INVERSION FOR BACKGROUND INHOMOGENEITY FROM PHASE DISTORTIONS OF ONE-DIMENSIONAL WAVE TRAINS

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ABSTRACT We examine a possible means for detecting large-scale structure in the solar convection zone which involves analysing the oscillation waveform and its Hilbert transform, rather than the eigenfrequencies of global modes. Using a simple one-dimensional model, we demonstrate strategies by which the deleterious effects of mode beating and noise can be overcome.

INTRODUCTION

An important goal of helioseismology is to detect lateral structure in the solar convection zone, such as convective flow on horizontal scales much larger than supergranules. The direct detection of such flow by tracer and Doppler measurements has proven difficult, suggesting that if it exists it is confined beneath the photosphere. However, subsurface convective flow would influence the propagation of acoustic waves, since bulk motion advects the waves, and temperature fluctuations produce horizontal variations in sound speed.

THE HILBERT-TRANSFORM TECHNIQUE

Our objective is to develop a means by which the form of a one-dimensional wave train can be analysed to reveal underlying flow and variation in sound speed. Ideally, we would like to follow the propagation of a single wave. However, without observational coverage of the entire solar surface, isolating individual modes would require very long strings of data, and may not be possible, even in principle. We therefore consider the propagation of wave
trains comprised of several waves having nearly identical temporal frequencies, and with closely spaced but differing values of the horizontal wave number, as did Gough, Merryfield and Toomre (1991). Such data might be obtained by observing a strip of the sun’s surface aligned with the solar equator. The Doppler measurements would be averaged in latitude to accentuate waves traveling nearly parallel to the equator. Then they would be filtered in the Fourier domain to isolate modes of like order \( n \) with frequencies close to some chosen value \( \omega \). We estimate that in practice about 10 modes might be present with appreciable amplitude in such a filtered wave train.

We demonstrate how the data would be analysed by considering initially a one-dimensional wave form \( \Psi(x; \omega) \) at a single instant in time, where \( x \) is a horizontal coordinate. Even though \( \Psi \) may consist of several superposed simple waves, we represent it as a single wave

\[
\Psi = A \cos \phi \tag{1}
\]

having variable amplitude \( A(x; \omega) \) and phase \( \phi(x; \omega) \).

The Hilbert transform is defined by

\[
\hat{\Psi} = \pi^{-1}P \int_{-\infty}^{\infty} \frac{\Psi(x)}{x - \xi} \, d\xi, \tag{2}
\]

where \( P \) denotes principal part, and may be computed by means of a simple operation involving the Fourier transform of \( \Psi \) (Bracewell 1978). From \( \Psi \) and its Hilbert transform we may compute

\[
\alpha = \sqrt{\Psi^2 + \hat{\Psi}^2}, \tag{3}
\]

\[
\psi = \tan^{-1}(-\Psi/\hat{\Psi}) \tag{4}
\]

as continuous functions of \( x \), which we identify with \( A \) and \( \phi \) respectively.

The wave form \( \Psi \) is presumed to obey the one-dimensional Helmholtz equation

\[
\frac{d^2\Psi}{dx^2} + \kappa^2(x)\Psi = 0, \tag{5}
\]

where \( \kappa(x) \) depends on the bulk flow velocity and sound speed; from it, these quantities can in principle be determined. A JWKB analysis indicates that \( \kappa(x) \) can be expressed in terms of \( A \) and \( \phi \) via

\[
\kappa(x) \simeq \frac{\delta \phi}{\delta x} \simeq A^{-2}(\Psi \hat{\Psi}' - \hat{\Psi} \Psi'), \tag{6}
\]

where the prime denotes differentiation with respect to \( x \), provided that \( \kappa^{-1} \) is much less than the spatial scale on which \( \kappa \) varies. We shall use equation (6) in inverting for \( \kappa \).

**NUMERICAL EXPERIMENT**

We examine how the Hilbert-transform technique might be applied to SOI or GONG data using the idealized model described above. We consider an
Fig. 1. (a) Imposed variation in $\kappa$ (for median $m$), (b) typical wave train.

Fig. 2. Continuous lines are: (a) raw inversion of the data in Fig. 1b, (b) inversion with mode beating effects removed. In both panels the dashed line is the actual $\kappa$.

observing window 1024 pixels across encompassing 1/3 of the solar circumference. The data are assumed to have been interpolated onto a grid that is equally spaced in longitude. We examine the propagation of a train of 9 waves, having azimuthal orders $m = 46, 47, ..., 54$ with random amplitudes and random initial phases, and like $\omega$. We suppose that $\kappa$, which depends on $m$, is independent of time, and impose on it a specific variation whose relative amplitude is 1 per cent (Figure 1a). The constituent wave forms are computed by solving equation (5).

**REMOVING INTERFERENCE EFFECTS**

The envelope of a typical wave form, illustrated in Figure 1b, varies substantially. The variation in $\kappa$ causes some slight amplitude variation, but that is overwhelmed by modulation due to interference (beating). Beating also contaminates the phase measurements, which we are using to diagnose $\kappa(x)$, and which must therefore be purged before useful results can be obtained.

Figure 2a shows a raw inversion for $\kappa(x)$ from the wave form of Figure 1b using equation (6). The outcome does not resemble the actual $\kappa$ (dashed line), being dominated by a spurious component due to mode beating. However,
Fig. 3. Inversions for data with noise: (a) unfiltered, (b) filtered.

one may use the fact that the spurious component of the inversion propagates nearly without distortion at the group velocity $v_g$ of the waves. We remove the contamination by obtaining a time series of inversions and then evaluating $\kappa$ at a point that travels across the observing domain at velocity $v_g$. The decontaminated inversion, shown in Figure 2b, quite accurately represents $\kappa$ to within an additive constant.

EFFECTS OF NOISE

We have repeated the computation after adding to each pixel random noise with relative rms amplitude 0.001. An inversion corresponding to Figure 2b is shown in Figure 3a, and is clearly quite unsatisfactory.

The inversion is improved by Fourier filtering the wave train. We apply a low-pass filter that eliminates variations in the signal characterized by $m > 250$. The resulting inversion (Fig. 3b) is corrupted somewhat by the noise. However, the errors in the inversion are now relatively small, and the gross variation of $\kappa$ is once again evident.

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