fibrils (see, e.g., discussion in Parker 1979 and DeLuca & Craig 1992). However, the dynamics of strong individual isolated fibrils, such as those considered in this paper, should be well described by the infinitely conducting limit.

Finally, we should emphasize that the fibrils considered in this paper are untwisted. While this assumption simplifies the fibril model considerably, it is probably not a reasonable assumption for the Sun. Untwisted fibrils are neutrally stable to shearing motions across the diameter of the fibril, yet we assume that the cross section remains circular. Cattaneo, Chiu, & Hughes (1990) have argued that twisted toroidal flux emerges more readily from the overshoot region than untwisted toroidal fields, even though the latter are linearly more unstable. We are in the process of formulating a more realistic thin flux tube model that will include twist. Thin flux tubes with small amounts of twist may well be approximated by untwisted flux tubes with circular cross-section.

This paper is organized as follows: the basic equations and numerical method are described briefly in § 2. In § 3, we use simple analytical models to show under what conditions an isolated flux ring will expand or collapse; the collapse time and the asymptotic field strength are also estimated. In § 4, we present the results of our numerical experiments. In § 5 we summarize the most important results we have found and consider the implications of these results for the solar cycle.

2. THE NUMERICAL MODEL

To carry out this investigation, we have developed a numerical algorithm for solving the equation of motion for a single, closed magnetic flux tube moving in three dimensions; the algorithm is based on solving the momentum equation for the magnetized fluid in a coordinate system where the mesh is always spaced uniformly along the tube. This numerical algorithm is described in detail in FFD. In this paper, we make some assumptions which simplify the FFD dynamics equations. First, because the focus of this study is on the expansion or contraction of flux tubes due to fluid shearing motions, we ignore from the outset the magnetic buoyancy and Coriolis forces. In addition, the flux tubes considered here have sufficiently weak flows along the tube that we have replaced the parallel component of the momentum equation with the condition of hydrostatic equilibrium in the direction along the tube.

In the directions normal to the flux tube, the momentum equation is given by

$$\frac{D \rho_i}{Dt} + \frac{B^2}{4\pi} \frac{\partial \rho_i}{\partial s} + C_D \rho_e \frac{|v_{fz} - v_s| (v_{fz} - v_s)}{\sqrt{\pi B / B}} = 0, \tag{1}$$

where $v_s$ is the velocity of flux tube plasma in the directions normal to the tube orientation, $v_{fz}$ is the velocity of the external fluid surrounding the flux tube in the normal directions, $B$ is the magnetic field strength, $C_D$ is the coefficient of the aero-
dynamic drag force for a flux tube with circular cross-section (see, e.g., Batchelor 1967) and is assumed to be unity, and $\Phi$ is the magnetic flux in the tube. $D / Dt$ refers to the full, Lagrangian time derivative (in Eulerian coordinates $D / Dt = \partial / \partial t + \mathbf{v} \cdot \nabla$). This balance of force is essentially the same as that considered by Parker (1982b). On the left-hand side, $\rho_i$ denotes the mass density within the tube, while $\rho_e$ represents the “added mass” of the plasma outside the tube being accelerated (see, e.g., the discussion in Parker 1985). By neglecting the buoyancy force, we implicitly assume that the flux tube is in “mechanical equilibrium” (i.e., that $\rho_e = \rho_i$) but see Fisher, Mclmont, & Chou (1991) for the effects of field amplification on flux tubes in mechanical equilibrium near the base of the convection zone of stellar interiors). The vector $r(s)$ defines the space curve delineated by the tube, where $s$ is arc-length measured along the tube from some reference mass point. The external flow field $v_s$ that we adopt in our simulations is described in § 4. Note that equation (1) assumes that the external fluid can affect the motion of the flux tube, but that the motion of the tube has no effect on the external velocity field.

The position update equation is given by

$$\frac{Dr}{Dt} = v_s, \tag{2}$$

where $v_s$ includes both the normal and parallel components of the velocity.

In addition to the momentum and position update equations, an equation for the magnetic field strength must be solved. The ideal MHD induction equation is

$$\frac{D B}{Dt} = \nabla \times \mathbf{E} - \mathbf{v} \times \nabla \mathbf{B} = \left( \frac{B}{\rho} \cdot \nabla \right) \mathbf{v}. \tag{3}$$

In this paper, we consider flux tube motion in the limit that both the external and internal media are incompressible. Because we also assume hydrostatic equilibrium in the direction along the tube, this results in an extremely simple solution for the induction equation: The field strength is uniform along the loop and reduces to

$$B(t)/L(t) = B_0/L_0, \tag{4}$$

where $B_0$ and $L_0$ are the initial values of the field strength and tube length, and where we have assumed that the flux tube forms a closed ring. If the length $L(t)$ of the flux ring is known, then the field strength is also known. In the more general case described in FFD, it is possible to have sausage-wave disturbances driven by fluctuations in the magnetic pressure superposed on this slower, secular trend, but here these have been suppressed.

The parallel component of the velocity $v_s$ needed for the position update equation can be determined from equation (3). Since the magnetic field strength and the density are assumed uniform along the loop, dotting equation (3) with the unit vector $\hat{s}$ along the flux tube results in the equation

$$\frac{1}{B} \frac{DB}{Dt} = \frac{\partial v_s}{\partial s} - \kappa \cdot \mathbf{v}. \tag{5}$$

By integrating equation (5) and using equation (4) it is straightforward to show that

$$v_s(u) = u \frac{dL}{dt} + L \int_0^u \kappa \cdot v \, du' + v_s(0), \tag{6}$$
where \( \kappa \equiv d\mathbf{x}/ds \) is the curvature vector, and \( u \) measures distance along the tube in normalized coordinates. The quantity \( v_\perp(0) \) is a constant of integration, but it is not arbitrary. It can be eliminated by noting that the circulation \( \int_0^l v_\perp(u)du \) should be conserved and is equal to zero, since the plasma in our flux rings is initially at rest (the circulation would not necessarily be conserved if we were to include a Coriolis force in eq. [1]). After some algebra this results in

\[
 v_\perp(u) = \left( u - \frac{1}{2} \right) \frac{dL}{dt} + L \int_0^u \kappa \cdot \mathbf{v} du'.
\]

(7)

The Lagrangian time derivatives on the left-hand sides of equations (1) and (2) are converted to partial time derivatives at fixed values of \( u \) as described in FFD. The resulting partial differential equations are then equivalent to a set of 6N coupled, first-order ordinary differential equations in time which are solved using a RK4 stepping algorithm (e.g., Press et al. 1986). Further details of the numerical method are described in FFD.

3. THE CRITICAL LENGTH SCALE AND ASYMPTOTIC FLUX RING BEHAVIOR

In order to interpret the results of our numerical simulations (to be discussed in § 4), we first consider some approximate solutions to the equation of motion for isolated flux rings in an imposed flow field. These solutions lead to simple scaling laws which can then be compared with the numerical solutions. In § 3.1, we examine the collapse of flux rings in a stationary medium. In § 3.2, we consider the evolution of flux rings in a flow field with a single characteristic length scale and its associated velocity amplitude. We argue for the existence of a “critical length scale” of a flux ring, which separates the behavior into collapse, if the size is less than the critical length, and expansion, if the size exceeds the critical length. For those rings which expand, we go on to derive asymptotic limits of the magnetic field strength and the total circumference of the stretched tube in terms of the scales of the flow field and the initial parameters describing the ring.

3.1. The Collapse of Magnetic Flux Rings

First, consider a circular ring of flux in an incompressible external fluid which is at rest. The only force acting on the ring is the tension force and the ring will begin to collapse. As the collapse proceeds, the aerodynamic drag of the ring on the motionless fluid will lengthen the collapse time, which we now process to estimate.

In the incompressible limit, the volume of the flux ring is preserved: \( 2\pi RA = 2\pi R_0 A_0 \), where \( R \) is the radius of the circular ring, \( A \) is the cross-sectional area, and the subscript 0 denotes the initial values. The flux is also conserved, \( BA = B_0 A_0 = \Phi \). The evolution of the ring is found by neglecting the unimportant inertial term and balancing tension and aerodynamic drag forces. Using the above conservation equations and recognizing that \( \frac{d}{dt}(R/R_0) \) from equation (1) then becomes

\[
 \frac{d}{dt} \left( \frac{R}{R_0} \right) = - \left( \frac{n\alpha_0 V_{A0}^2}{C_p R_0^2} \right)^{1/2} \left( \frac{R}{R_0} \right)^{1/4},
\]

where \( \alpha_0 \) is the initial cross-sectional radius \( (A_0 = \pi\alpha_0^2) \), and \( V_{A0} = [B_0^2/(4\pi\rho)]^{1/2} \) is the initial value of the Alfvén speed. The solution shows that the radius collapses in a finite time:

\[
 R(t) = R_0 \times \left( 1 - \frac{t}{\tau} \right)^{4/3},
\]

(9)

where the collapse time \( \tau = 4/[C_p R_0^2(n\alpha_0 V_{A0})]^{1/2} \). For a tube with magnetic flux comparable to that in a very small active region \( (\Phi \sim 10^{10} \text{ Mx}) \), a relatively high field strength \( (10^5 \text{ G}) \), a density of \( \rho \sim 0.2 \text{ g cm}^{-3} \) (consistent with that near the base of the convection zone), and a fairly large radius \( (R_0 \sim 10^8 \text{ cm}) \), about half the depth of the convection zone, the collapse time is \( \tau \sim 32 \text{ days} \). For smaller, “elemental” flux tubes with \( \Phi \sim 10^{18} \text{ Mx}, \tau \sim 103 \text{ days} \).

A real flux ring cannot collapse to a singularity as equation (9) suggests. Eventually the thin flux tube approximation breaks down, and further evolution of the isolated ring involves the interaction of the thick tube with itself. For a circular ring, an O-type neutral point will form when the tube cross-sectional radius \( a \) approaches the ring radius \( R \). The field strength, \( B_{\text{sat}} \), at this point is related to the flux \( \Phi \), initial field strength \( B_0 \) and radius \( R_\text{sat} \) by \( B_{\text{sat}} = \Phi/(\pi R_\text{sat}^2) \), where \( R_\text{sat} = (R_0 A_0/\Phi)^{1/3} \) is the ring radius of the fat flux tube. For a 10^8 G flux ring with 10^18 Mx and an initial radius 10^16 cm, this “fat flux blob” has a field strength of 316 G and a radius of 3 \times 10^7 cm. The O-type neutral point is not susceptible to any type of fast reconnection; left undisturbed it will decay on the ohmic time scale, \( \tau_{\text{ohmic}} = d^2/\eta \approx 3 \times 10^3 \text{ yr} \) for \( \eta = 10^6 \text{ cm}^2 \text{ s}^{-1} \).

Finally, we note that if the flow field is not zero, as assumed above, but that it changes only slightly over the region sampled by the flux ring, then the flux ring will still collapse, while it is being advected at the average flow velocity.

3.2. The Critical and Asymptotic Length Scales

Next, we consider a flux ring in a more complex flow. One can develop some understanding for how the ring will evolve by examining how the aerodynamic drag and magnetic tension forces vary as the tube stretches or contracts. In this exercise, there are two invariants: the mass in the tube \( M = \rho AL \) and the flux \( \Phi = BA \). For flux tubes in incompressible fluids, the magnetic field is linearly dependent on the total length of the tube, \( B(L) = B_0 L/L_0 \) (eq. [4]), where \( B_0 \) and \( L_0 \) are the initial values of the field strength and length. We assume that there is a relationship between the total length \( L \) of the tube and the quantity \( R(L) \), which measures the characteristic minimum “radius of curvature” describing the tightest bends in the flux tube. We also assume that there is a similar relationship between \( L \) and the characteristic maximum flow velocity difference \( V(L) \) encountered by two different points on the flux tube. To determine whether a flux ring of a given length \( L \) will stretch or contract, we compare the drag and tension forces. The drag force can be approximated by \( D(L) = C_d V^2(L)/[\text{v/\Phi/(B(L))}]^{1/2} \); the tension force by \( T(L) = B^2(L)/[4\pi R(L)] \).

If \( D(L) > T(L) \), then the tube will expand, whereas if \( D(L) < T(L) \) the tube will contract. When \( D(L) = T(L) \) the loop is in equilibrium; from the preceding it is clear that an equilibrium will be stable if \( dT(L)/dL > dD(L)/dL \) and unstable if \( dT(L)/dL < dD(L)/dL \). We now proceed to estimate behaviors for \( V(L) \) and \( R(L) \) (and hence \( D(L) \) and \( T(L) \)) for a flow field with a single characteristic size scale \( L \) and velocity amplitude \( V_\circ \). The comparison of the tension and drag terms in the fibril evolution equation (1) is not new. Parker (1982b) equated these terms to estimate the slip velocity of a fibril through a fluid in its derivation of the fibril version of the mean field equations.
For any realistic subsonic flow, the flow field will be everywhere continuous. Thus the velocity difference between two points will generally vanish linearly with the separation distance. Therefore, for loops with $L \ll l$, we expect $V(L) \propto L/l$; for a circular loop with diameter $d$, one might expect $V(L) \propto V_m d/l = V_m L/(\pi l)$. For a loop which is sufficiently long that it traverses many flow size scales ($L \gg l$), on the other hand, the expected maximum velocity difference will be the characteristic amplitude of the flow, $V_m$. We can incorporate both limiting behaviors and adopt a continuous transition between them by assuming $V(L) = V_m \tanh [L/(\pi l)]$.

In § 3.1, we argued that if the flow field is nearly uniform in the region of a closed flux tube, the tube will assume the form of a shrinking, circular ring. Thus the radius of curvature $R(L)$ might be expected to vary linearly with $L$ [i.e., $R(L) \approx L/(2\pi n)$] when $L \ll l$. However, if $L \gg l$, and the tube snakes its way through the flow field in some complex manner, we would expect that the tightest radius of curvature would be roughly equal to the flow size scale itself scale itself. $l$. Both of these limits are satisfied by assuming $R(L) = l \tanh [L/(2\pi n)]$.

Evaluating the tension and drag forces with the above expressions for $R(L)$ and $V(L)$ we find

$$T(L) = \frac{B^2_0}{4\pi l} \left( \frac{L}{l_0} \right)^2 \frac{1}{\tanh \left[ L/(2\pi n) \right]},$$

and

$$D(L) = \frac{C_d \rho V^2_m}{\sqrt{\pi A_0}} \left( \frac{L}{l_0} \right)^{1/2} \tanh \left[ \frac{L}{(\pi l)} \right],$$

where $B_0$ is the initial field strength, $L_0$ the initial length, and $A_0 = \Phi/B_0$ the initial cross-sectional area. We can find the approximate length $L$ of equilibrium flux tube solutions by balancing the drag and tension forces. Doing this yields the following equation for equilibrium values of $L$:

$$\left( \frac{L}{L_0} \right)^{3/2} = \frac{l}{\lambda} \tanh \left[ \frac{L}{(\pi l)} \right] \tanh \left[ \frac{L}{(2\pi n)} \right],$$

where

$$\lambda = \frac{B^2_0 A^{1/2}}{4\pi^{1/2} C_d \rho V^2_m},$$

is the radius of curvature for a flux tube in equilibrium with $L = L_0 \gg nl$.

Now consider two limiting forms of equation (13). For large values of $x$, $\tanh (x) \to 1$, so in the limit of large $L (L \gg 2\pi n)$ the following equilibrium is reached:

$$L_{as} = \left( \frac{1}{\lambda} \right)^{2/3} L_0.$$  \hspace{1cm} (14)

This equilibrium is stable, because in the limit of large $L$ the tension varies as $L^2$ while the drag varies as $L^{1/2}$ and thus $dT(L)/dL > dD(L)/dL$. $L_{as}$ is the asymptotic limit of the fibril length. If $\lambda = l$ the tube length will be unchanged by the flow.

For small values of $x$, $\tanh (x) \to x$, so in the limit of small $L (L \ll nl)$ one can solve equation (12) to find a different, smaller equilibrium length of the loop:

$$L_c = \left( \frac{2\pi^2 l^2 \lambda^2}{l_0^2} \right)^{2/3} L_0.$$  \hspace{1cm} (15)

In this case, the equilibrium is unstable: For small values of $L$, the tension varies as $L$, while the drag varies as $L^{5/2}$, and therefore $dT(L)/dL < dD(L)/dL$. We denote that equilibrium loop length $L_c$ as the "critical length.''

From the existence of the two equilibrium solutions, one stable and one unstable, we can draw some rather general conclusions: for loops with $L < L_c$, the only possible fate is a collapse driven by magnetic tension, since the tension force will dominate the stretching driven by shearing motions; for loops with $L > L_c$, on the other hand, expansion driven by shearing motions overcomes the tension force and the loop continues to lengthen. Eventually, as the loop stretches, its length will approach $L_{as}$. Because this equilibrium is stable, the loop will then tend to remain near this equilibrium, even if perturbations push the tube away from equilibrium. For a loop which happens to have a length near the unstable equilibrium value $L_c$, and small perturbation could push it either into the collapsing or expanding regime. The overall situation is illustrated in Figure 1, where we plot the amplitudes of the drag and tension forces from equations (10) and (11) as a function of the flux tube length $L$; at the top of the figure we have indicated with arrows the direction of loop evolution for various ranges of $L$. The physical parameters $B_0$, $l_0$, and $\Phi$ are chosen to coincide with the "model I" set of numerical simulations described in § 4.

It is important to note that the magnetic field strength achieved by the asymptotic solution is independent of the initial length $L_0$ and initial field strength $B_0$, and depends only on parameters describing the flow field ($\rho$, $V_m$, and $l$), and the total magnetic flux $\Phi$. We can recast equation (14) in terms of the expected field strength of expanded loops using the linear relation between $L$ and $B$ resulting in

$$B_{as} = B_0 \left( \lambda \right)^{2/3};$$

$$= 3.69 \frac{C_d^{2/3} \rho^{2/3} V_{m}^{2/3} l_0^{4/3}}{\Phi^{1/3}}.$$  \hspace{1cm} (16)

Notice that the second form of equation (16) could have been derived from equation (13) with $\lambda = l$ and $B_0 = B_{as}$.

In a similar way we can recast the critical length equation (15) in terms of the magnetic field strength:

$$B_c = \frac{B_0(2\pi^2 l^2 \lambda^2)^{2/3}}{l_0^2};$$

$$= 4.25 \frac{B_0^{4/3} \Phi^{1/3} \rho^{2/3} V_{m}^{2/3}}{l_0^{4/3}}.$$  \hspace{1cm} (17)

We shall see in the numerical simulations described in § 4 both of the qualitative behaviors predicted by this simple model: two different evolutionary paths, collapse or expansion to a stable state. The existence of these different evolutionary paths for magnetic fibrils has not been, to our knowledge, previously identified. Furthermore, the quantitative prediction of the asymptotic field strength in equation (16) is well supported by the simulations. However, the value of the critical field strength $B_c$ is not well predicted by our simple model (eq. [17] predicts too small a value compared to that we estimate from the numerical simulations); this will be discussed further in § 4.

4. RESULTS OF THE NUMERICAL EXPERIMENTS

To test the predictions of § 3, and to gain additional insight into the behavior of fibril magnetic fields in a vigorous flow field, we consider the evolution of magnetic flux loops embedded in a steady, spatially complex flow with a single predomi-
nant length scale. We choose the “ABC” flow (Arnold 1965; Dombre et al. 1986) because it is a realizable flow, it is a solution of the Euler equations, it has regions of chaotic streamlines, and it is a Beltrami flow (the vorticity is parallel to the velocity). Because of these properties, the ABC flow is widely used in the studies of dynamos operating in the limit of large magnetic Reynolds number, the so-called fast dynamos (see Childress 1979; Galloway & Frisch, 1984, 1986; Gilbert 1991). The presence of only a single length scale in the ABC flow is computationally convenient because there are only a few free parameters.

The ABC velocity field is defined by the following equations:

\[ V_x = V_m [B \cos(y/l) + C \sin(z/l)] \]  \hspace{1cm} (18a)

\[ V_y = V_m [C \cos(z/l) + A \sin(x/l)] \]  \hspace{1cm} (18b)

\[ V_z = V_m [A \cos(x/l) + B \sin(y/l)] \]  \hspace{1cm} (18c)

For all of the cases discussed in this paper we choose \( A = B = C = 1 \), \( l = 10^{10} \) cm, and \( V_m = 10^4 \) cm s\(^{-1}\), being values that are roughly representative of what one might expect to find in the solar convection zone.

To keep the number of free parameters in our simulations as small as possible, we restrict our initial flux ring configurations to be circles of radius \( R_0 \) oriented parallel to the \( y-z \) plane. Each ring has an initial field strength \( B_0 \), a total magnetic flux \( \Phi \), and an initial cross-sectional area \( A_0 = \Phi B_0 \). For a fixed set of \( \Phi, R_0, \) and \( B_0 \) we evolve 50 loops with different initial positions. The initial center points are chosen from a random distribution of points within the cube bounded by \( x, y, z \) between 0 and \( \pi l \). Each loop samples different parts of the ABC flow and therefore evolves in different ways. We denote each set of 50 simulations corresponding to fixed values of \( \Phi, B_0, \) and \( R_0 \) as a “model.” While the ABC flow is periodic in all spatial directions, in the present calculations we consider the flux rings to be in an infinite domain \( [-\infty \leq x, y, z \leq \infty] \).

Table 1 gives a summary of the six different models we have run (the same 50 initial ring center positions are used in each of the models). The initial conditions of the flux rings (shown in cols. [2]–[4]) are chosen to be representative of flux rings near the base of the convection zone. However, numerical experi- ence has also played a role. The flux rings considered here are truly of global size. Smaller flux rings, with weaker field strengths, can be modeled, but the dramatic increases in length expected for such calculations introduce computational difficulties as discussed below. Column (6) contains the average field strength of the evolved solutions \( \langle B \rangle \); column (5) gives the predicted asymptotic field strength \( B_m \) from equation (16).

Column (7) gives the average minimum radius of curvature of the evolved solutions. Column (8) contains the number of solutions (out of 50) that were successfully evolved to \( t = 3 \times 10^7 \) s (about 1 yr); only these are used in computing the averaged quantities in columns (6) and (7). The numerical simulations cease if the radius of curvature becomes singular. In model 6 this occurs because all of the solutions, save one, collapse in a finite time. For models 1, 2, and 5, singular radii can occur when the tube passes through itself or when the flux ring is stretched into a very thin cigar shape by the ABC flow. A more realistic, turbulent flow field together with the implementation of a crossing detection and reconnection algorithm may eliminate these singularities. There were no singularities in models 3 and 4 because the large value of the flux and field strength make these solutions relatively difficult to stretch, and as a result there is less chance of the field becoming singular. Column (9) shows the collapse time \( \tau \) (§ 3.1) for a flux ring with each model’s initial conditions, computed for a fluid at rest.

The models shown can be grouped in several ways to see how the variation of different parameters affects the solutions. In Figure 2 we show the time series of the ring length for each of the solutions in models 1, 2, and 3. These models have the same initial field strength and size, but different flux and as a result different initial cross-sectional areas. All other parameters are fixed. The morphology of the evolution is similar for all three models, although the tubes with the smallest flux are more readily stretched than the thicker flux tubes. The reason for this is that the smaller flux (thinner) fibrils have a larger
TABLE 1
SUMMARY OF SOLUTIONS

<table>
<thead>
<tr>
<th>Model</th>
<th>(10^{18} Mx)</th>
<th>(10^{15} cm)</th>
<th>(10^5 G)</th>
<th>(10^5 G)</th>
<th>(10^5 G)</th>
<th>(10^3 cm)</th>
<th>Number</th>
<th>t* (yr)</th>
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<td>3.2</td>
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NOTE.—For each model 50 flux rings with different initial positions were evolved for 3 \times 10^7 s.

* Flux.

\( R \): Initial ring radius.

\( B \): Initial field strength.

\( B' \): Asymptotic field strength given by eq. (16).

\( \langle B \rangle \): Average final field strength.

\( R' \): Average minimum radius of curvature.

\( N \): Number of flux rings that were successfully evolved to 3 \times 10^7.

\( t' \): Collapse time given by eq. (9).

Fig. 2.—Time evolution of models with different flux. These solutions have the same initial radius and field strength: \( R_0 = 3.2 \times 10^{15} \) cm and \( B_0 = 10^5 \) G. \( B_{\infty} \) is plotted as a horizontal line.
surface to volume ratio, meaning that the aerodynamic drag force per unit volume is larger for these thin fibrils than for the thick (large flux) fibrils; the tension force per unit volume, on the other hand, is independent of the cross-sectional area (we assume that the flux tubes always have circular cross sections). For comparison with the predictions of § 3.2, the asymptotic field strength from equation (16) is shown as a horizontal line in each of plots.

Table 1 shows that the average final field strength $\bar{B}_f$ is generally consistent with $B_{\infty}$, which is remarkable given the approximate nature of the assumptions used to derive equation (16). There are also systematic differences, however, as can be seen by comparing the $B_{\infty}$ lines (horizontal) in Figure 2 with the set of simulations in each of the three panels. As the flux increases from $10^{18}$ to $10^{19}$ Mx, the simulations change from being below $B_{\infty}$ to being distributed about $B_{\infty}$. Why is this? The answer may be found by examining some of the morphological differences between models 1, 2, and 3. Because model 1 has a small value of $\Phi$, the evolved flux tubes experience a much greater increase in their length than do the model 3 tubes (a factor of 10 versus a factor of 5). A result of this is that the model 1 tubes, being more deformed, generally have a slightly smaller minimum radius of curvature than do the model 3 tubes (see col. [7], Table 1). In our derivation of equations (16), however, we have assumed that in the asymptotic limit the radius of curvature, which determines the amplitude of the tension force, is equal to the length scale $l$ of the flow field, independent of $\Phi$. It is the small increase in the minimum radius of curvature with $\Phi$ that causes the systematic change with $\Phi$ in the difference $B_{\infty}$ and $\bar{B}_f$ seen in Table 1 and Figure 2. To illustrate this effect more clearly, we compute the distribution of the final magnetic field strength achieved versus the minimum radius of curvature for each simulation. Using equation (16) as a guide, we factor out the explicit $\Phi^{-1/3}$ dependence on flux of the expected field strength. Figure 3 shows the dependence of $B_f \times (\Phi/10^{18} \text{ Mx})^{1/3}$ on the minimum radius of curvature for the solutions in models 1, 2, 3, and 5 that were successfully evolved for $3 \times 10^7$ s. Although there is some scatter, the figure clearly indicates that the dependence of $B_f(\Phi/10^{18} \text{ Mx})^{1/3}$ on $l_{\infty}$ is consistent with the $l^{2/3}$ power law (solid line) expected from equation (16). Further, the model 3 solutions (triangles) do in fact have larger minimum radii of curvature than do the model 2 solutions (asterisks) and the model 1 solutions (diamonds), as noted above. The model 5 solutions (plus sign) are discussed below.

In Figure 4 we show the effect of varying the initial radius on the evolution of the flux rings. For these models, the flux is fixed at $\Phi = 10^{18}$ Mx, and the initial field strength is $B_0 = 10^5$ G. Shown are the simulations of 6, 3, and 4, with initial flux ring radii of 1, 3.16, and $10 \times 10^{16}$ cm, respectively. Model 6 has the smallest initial radius, and in this case all of the flux rings collapse in a finite time, with one exception. This collapse is consistent with the behavior of loops less than the critical length, as discussed in § 3 and illustrated schematically in Figure 1. Nearly all of the tubes find themselves in a situation where the tension exceeds drag force and thus the tubes collapse. Notice that the flux tubes continue to collapse even when the field strength drops below the equipartition value, $B_{\infty}[B_{\infty}^2/(8x) = 1/2 \rho V^2 Z]$. For these solutions, the equipartition strength provides neither an upper nor a lower limit. Note also that the collapsing solutions collapse within a time which is consistent the time scale $\tau$ predicted in § 3.1 (see Table 1 for estimates of $\tau$ for each model). When the initial radius is larger, as in models 3 and 4, none of the flux rings collapse; they all evolve to a stable solution with an amplitude well represented by the asymptotic field strength (eq. [16]). This is just what we expect from § 3: Model 6 illustrates loop behavior below the critical length, while models 3 and 4 demonstrate loop behavior above the critical length. While the analysis of § 3 correctly predicts the qualitative behavior, it does a poor job of predicting the actual value of the critical length. The critical magnetic field strength $B_c$ from equation (17) is plotted for model 6 as a

![Graph](https://example.com/graph.png)

**Fig. 3.—**The dependence of the normalized field strength $[B_f(\Phi/10^{18} \text{ Mx})^{1/3}]$ on the minimum radius of curvature for models 1, 2, 3, and 5. The solid line has the form of a power law: $B_f(\Phi/10^{18} \text{ Mx}) \propto r_{\infty}^{2/3}$ with arbitrary normalization. The dashed line shows the asymptotic field for these normalized solutions. The data are consistent with the $(\frac{1}{3})$ power law. The asymptotic field strength, though slightly larger than the densest cluster of points, is a reasonable approximation for most of the points.
horizontal line ($B_c$ for the other models are less than $10^7$ G). It is clear that the amplitude predicted by equation (17) is more than an order of magnitude too small. The discrepancy is not surprising since in § 3 we made simplifying assumptions about the shape (i.e., a circle) for loops near the critical length; the numerical simulations show that these loops have a long, elliptical shape with their minimum radius of curvature considerably shorter than $L/(2\pi)$. As a result the tension force in the simulations is substantially larger than that estimated in § 3.

Figure 4 also shows that the distribution of solutions around the asymptotic field strength is smaller for the initially larger rings (model 4) than the initially smaller rings (model 3). The large rings sample more of the steady flow and so have smaller variations from one initial position to another, as compared to the smaller rings. This effect is also noticeable in the large variation in evolution seen in model 6, where some rings collapsed steadily, while others which initially grew were advected into a region of small local shear and then collapsed. One ring was sufficiently lucky to move through a region of high shears, into the expanding regime; the ring then attained a field strength consistent with $B_{eq}$.

Figure 5 shows the effect of varying the initial field strength on the evolution of the flux rings. Models 6 and 5 have the same flux, $\Phi = 10^{19}$ Mx, and initial radius $R_0 = l = 10^{10}$ cm, but model 5 has a smaller initial field strength: $B_0 = 10^4$ G. There are no collapsing solutions in model 5, but the range of variation is much larger than we have seen in the other models that evolve to the steady state. The solutions no longer collapse because the tension force has been reduced for the lower field strength rings, and the local shear is always large enough to stretch the ring. However, the final evolution is sensitive to the initial ring position. For the first time, we see some solutions that appear to remain time dependent (the lowest oscillatory curves in model 5). Close examination of these solutions reveals that the rings are confined to small jets in the ABC flow. The shear across these jets is smaller than the maximum shear in the flow, so that the final field strength is lower. As the rings are advected in the jets, the local shear flow encountered by the ring changes periodically, causing the ring length to oscillate. The steady solutions for rings with model 5 initial conditions are bounded below by the equipartition strength and above by the asymptotic field strength. In Figure 3 the model 5 solutions are shown as plus signs. While these solutions have a wide range in $l_{\text{min}}$ and $B_{f} \times (\Phi/10^{14})^{1/3}$, they are
still consistent with the $l^{2/3}$ power law predicted by our scaling law (eq. [16]). The three points near the bottom of Figure 3 are the time-dependent solutions. It appears that they do not obey the scaling law of §3.

Figure 6 shows the evolved steady state of a flux ring with Model 1 initial conditions. The thick tube is the flux ring plotted with an artificially large cross section so that it is easier to view. The thin tubes show the trajectories of particles in the ABC flow. The particles are chosen to initially lie in a plane around the tube; the converging paths of the particles show the structure of the ABC flow. Notice how the particles flow along the tube on the straight portion and then diverge at the corners. The field amplification stops when the tension force balances the aerodynamic drag force everywhere. The tension force is largest at the tightest bends in the tube; this is where the small jets in the ABC flow are pulling hardest on the flux tube. The size of the tightest bends can be compared with the cube in the lower right-hand corner of the figure; the cube has a side equal to the characteristic flow length $l = 10^{10}$ cm. One can see that the smallest radii of curvature are somewhat smaller than $l$. The lengths of the $x$, $y$, and $z$ axes in Figure 6 are $1.7 \times 10^{11}$ cm.

Finally, we note that occasionally during the evolution of one of the more complex flux ring configurations, the ring will cross itself. In the present code the evolution proceeds without recognizing that an unphysical process has occurred. An important modification to the present code will allow the tubes to reconnect and change their topology. Under such conditions, the evolution may be quite different. Consider the long thin loop above the cube in Figure 6. The streamlines show how this loop is stretched and twisted by the ABC flow. If this loop crosses itself and reconnects we will have two flux ropes: one large and one small. Each flux rope will have the same field strength and contain the same flux, but their subsequent evolution will be substantially different. In the case where the tube pinches off when it has nearly reached its steady state configuration, the small loop in the jet indicated by the streamlines will be advected along and will collapse until its field strength is consistent (via eq. [16]) with the local shear in the jet. If it happens that the field strength of the small flux ring is too large (or the size of the ring formed by reconnection too small), the ring will collapse indefinitely. The evolution of the larger ring will be less dramatic; as we have supposed that the reconnection has taken place when the flux ring has nearly reached its steady state configuration, the field strength of the large ring is nearly in equilibrium with the large-scale flow, so the large flux ring will adjust to a new steady equilibrium with about the same magnetic field strength, but a shorter overall length.

5. SUMMARY

In this paper, we have examined the evolution of individual closed magnetic flux tubes embedded in an "ABC" flow field; velocity amplitudes and length scales of the flow were chosen to be roughly consistent with what one might find in the lower part of the solar convection zone. The flux tubes are assumed to couple to the flow field through an aerodynamic drag force, a magnetic tension force opposes the tendency of the flow field to stretch and distort the flux tube configuration indefinitely. We studied flux tube evolution in two different ways, first, through the use of a simple analytical scaling treatment of the forces, and second, through a series of numerical simulations of flux tube motion. Both approaches have led us to conclude the following:

1. There is a critical length scale which separates two different types of behavior—collapse of flux tubes, or flux tube amplification. Flux tubes which collapse seem to do so on the time scale $\tau$ estimated in §3.1.
2. Flux tubes which are stretched (amplified) reach an asymptotic field strength $B_{\infty}$ which depends only on the parameters of the flow field and the total flux in the tube. The quantity $B_{\infty}$ can in some cases be far greater than the equipartition field strength.

The role of turbulent flows with multiple length scales needs to be studied in some detail. The present work suggests that small-scale flows will not have a strong effect on large-scale, strong fields. However, this needs to be demonstrated. Furthermore, the fate of our collapsing solutions will be strongly influenced by the presence of small scale flows. We are in the process of modifying our code to study the evolution of isolated flux rings in a forced-damped fluid system (for example convection, and homogeneous turbulence).

The role of turbulent diffusion in the operation of the solar cycle dynamo has recently been called into question. In the classic kinematic solar dynamo models small-scale turbulent motions are assumed to lead to both turbulent diffusion of the large-scale fields, and to an "$\alpha$ effect" which generates a large-scale poloidal field from the large-scale toroidal field. In the case of diffuse fields, Cattaneo & Vainshtein (1991) and Vainshtein & Cattaneo (1992) have now demonstrated that Lorentz force feedback completely inhibits turbulent diffusion for all but the weakest fields. As a result of this work, Vainshtein, Parker, & Rosner (1993), have modified the mean field dynamo equations to account for the limitations on turbulent diffusion imposed on the system by feedbacks of the Lorentz force on the velocity field. For a strong toroidal field, turbulent mixing will not be suppressed in the plane perpendicular to the field. Two-dimensional turbulent mixing could then help diffuse the toroidal field and create loops which could be rotated into the meridional plane. These loops are used by the authors to maintain the weak poloidal field. Our results strongly suggest that their model will have serious problems in the fibril limit, because the loops formed by reconnecting oppositely directed toroidal loops will have a strong tendency to collapse if the toroidal field is strong. The collapse of these poloidal loops will quench the propagating dynamo wave.

We have found that under the assumption that magnetic fields are of fibril form (1) long flux tubes are unable to be bent on size scales significantly shorter than the dominant flow length scale, and (2) short flux tubes collapse and become dynamically unimportant. These two results together imply that turbulent motions on small scales are unimportant in determining the evolution of fibril magnetic fields. We believe that any model of the solar cycle dynamo which involves fibril fields must accommodate these constraints.

Recently, there have been a number of notable successes in using models of flux tube dynamics to explain many observed properties of solar active regions (D'Silva & Choudhuri 1991, 1992; FFD). These successes lead us to conclude that the fundamental magnetic components of the solar active regions are in fact long (large-scale), deeply rooted, magnetic flux tubes. We also know from the Hale polarity laws and the polarity reversals from one cycle to the next that the magnetic topology of the solar cycle fields must change. A challenge for the future will be to develop a model of the solar cycle dynamo from the dynamics of magnetic fibrils. This will require the development of models which allow magnetic flux tubes to collide and reconnect, which our present model does not do. We believe
that understanding the dynamic interactions of fibril fields will further our understanding of how the solar cycle works.

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