MAGNETOACOUSTIC-GRAVITY SURFACE WAVES

II. Uniform Magnetic Field

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Abstract. The linearized theory for the parallel propagation of magnetoacoustic-gravity surface waves is developed and a dispersion relation obtained for the case of an isothermal interface of a uniform horizontal magnetic field residing above a field-free medium. The transcendental dispersion relation is solved numerically for a range of parameters and the resulting dispersion curves and corresponding eigenfunctions plotted. As in the case of a uniform Alfvén speed (Paper I), the existence of the fast and slow magnetoacoustic-gravity surface modes and the $f$-mode (modified by the presence of the uniform magnetic field) is determined by the relative temperatures of the two media either side of the interface. If the lower field-free region is cooler than the upper magnetic atmosphere only the slow magnetoacoustic-gravity surface mode may propagate. In addition to these three surface modes we find higher harmonic-type trapped modes. The existence of these modes also depends on the temperatures either side of the interface. They propagate only when both the field-free region is warmer than the magnetic field region and the Alfvén speed is greater than the corresponding sound speed in the magnetic atmosphere.

1. Introduction

In Paper I (Miles and Roberts, 1992) we derived the dispersion relation for magnetoacoustic-gravity surface waves for the case of an isothermal interface with a uniform Alfvén speed residing above a field-free region. In the present paper we once again consider such an interface but now with the upper region permeated by a uniform, horizontal magnetic field. The Alfvén speed is then no longer a constant but increases exponentially with height at a rate half that of the exponentially decreasing density.

As in the uniform Alfvén speed case of Paper I, when the field-free region is warmer than the upper magnetic atmosphere we find that three surface modes may propagate on the interface. These modes are the fast and slow magnetoacoustic-gravity surface modes and the $f$-mode modified by the presence of the uniform magnetic field. However, in contrast to Paper I we also find harmonic modes, somewhat akin to the usual $p$-modes in helioseismology but here confined to the magnetic chromosphere. These modes again only occur when the field-free region is warmer than the magnetic medium. These are trapped modes resulting from the increasing Alfvén speed with height into the atmosphere.

Nye and Thomas (1976b) similarly found such trapped modes in their study of running penumbral waves, which they modelled as modes trapped by an increasing sound speed into the convection zone and by an increasing Alfvén speed into the penumbral atmosphere. We view the modes discussed by Nye and Thomas as $p$-modes influenced by the presence of a horizontal magnetic field (see also Evans and Roberts,
1990). These modes are absent in the case of zero gravity whereas the surface modes considered in this paper exist even in the absence of gravity and with no trapping (i.e., wave reflection or refraction) being required. The surface modes owe their existence solely to the presence of the discontinuity (or rapid variation) in the magnetic field.

The next section of the paper will deal with the analytical formulation of the problem using the linearized theory of ideal MHD to derive a dispersion relation. We then consider a numerical investigation of the dispersion relation.

2. The Dispersion Relation

2.1. The Equilibrium

Consider a single magnetic interface, situated at \( z = 0 \), in an atmosphere stratified by gravity acting in the negative \( z \)-direction. We assume an equilibrium state in which the temperature and magnetic field are given by

\[
T(z), B(z) = \begin{cases} 
T_0, & B_0, & z > 0, \\
T_e, & 0, & z < 0, 
\end{cases}
\]

(1)

where \( T_0, B_0, \) and \( T_e \) are constants. As in Paper I, we adopt the notation of denoting quantities above the interface (in \( z > 0 \)) by a subscript '0' and those below (in \( z < 0 \)) by a subscript 'e'. See Figure 1. In contrast to Paper I, the magnetic field in this case provides no support against gravity. The equilibrium therefore is one of hydrostatic

![Figure 1](https://example.com/figure1.png)  
Fig. 1. The equilibrium model of a single magnetic interface \( z = 0 \) in a stratified atmosphere with a uniform horizontal magnetic field in \( z > 0 \).
balance determined by
\[ \frac{dp(z)}{dz} = -\rho(z)g \]  
(2)
and the ideal gas law,
\[ p(z) = \frac{k_B}{m_{av}} \rho(z)T(z), \]  
(3)
where \( p(z) \) and \( \rho(z) \) are the gas pressure and density, \( k_B \) Boltzmann’s constant, and \( m_{av} \) the mean particle mass of the plasma.

Pressure balance at the interface \( z = 0 \) means that the total (magnetic plus gas) pressure above the interface must equal the gas pressure in the field-free region below. This requirement of pressure balance, together with the ideal gas law, leads to a relationship between the densities on either side of the interface: namely,
\[ \frac{c_e^2}{c_0^2} \frac{\rho_e(0-)}{\rho_0(0+)} = 1 + \frac{\gamma}{2} \beta, \]  
(4)
where \( \beta = c_0^2/v_A^2 \) is the squared ratio of the sound speed \( c_0 (= (\gamma p_0/\rho_0)^{1/2}) \) to the Alfvén speed \( v_A (= B_0/(\mu_0 \rho_0(0+))^{1/2}) \) at the base of the magnetic atmosphere, and \( c_e (= (\gamma p_e/\rho_e)^{1/2}) \) is the sound speed in the field-free region.

The density and Alfvén profiles are
\[ \rho(z), v_A(z) = \begin{cases} \rho_0 e^{-z/H_0}, & v_A e^{z/2H_0}, & z > 0, \\ \rho_e e^{-z/H_e}, & 0, & z < 0, \end{cases} \]  
(5)
where \( H_0 (= c_0^2/\gamma g) \) and \( H_e (= c_e^2/\gamma g) \) are the isothermal scale-heights above \( (z > 0) \) and below \( (z < 0) \) the interface, respectively.

2.2 THE MAGNETIC ATMOSPHERE

Two-dimensional, linear, isentropic perturbations of the form
\[ \mathbf{v}(x, z, t) = (v_x(z), 0, v_z(z)) e^{i(\omega t - k_x x)}, \]  
(6)
for frequency \( \omega \) and horizontal wavenumber \( k_x \), satisfy the second-order ordinary differential equation (Paper I; see also Adam, 1977; Small and Roberts, 1984; Roberts, 1985; cf. Goedbloed, 1971)
\[ \frac{d}{dz} \left\{ \frac{\rho(z)(c_s^2(z) + v_A^2(z))}{(\omega^2 - k_x^2 c_s^2(z))} \frac{dv_z}{dz} \right\} + \frac{\rho(z)}{(\omega^2 - k_x^2 c_s^2(z))} \frac{d}{dz} \left( \frac{\rho(z) c_s^2(z)}{(\omega^2 - k_x^2 c_s^2(z))} \right) v_z = 0, \]  
(7)
where \( c_s^2 = c_T^2 v_A^2/(c_s^2 + v_A^2) \) is the square of the cusp speed.
For the profile (5) the governing ordinary differential Equation (7) for the magnetic field region (i.e., $z > 0$) reduces to (Nye and Thomas, 1976a; Evans and Roberts, 1990; see also Miles and Roberts, 1991)

\[
\{c_0^2 \omega^2 + (\omega^2 - k_x^2 c_0^2) v_A^2 e^{z/H_0}\} \frac{d^2 v_z}{dz^2} - \frac{c_0^2 \omega^2}{H_0} \frac{dv_z}{dz} + \\
+ \{(\omega^2 - k_x^2 c_0^2) (\omega^2 - k_x^2 v_A^2 e^{z/H_0}) - g k_x^2 (g - c_0^2 / H_0)\} v_z = 0 .
\] (8)

This second-order differential equation with non-constant coefficients may be cast in the form of a hypergeometric differential equation by transforming the dependent ($v_z(z)$) and independent ($z$) variables according to (Adam, 1975; Nye and Thomas, 1976a)

\[
V_z = v_z e^{z K_0 / H_0}, \quad X = X_0 e^{-z / H_0},
\] (9)

where

\[
X_0 = \frac{\beta \Omega_0}{(K_0^2 - \Omega_0^2)}.
\] (10)

Here $\Omega_0 = H_0 \omega / c_0$ and $K_0 = H_0 k_x$ are the non-dimensional frequency and horizontal wavenumber, respectively.

Using the transformations (9), Equation (8) reduces to

\[
\left(\frac{X}{H_0}\right) e^{-z K_0 / H_0} \left\{X (1 - X) \frac{d^2 V_z}{dX^2} + [r - (p + q + 1)X] \frac{dV_z}{dX} - pq V_z \right\} = 0
\] (11)

and so

\[
X (1 - X) \frac{d^2 V_z}{dX^2} + [r - (p + q + 1)X] \frac{dV_z}{dX} - pq V_z = 0 ;
\] (12)

this is the hypergeometric differential equation (see, for example, Abramowitz and Stegun, 1965). The hypergeometric differential equation has regular singularities at $X = 0, 1, \text{ and } \infty$. For the specific problem we are interested in here the parameters of the hypergeometric differential Equation (12) satisfy

\[
p + q = r = 2K_0 + 1 ,
\] (13)

\[
p, q = \frac{1}{2} \pm K_0 \pm \frac{1}{2} \left\{1 - \left(\frac{\omega}{\omega_{ao}}\right)^2\right\} \pm 4K_0^2 \left[1 - \left(\frac{\omega_{go}}{\omega}\right)^2\right]^{1/2} ,
\] (14)

where $\omega_{ao} = c_0 / 2H_0$ and $\omega_{go} = (\gamma - 1)^{1/2} c_0 / \gamma H_0$ are the acoustic and buoyancy (Brunt–Väisälä) frequencies for the isothermal, magnetic medium. We note that if $\omega$ lies
in the range $\omega_{g_0} < \omega < \omega_{a_0}$, then the parameters $p$ and $q$ are purely real. However, in general $p$ and $q$ will be complex conjugates.

The non-trivial solution to Equation (11) is the general solution to the hypergeometric differential Equation (12), which in terms of the original variable $v_z(z)$ about $X = 0$ is given by (Abramowitz and Stegun, 1965)

$$v_z(z) = d_1 F(p, q; r; X) e^{-zK_0/H_0} +$$
$$+ d_2 X_0^{-2K_0 F(1 - p, 1 - q; 2 - r; X) e^{zK_0/H_0}, \quad z > 0,}$$

where $d_1$ and $d_2$ are arbitrary constants and $F$ is the hypergeometric function defined by the Gauss hypergeometric series

$$F(p, q; r; X) = 1 + \frac{pq}{r} X + \frac{p(p + 1)q(q + 1)}{r(r + 1)} \frac{X^2}{2!} + \cdots.$$  

(16)

The series (16) is absolutely convergent in $|X| < 1$. For $X \geq 1$ there is a logarithmic singularity in $v_z$ at the regular singularity $X = 1$ (Abramowitz and Stegun, 1965). The region $X \geq 1$ corresponds to

$$\frac{\beta \omega_0^2}{(K_0^2 - \omega_0^2)} \geq 1,$$

(17)

that is, to the region

$$c_T^2 \leq \frac{\omega^2}{k_x^2} \leq c_0^2.$$  

(18)

In this range of horizontal phase-speed $\omega/k_x$ the solution for $v_z$ is singular and so resonant absorption may occur, making resistive and viscous effects important. Provided the phase-speed lies outside the range given by (18) the solution for $v_z$ will be everywhere bounded. We note that the regular singularity at $X = \infty$ corresponds to $\omega^2/k_x^2 = c_0^2$ and is therefore included in the range (18).

2.3. The Field-Free Region

In the lower half-plane (i.e., in $z < 0$) there is no magnetic field and the governing ordinary differential Equation (7) reduces to one with constant coefficients:

$$\frac{\rho \omega^2}{(\omega^2 - k_x^2 c_e^2)} \left\{ \frac{d^2 v_z}{dz^2} - \frac{1}{H_e} \frac{dv_z}{dz} + A_e v_z \right\} = 0, \quad z < 0,$$

(19)

where

$$A_e = \frac{(\gamma - 1)g^2 k_x^2 + \omega^2(\omega^2 - k_x^2 c_e^2)}{\omega^2 c_e^2} = k_x^2 \left( \frac{\omega_{g_0}^2}{\omega^2} \right) - m_e^2,$$

(20)
with

\[ m_e^2 = \frac{k^2 c_e^2 - \omega^2}{c_e^2}; \]  

(21)

\[ \omega_{ke} = (\gamma - 1)^{1/2} c_e / \gamma H_e \] is the buoyancy frequency in the isothermal, non-magnetic region.

Equation (19) possesses the general solution

\[
v_z(z) = \left( d_3 \exp \left( z \frac{1 - 4 A_e H_e^2}{2 H_e} \right)^{1/2} + d_4 \exp \frac{-z(1 - 4 A_e H_e^2)^{1/2}}{2 H_e} \right) \times \\
\times \exp \left( \frac{z}{2 H_e} \right), \quad z < 0,
\]  

(22)

where \( d_3 \) and \( d_4 \) are arbitrary constants.

The linearized form of the magnetic energy density of the perturbations is given by \( B_0 \cdot B / \mu_0 \), leading to a magnetic energy density of \( B_0 B_x / \mu_0 \) for the perturbed field \( B = (B_x, 0, B_z) \). The \( x \)-component of the linearized induction equation, when Fourier decomposed, gives

\[
B_x = -\frac{B_0 \ dv_z}{i \omega \ dz}.
\]  

(23)

Thus, the magnetic energy density is related to

\[
\frac{B_0 B_x}{\mu_0} = \frac{i}{\omega} \frac{B_0^2}{\mu_0} \frac{dv_z}{dz}.
\]  

(24)

We impose the condition that the total (kinetic \( \sim \rho v_z^2 \)) plus magnetic \( \sim B_0^2 \ dv_z / dz \) energy density remains finite as \( |z| \to \infty \), and restrict attention to the circumstances of \( 4 A_e H_e^2 < 1 \). These two conditions imply that \( d_2 = d_4 = 0 \). Thus, the vertical velocity component \( v_z(z) \) in the two regions is given by

\[
v_z(z) = \begin{cases} 
    d_1 F(p, q ; r; X) e^{-z \kappa_0 / H_0}, & z > 0, \\
    d_3 \exp \left( \frac{1}{2 H_e} + M_e \right) z, & z < 0,
\end{cases}
\]  

(25)

where

\[
M_e = \frac{(1 - 4 A_e H_e^2)^{1/2}}{2 H_e}
\]  

(26)

and we have taken the positive square root of \( (1 - 4 A_e H_e^2)^{1/2} \), so \( M_e > 0 \). It may be shown from Equation (25) that \( v_z \to 0 \) as \( |z| \to \infty \) (see Evans and Roberts, 1990).
2.4. The General Dispersion Relation and Some Special Cases

Matching the normal component of velocity across the interface gives

\[ d_1 F(p, q; r; X_0) = d_2 . \]  

(27)

Integrating Equation (7) about the interface \( z = 0 \) yields the second matching condition (see also Paper I), namely that

\[
\frac{\rho(z) \left( c_s^2(z) + v_A^2(z) \right) \left( \omega^2 - k_x^2 c_T^2(z) \right)}{\left( \omega^2 - k_x^2 c_s^2(z) \right)} \frac{dv_z(z)}{dz} = \frac{k_x^2 g \rho(z) c_s^2(z)}{\left( \omega^2 - k_x^2 c_s^2(z) \right)} v_z(z)
\]

(28)

is continuous across \( z = 0 \). This quantity is in fact \( i \omega p_T(z) = g \rho(z) v_z(z) \), where

\[
p_T(z) = \frac{i \rho(z) \left( c_s^2(z) + v_A^2(z) \right) \left( \omega^2 - k_x^2 c_T^2(z) \right)}{\left( \omega^2 - k_x^2 c_s^2(z) \right)} \frac{dv_z(z)}{dz} - \frac{i \omega \rho(z) g}{\left( \omega^2 - k_x^2 c_s^2(z) \right)} v_z(z)
\]

(29)

is the perturbation in total (gas plus magnetic) pressure.

Applying the two matching conditions to the solution (25) determines the transcendental dispersion relation (Small and Roberts, 1984):

\[
\frac{\rho_0 (c_0^2 + v_A^2) \left( \omega^2 - k_x^2 c_T^2 \right)}{\left( \omega^2 - k_x^2 c_0^2 \right)} \left\{ k_x + \frac{pq}{r} \frac{X_0}{H_0} \frac{F(p + 1, q + 1; r + 1; X_0)}{F(p, q; r; X_0)} \right\} + \frac{k_x^2 g \rho_0 \rho e}{\left( \omega^2 - k_x^2 c_0^2 \right)} = \frac{\rho e c_e^2}{\left( \omega^2 - k_x^2 c_e^2 \right)} \left\{ k_x^2 g - \frac{1}{2H_e} + M_e \right\} \omega^2 .
\]

(30)

The dispersion relation (30) describes the parallel propagation of surface waves at a single magnetic interface in a gravitationally stratified atmosphere under the assumption of a uniform magnetic field and isothermal upper and lower atmospheres. As in Paper I, the Lamb modes \( \omega = k_x c_0 \) and \( \omega = k_x c_e \) (Lamb, 1932) do not satisfy the dispersion relation (30).

We note briefly two special cases of Equation (30). In the limit of zero gravity, \( A_e \rightarrow -m_e^2 \), and then Equation (30) reduces after much algebra (see the Appendix) to

\[
\rho_0 \left( k_x^2 v_A^2 - \omega^2 \right) m_e - \omega^2 \rho_e m_0 = 0 ,
\]

(31)

where \( m_e (> 0) \) is defined by Equation (21) and \( m_0 (> 0) \) by

\[
m_0^2 = \frac{\left( k_x^2 c_0^2 - \omega^2 \right) \left( k_x^2 v_A^2 - \omega^2 \right)}{\left( c_0^2 + v_A^2 \right) \left( k_x^2 c_T^2 - \omega^2 \right)} .
\]

(32)

Thus, we recover the dispersion relation describing the parallel propagation of surface waves on a magnetic interface, one side of which is field-free (Wentzel, 1979; Roberts, 1981; Miles and Roberts, 1989).
Also of interest is the incompressible limit \((c_0, c_e \to \infty)\). Consider the special case of a fluid having a uniform density distribution so that \(\rho_0(z) = \rho_0\) and \(\rho_e(z) = \rho_e\). Then, in the incompressible limit \(m_e \to k_x\), while the quantities \(F(p + 1, q + 1; r + 1; X_0)\), \(F(p, q; r; X_0)\) and \(X_0\) remain finite. However, \(pq/rH_0 \to 0\) and so Equation (30) reduces to the well-known result (see discussion in Paper I)

\[
\frac{\omega^2}{k_x^2} = \frac{c_e^2}{c_k^2} - g \frac{\rho_0 - \rho_e}{k_x (\rho_0 + \rho_e)},
\]

(33)

where

\[
c_e^2 = \frac{\rho_0}{(\rho_0 + \rho_e)} \nu_A^2.
\]

(34)

Thus, we recover the familiar dispersion relation for surface modes between two uniform incompressible fluids.

3. Numerical Solution of the Dispersion Relation

3.1. Cutoff curves

The transcendental dispersion relation (30) is solved subject to the constraints that \(\omega \neq k_x c_0\), \(\omega \neq k_x c_e\) and that \((1 - 4A_e H_e^2) > 0\). As in Paper I, this latter constraint gives rise to non-horizontal dashed curves, labelled \(R_1\) and \(R_2\) in Figure 2. We note that the confining curves \(R_3\) and \(R_4\) of Paper I no longer occur because the constraint they are trying to impose is automatically satisfied. The curves \(R_1\) and \(R_2\) denote cutoff curves for the propagation of the modes; beyond these curves, i.e., for \((1 - 4A_e H_e^2) < 0\), complex solutions may arise and correspond to internal modes. In this respect the plots in Figure 2 can be compared with the standard diagnostic diagram for acoustic-gravity in an isothermal atmosphere, dividing the \(\omega - k_x\) plane into regions of evanescence and propagation (see, for example, Roberts, 1985). The plots in Figure 2 are the equivalent diagrams when a magnetic field is included. The area above the upper dashed curve \(R_2\) corresponds to a region where magnetoacoustic waves may propagate, whilst that below the lower dashed curve \(R_1\) is a region in which magnetogravity waves may exist. The area bounded by the two curves is one in which magnetoacoustic surface waves (i.e., evanescent modes) are permitted to propagate.

The form of these constraints depends upon whether \(\omega > k_x c_T\) or \(\omega < k_x c_T\). Thus, if \(\omega > k_x c_T\), then the dimensionless longitudinal phase-speed, \(\omega/k_x c_e\), must satisfy

\[
\max \left(\frac{c_T}{c_e}, R_1\right) \leq \frac{\omega}{k_x c_e} \leq R_2,
\]

(35)

when a solution exists. If \(\omega < k_x c_T\), then the constraint becomes

\[
R_1 \leq \frac{\omega}{k_x c_e} \leq \min \left(\frac{c_T}{c_e}, R_2\right).
\]

(36)
The variation of the non-dimensional longitudinal phase-speed for magnetoacoustic-gravity modes with the non-dimensional horizontal wavenumber for the cases (a) $c_0/c_e = 0.9$, $v_A/c_e = 0.5$, (b) $c_0/c_e = 0.9$, $v_A/c_e = 0.75$, (c) $c_0/c_e = 0.9$, $v_A/c_e = 1.0$, and (d) $c_0/c_e = 1.4$, $v_A/c_e = 0.75$. The dashed curves cut-off curves for the propagation of the modes determined by the assumption that $4A_eH_\parallel < 1$ (see text). A slow surface mode is present in all four cases. In cases (a) and (b), for which $c_0 < c_e$ and $v_A < c_0$, harmonic modes are also present; the harmonics evolve to a fast surface wave for $v_A > c_0$, $c_0 < c_e$ (see case (c)). When $c_0 > c_e$ (case (d)), only a slow surface wave occurs.

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Here $R_1^2$ and $R_2^2$ are given by (see also Paper I)

$$R_{1,2}^2 = \frac{(1 + 4k_x^2H_e^2)(1 + 4k_x^2H_e^2 - 64k_x^2H_e^2 \gamma^2)}{8k_x^2H_e^2}.$$  

(37)
In the limit as \( k_x H_e \to \infty \), which includes the case of zero gravity, we obtain
\[
R_1 \to 0, \quad R_2 \to 1.
\] (38)

In the limit \( k_x H_e \to 0 \):
\[
R_1 \to \frac{2(\gamma - 1)^{1/2}}{\gamma}, \quad R_2 \to \infty;
\] (39)

for \( \gamma = \frac{5}{3} \), \( R_1 \to 0.98 \) as \( k_x H_e \to 0 \).

3.2. MAGNETOACOUSTIC SURFACE WAVES

Consider now the behaviour of the modes determined through a numerical investigation of the dispersion relation (30). We set \( \gamma = \frac{5}{3} \) throughout. The variation of the longitudinal phase-speed (non-dimensionalised with respect to the sound speed in the field-free region), \( \omega/k_x c_e \), with the dimensionless horizontal wavenumber \( k_x H_e \) is shown in Figure 2. In Figures 2(a, b), for which \( c_e > c_0 \) and \( v_A > c_0 \), the upper horizontal dashed line corresponds to \( \omega = k_x c_0 \). The lower horizontal dashed line is the asymptote to which the slow magnetoacoustic-gravity surface mode tends as \( k_x H_e \to \infty \), and is determined by Equation (31). The other horizontal dashed line in Figures 2(a, b) is the line \( \omega = k_x c_T \). Notice the occurrence of harmonic modes in Figure 2(a); harmonic modes also arise in Figure 2(b) but only the first harmonic is visible. The harmonics asymptote to \( \omega/k_x c_e = c_0/c_e \) (\( = 0.9 \)) as \( k_x H_e \to \infty \); however, as noted earlier, the solution \( \omega = k_x c_0 \) is not itself permitted. Thus, in Figures 2(a) and 2(b) as \( k_x H_e \to \infty \) only the slow mode exists. This result is as expected from the non-gravity case (Roberts, 1981; Miles and Roberts, 1989), for which both the fast and slow magnetoacoustic surface modes propagate only when \( c_e > c_0 \) and \( v_A > c_0 \). Thus, in Figure 2(c), for which these conditions are satisfied, we see that as \( k_x H_e \to \infty \) both the slow and fast magnetoacoustic surface modes propagate. The upper horizontal dashed line in Figure 2(c) is the limit to which the fast magnetoacoustic-gravity surface mode asymptotes as \( k_x H_e \to \infty \), and is determined by the \( g = 0 \) dispersion relation (31). Finally, in Figure 2(d), for which \( c_e < c_0 \) and \( v_A < c_0 \), we find that only the slow magnetoacoustic surface mode propagates as \( k_x H_e \to \infty \); again this is consistent with the zero gravity case. We note that none of the modes lie in the region defined by Equation (18), where the solution for \( v_z \) is singular.

It is evident from Figures 2(a, b) that the harmonic modes are confined to the region defined by
\[
\frac{c_0}{c_e} \leq \frac{\omega}{k_x c_e} \leq R_2.
\] (40)

We note as the value of \( v_A/c_e \) is increased, the second and subsequent harmonics originate at higher values of the non-dimensional wavenumber \( k_x H_e \). In Figure 2(b), for example, only the first harmonic is seen for the range of \( k_x H_e \) shown. We observe
also the absence of the harmonic modes in Figure 2(d), where \( c_0/c_e > 1 \), i.e., \( c_0/c_e > R_2 \) as \( k_z H_e \to \infty \), and so the inequality given by Equation (40) is violated.

We identify the first of the harmonic modes as the \( f \)-mode modified by the presence of a uniform horizontal magnetic field. In Paper I we pointed out that as the magnetic field strength is increased the \( f \)-mode develops into the fast magnetoacoustic-gravity surface mode, eventually becoming the fast magnetoacoustic surface mode at very small

\[
\begin{align*}
(a) & \quad c_0/c_e = 0.9 \quad v_A/c_e = 0.5 \\
& \quad k_z H_e = 2.0 \\
(b) & \quad c_0/c_e = 0.9 \quad v_A/c_e = 0.5 \\
& \quad k_z H_e = 10.0
\end{align*}
\]

Fig. 3. The eigenfunction of the vertical velocity \( v_z \) for the harmonics in Figure 2(a) (for which \( c_0/c_e = 0.9 \) and \( v_A/c_e = 0.5 \)) for (a) \( k_z H_e = 2.0 \) and (b) \( k_z H_e = 10.0 \). The modes are plotted in units of \( v_z(0) \).
wavelengths \((k_x H_e \gg 1)\). Similarly, here we note that as the magnetic field strength is increased from \(v_A/c_e = 0.5\) (Figure 2(a)) to \(v_A/c_e = 1.0\) (Figure 2(c)) the first harmonic mode in Figure 2(a) develops into the upper mode in Figure 2(c). That is, the \(f\)-mode (modified by the presence of the magnetic field) in Figure 2(a) changes in character as \(v_A/c_e\) is increased, becoming the fast magnetoacoustic-gravity surface wave in Figure 2(c).

The eigenfunctions \(v_z\) of the first two harmonics in Figure 2(a) are displayed in Figure 3. Figure 3(a) shows the eigenfunction for the first or fundamental harmonic in Figure 2(a) (at \(k_x H_e = 2.0\)). We note that the eigenfunction has no nodes and is therefore characteristic of an eigenfunction of a surface mode. We recognize this eigenfunction as that of the \(f\)-mode in the presence of a magnetic field. Figure 3(b) gives the eigenfunction for the second harmonic in Figure 2(a) (at \(k_x H_e = 10.0\)). We observe that this eigenfunction has one node, similar to that of the eigenfunction of the first \((n = 1)\) \(p\)-mode. The second harmonic’s eigenfunction, then, is characteristic of a trapped mode, the trapping in this case arising due to refraction caused by the increasing Alfvén speed (and therefore increasing magnetoacoustic speed) with height into the atmosphere. In Paper I, the Alfvén speed is constant and so no trapping occurs; consequently there are no harmonic modes present. A plot of the eigenfunction for the third harmonic (not shown) from Figure 2(a) exhibits a trapped mode with two nodes.

Finally, in Figure 4, we compare and contrast the eigenfunctions of the \(f\)-mode (in

\[ \frac{c_0}{c_e} = 0.9 \quad \frac{v_A}{c_e} = 1.0 \]

\[ Z/H \]

Fig. 4. The eigenfunctions for the \(f\)-mode (modified by the magnetic field) with \(k_x H_e = 1.0\) and the slow and fast magnetoacoustic-gravity surface modes with \(k_x H_e = 5.0\), when \(c_0/c_e = 0.9\) and \(v_A/c_e = 1.0\). The eigenfunctions are plotted in units of \(v_z(0)\).
the presence of a magnetic field) with the fast and slow magnetoacoustic-gravity surface waves. The eigenfunction of the $f$-mode (modified by the presence of the magnetic field) at $k_x H_e = 1.0$ in Figure 2(c) is shown together with those of the fast and slow magnetoacoustic-gravity surface modes at $k_x H_e = 5.0$ in Figure 2(c). The curves are for the case $c_0/c_e = 0.9$ and $v_A/c_e = 1.0$. We note, as in Paper I, that the $f$-mode and the fast magnetoacoustic-gravity surface mode have similar profiles, demonstrating the association which exists between them.

5. Conclusions

We have derived a dispersion relation describing the parallel propagation of magnetoacoustic-gravity surface modes and the $f$-mode at a single magnetic interface. Both the model of Paper I and that discussed here yield similar features for some of the wave structure. For example, in the presence of a magnetic field, the $f$-mode fails to propagate if the magnetic region is warmer than the field-free medium. Also, the $f$-mode changes its character as the magnetic field strength is increased, and eventually develops into the fast magnetoacoustic-gravity surface wave. Dispersion curves for the fast and slow magnetoacoustic-gravity surface waves show that both modes propagate with phase-speeds which decrease as the horizontal wavenumber is increased, gradually asymptoting to distinct limits as $k_x H_e \rightarrow \infty$, provided both $c_e > c_0$ and $v_A > c_0$. If either of these latter two conditions is not met, then the propagation of the fast magnetoacoustic-gravity surface mode is restricted and only the slow magnetoacoustic surface wave exists as $k_x H_e \rightarrow \infty$.

However, in contrast to the constant Alfvén speed model of Paper I, the increasing Alfvén speed model discussed here results in the trapping of modes due to refraction. These trapped modes are in some respects similar to the usual $p$-modes in helioseismology and may be relevant to the recent discussions of the possibility of a chromospheric cavity (see, for example, Kumar et al., 1991; Woodard and Libbrecht, 1991). Trapped modes only occur when the field-free region is warmer than the magnetic medium (i.e., when $c_e > c_0$).

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Appendix. The Limit of Zero-Gravity

In the limit as $g \rightarrow 0$ the transcendental dispersion relation Equation (30) becomes

$$\rho_0 \left( \frac{k_x^2 v_A^2}{m_e^2} - \omega^2 \right) \left\{ k_x + \frac{p q}{r} \frac{1}{H_0} \frac{F(p + 1, q + 1; r + 1; X_0)}{F(p, q; r; X_0)} \right\} = \frac{\rho_e}{m_e} \omega^2. \tag{A1}$$
Thus, in order to determine the dispersion relation in the limit as $g \to 0$ requires a knowledge of the hypergeometric functions in this limit. However, directly taking the limit as $g \to 0$ of the hypergeometric functions is not possible since all three parameters $p$, $q$, and $r$ become infinite in this limit. Instead, following a suggestion by Dr A. W. Hood, we work from the hypergeometric differential Equation (12), viz.,

\[
X(1 - X) \frac{d^2 V_z}{dX^2} + \left[ r - (p + q + 1)X \right] \frac{d V_z}{dX} - pq V_z = 0 ,
\]

(A2)

where

\[
X = X_0 e^{-z/H_0} = \frac{C_0^2 \omega^2}{v_0^2 (k_x^2 c_0^2 - \omega^2)} e^{-z/H_0} , \quad z > 0 .
\]

(A3)

This may be rewritten as

\[
\frac{d^2 V_z}{dX^2} + \left\{ \frac{r}{X} + \frac{1}{(X - 1)} \right\} \frac{d V_z}{dX} - \frac{pq}{X(1 - X)} V_z = 0 .
\]

(A4)

Introduce the transformation

\[
V_z = \frac{Y}{(X - 1)^{1/2} X^{r/2}} .
\]

(A5)

Then Equation (A4) reduces to

\[
\frac{d^2 Y}{dX^2} - H_0^2 h(X, H_0) Y = 0 ,
\]

(A6)

where

\[
H_0^2 h(X, H_0) = \frac{X^2 (r^2 - 4pq - 1) - 2X(r^2 - 2pq - r) + r(r - 2)}{4X^4 (X - 1)^2} .
\]

(A7)

Now, with $p$, $q$, and $r$ defined by Equations (13) and (14), we have

\[
r^2 - 4pq - 1 = 4H_0^2 \left\{ \frac{k_x^2 c_0^2 - \omega^2}{c_0^2} - \frac{(\gamma - 1) k_x^2 c_0^2}{H_0^2 \omega^2} \right\} ,
\]

(A8)

\[
r^2 - 2pq - r = 2H_0^2 \left\{ \frac{2k_x^2 c_0^2 - \omega^2}{c_0^2} - \frac{(\gamma - 1) k_x^2 c_0^2}{H_0^2 \omega^2} \right\} ,
\]

(A9)

and

\[
r(r - 2) = 4k_x^2 H_0^2 - 1 .
\]

(A10)

We seek a solution to Equation (A6) of the form

\[
Y = \exp \{ H_0 G(X, H_0) \} , \quad (A11)
\]
where

\[ G(X, H_0) = G_1(X) + \frac{1}{H_0} G_2(X) + \cdots \]  (A12)

is such that as \( H_0 \to \infty \) (i.e., \( g \to 0 \)), \( G(X, H_0) \to G_1(X) \). To determine \( G_1(X) \) we substitute (A11) into (A6). Then

\[ \frac{1}{H_0} \frac{d^2 G}{dX^2} + \left( \frac{dG}{dX} \right)^2 - h(X, H_0) = 0. \]  (A13)

In the limit as \( H_0 \to \infty \), Equation (A13) gives \( G_1(X) \) as

\[ \left( \frac{dG_1}{dX} \right)^2 = \lim_{H_0 \to \infty} h(X, H_0), \]  (A14)

that is,

\[ \left( \frac{dG_1}{dX} \right)^2 = \frac{X(k_x^2c_0^2 - \omega^2) - k_x^2c_0^2}{c_0^2X^2(X - 1)}. \]  (A15)

Now, from the transformation (A5), we have

\[ \frac{1}{V_z} \frac{dV_z}{dX} = \frac{1}{Y} \frac{dY}{dX} - \frac{1}{2} \left( \frac{X + r(X - 1)}{X(X - 1)} \right). \]  (A16)

With

\[ V_z(z) = d_1 F(p, q; r; X) e^{-zK_0/H_0}, \quad z > 0, \]  (A17)

we obtain

\[ \frac{1}{V_z} \frac{dV_z}{dX} \bigg|_{z=0} = \frac{pq}{r} \frac{F(p + 1, q + 1; r + 1; X_0)}{F(p, q; r; X_0)}. \]  (A18)

So Equation (A16) in the limit as \( H_0 \to \infty \) (i.e., \( g \to 0 \)) becomes

\[ \lim_{H_0 \to \infty} \left[ \frac{pq}{r} \frac{1}{H_0} \frac{F(p + 1, q + 1; r + 1; X_0)}{F(p, q; r; X_0)} \right] = \]

\[ = \lim_{H_0 \to \infty} \left\{ \frac{1}{H_0Y} \frac{dY}{dX} \right\}_{z=0} - \frac{1}{2} \lim_{H_0 \to \infty} \left\{ \frac{X + r(X + 1)}{H_0X(X - 1)} \right\}_{z=0}. \]  (A19)

Now

\[ \lim_{H_0 \to \infty} \left\{ \frac{1}{H_0Y} \frac{dY}{dX} \right\}_{z=0} = \lim_{H_0 \to \infty} \frac{dG_1}{dX} \bigg|_{z=0}. \]  (A20)
where $dG/dX$ is given by Equation (A15). Thus Equation (A19) reads

$$
\lim_{H_0 \to \infty} \left[ \frac{pq}{r} \frac{1}{H_0} \frac{F(p + 1, q + 1; r + 1; X_0)}{F(p, q; r; X_0)} \right] = \\
= \pm \left\{ \frac{X_0(k_x^2c_0^2 - \omega^2) - k_x^2c_0^2}{c_0^2X_0^2(X_0 - 1)} \right\}^{1/2} - \\
- \frac{1}{2} \lim_{H_0 \to \infty} \left\{ \frac{X_0/H_0 + (2k_x + 1/H_0)(X_0 - 1)}{(X_0 - 1)} \right\}.
$$

(A21)

Finally, after some algebra this reduces to give

$$
\lim_{H_0 \to \infty} \left[ \frac{pq}{r} \frac{1}{H_0} \frac{F(p + 1, q + 1; r + 1; X_0)}{F(p, q; r; X_0)} \right] = \pm \frac{1}{X_0} \left[m_0 - k_x\right].
$$

(A22)

Taking the positive root of Equation (A22) and substituting into Equation (A1) yields the result

$$
\rho_0(k_x^2v_A^2 - \omega^2)m_e - \omega^2\rho_em_0 = 0.
$$

(A23)

This is the dispersion relation applicable in the absence of gravity (i.e., Equation (31)).

References


