A MODIFIED BOHR-SOMMERFELD CONDITION FOR P-MODES

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INTRODUCTION

The approximate result

\[ \int_{z_1}^{z_2} \kappa(z) \, dz \approx (n + \alpha)\pi, \]  

the Bohr-Sommerfeld condition, has already shown its use in the study of p-modes (Christensen-Dalsgaard et al 1985). The quantity \( n \) is an integer, describing the overtone number of the mode, and \( \alpha \) is a phase constant, whose precise value depends upon the physics of the boundary, the chromosphere. When the area of interest is the interior of the Sun, \( \alpha \) is usually taken to be \(-1/2\). This approach, however, allows us to say nothing about frequency shifts due to changes in the state of the chromosphere.

Another branch of work (Campbell and Roberts 1989, Evans and Roberts 1990, 1991, 1992) has progressed by finding analytical solutions to the wave equation in the interior and chromosphere for various simple models, and producing a dispersion relation by matching these solutions across an interface at the photosphere. This approach has had some successes, such as the prediction of p-mode frequency changes over the solar cycle as a consequence of changes in magnetism (Libbrecht and Woodard 1990, 1991). However, the requirement that the model chosen provides an analytical solution greatly restricts the range of models which can be studied.

A further approach (Wright and Thompson 1992) has been to obtain a numerical solution in the chromosphere, and to provide from this a quantity \( Z \), proportional to the ratio of the two quantities that are continuous across the interface. This value is then used through a perturbation method to determine p-mode frequency shifts.

Our aim here is to combine elements of each of these approaches, formulating a method which will allow us to examine the effect on frequencies for general chromospheres through a modified Bohr-Sommerfeld condition.

DERIVATION

We consider a plane-parallel atmosphere, free of magnetic field and stratified by gravity acting in the positive \( z \)-direction. We consider linear perturbations of the form \( \mathbf{v} = (v_x(z), 0, v_z(z)) \exp[i(\omega t - k_x x)] \), for frequency \( \omega \) and horizontal wavenumber \( k_x \). Setting

\[ Q = \rho^{1/2} c_s^2 \nabla \cdot \mathbf{v}, \]  

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where $c_s(z)$ is the sound speed in the gas of density $\rho$, we find that motions of the plasma are governed by the differential equation

$$\frac{d^2Q}{dz^2} + \kappa^2(z)Q = 0,$$

(3)

where

$$\kappa^2(z) = \frac{\omega^2}{c_s^2} - \frac{1}{4H^2}(1 + 2H') - k_z^2 + \frac{k_x^2}{\omega^2} \left( \frac{g}{H} - \frac{g^2}{c_s^2} \right).$$

(4)

Here $H$ is the density scale height.

It is convenient to consider the case for which the buoyancy frequency is identically zero, thus removing the last term from equation (4). This allows us to focus on p-modes.

Examining the variation of $\kappa^2$ with depth near the solar surface, we find that there is always a zero of $\kappa^2$ at or near the surface. This is due to the increase in size of the acoustic cut-off term (the second term in equation (4)). There is another zero of $\kappa^2$ deep inside the Sun, where the increasing temperature acts to refract the wave back to the surface. Where we have two such zeros, or turning points, a cavity capable of containing p-modes exists between them.

This situation is amenable to WKB analysis (Nayfeh 1983). We proceed by finding the Langer solution around each turning point. A Langer solution has the advantage of being valid both close to and away from a turning point, although it is not valid near another turning point. If we assume that each of these solutions is evanescent at large distances from the cavity, we can derive the Bohr-Sommerfeld eigenvalue condition (1), with $\alpha = -1/2$.

We wish to consider the effect of a discontinuity near the upper turning point $z_1$; we thus allow the Langer solution around $z_1$ to remain general, and take

$$Q_1(z) \sim \left[ -\frac{t_1(z)}{\kappa^2(z)} \right]^{1/4} \{c_1 \text{Ai}(t_1) + d_1 \text{Bi}(t_1) \},$$

(5)

where $\text{Ai}$ and $\text{Bi}$ are Airy functions and

$$\frac{2}{3}(-t_1)^{3/2} = \int_{z_1}^{z} \kappa(s) \, ds.$$ 

(6)

The Langer solution valid around the lower turning point $z_2$ is given by

$$Q_2 \sim c_2 \left[ -\frac{t_2}{\kappa^2} \right]^{1/4} \text{Ai}(t_2),$$

(7)

where

$$\frac{2}{3}(-t_2)^{3/2} = \int_{z}^{z_2} \kappa(s) \, ds.$$ 

(8)

The Airy functions $\text{Ai}(t)$ and $\text{Bi}(t)$ have the property that for $t > 0$ their behaviour is exponential; this corresponds to the evanescent tail of p-modes outside the cavity. For $t < 0$, the Airy functions are oscillatory, representing the propagation of sound waves inside the cavity. We form an eigenvalue condition by matching the asymptotic forms for $Q_1$ and $Q_2$ inside the cavity.
Define an angle $\phi$ by writing $\tan \phi = d_1/c_1$. Then imposing $Q_1 \approx Q_2$ in the cavity leads to the two relations

$$(-1)^{n+1}c_1 \sec \phi = c_2$$

and

$$\int_{z_1}^{z_2} \kappa(z) \, dz \approx \left(n - \frac{1}{2}\right)\pi - \phi.$$  \hspace{1cm} (10)

Equation (10) is our modified Bohr-Sommerfeld condition. The effect of the boundary, and thus of the chromosphere, is expressed through the value of the phase angle $\phi$. In terms of the phase $\alpha$, we have $\alpha = -1/2 - \phi/\pi$; in general, $\phi$ depends upon the frequency $\omega$.

To evaluate $\phi$, we must consider in greater detail the boundary condition on the wave disturbance at the surface. We impose the boundary condition as follows. A solution of the wave equation in the chromosphere provides us with the value of a quantity $Z$, defined by

$$Z = -\frac{v_z}{\rho c_s^2 \nabla \cdot \mathbf{v}},$$

immediately above the interface; this quantity differs by a constant only from the quantity $\delta L$, the Lagrangian total pressure perturbation (Wright and Thompson 1992). Since $Z$ is continuous across the boundary, and since we may derive a functional relationship between $Q'/Q$ and $Z$, we can calculate the value of $Q'/Q$ immediately below the interface.

From the Langer solution (5) we derive the functional form of $Q_1'/Q_1$ near the boundary, and then equate the value of this function at $z = 0$ with its actual value as provided by the boundary condition, giving an expression for $\phi$. We obtain

$$\tan \phi = -\frac{X(0)\text{Ai}(t_1(0)) + \left[ -\frac{\kappa^2(0)}{t_1(0)} \right]^{1/2} \text{Ai}'(t_1(0))}{X(0)\text{Bi}(t_1(0)) + \left[ -\frac{\kappa^2(0)}{t_1(0)} \right]^{1/2} \text{Bi}'(t_1(0))},$$

where

$$X(0) = \frac{Q_1'}{Q_1}_{z=0^+} + \frac{1}{4t_1(0)} \left[ -\frac{\kappa^2(0)}{t_1(0)} \right]^{1/2} + \frac{1}{4} \left( \frac{\kappa^2}{4} \right)'.$$

The result (12) gives the value of $\phi$ to be used in equation (10).

APPLICATION AND DISCUSSION

We now apply the above method to a simple model. We model the solar interior by a linear polytrope, with a photospheric temperature of 4170$K$. The chromosphere is chosen to be the simplest possible: isothermal and non-magnetic.

We find that the p-mode frequencies obtained agree well with those found from an exact treatment of the same problem (Campbell and Roberts 1989). Of more interest is the effect on frequencies of changing the chromospheric temperature. To produce the frequency shifts depicted in Figure 1, we have raised the chromospheric temperature from 4170$K$ to 4400$K$. The degree $l$ of the mode has
been taken to be 50, and the order $n$ varies from 1 to 42. The dotted line represents the frequency shift obtained from an exact solution for the same profiles. Results from the two methods correspond to an accuracy of about 10-20%.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Frequency shift $\Delta \nu$ as a function of frequency $\nu (\equiv \omega/2\pi)$}
\end{figure}

Thanks are due to Rekha Jain for the exact results here used.

In conclusion, we have seen that the presence of a boundary alters the form of the Bohr-Sommerfeld condition, by allowing the existence of the Bi($t_1$) solution around the upper turning point. The boundary condition specifies the value of $Q'/Q$ at the boundary, and thus the relative amplitudes of the two upper Airy solutions.

This process is expressed through the amount of phase contained between the turning points. The existence of the Bi($t_1$) solution decreases the amount of phase contained; this corresponds to a non-zero $\phi$ in the modified Bohr-Sommerfeld condition.

We intend to extend this formulation to examine the effects on p-mode frequencies of non-isothermal, magnetic chromospheres.

REFERENCES