MAGNETOACOUSTIC-GRAVITY SURFACE WAVES

I. Constant Alfvén Speed

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Abstract. The linearized theory for the parallel propagation of magnetoacoustic-gravity surface waves is developed for an interface of a horizontal magnetic field above a field-free medium. The media either side of the interface are taken to be isothermal. The dispersion relation is obtained for the case of a constant Alfvén speed. In the absence of gravity the interface may support one or two surface modes, determined by the relative temperatures and magnetism of the two media. The effect of gravity on the modes is examined and dispersion diagrams and eigenfunctions are given. In the usual $\omega - k_z$ diagnostic diagram, the domain of evanescence is shown to be divided into two distinct regions determining whether a given mode will have a decaying or growing vertical velocity component. In the absence of a magnetic field the transcendental dispersion relation may be rewritten as a polynomial. This polynomial possesses two acceptable solutions only one of which may exist in any given circumstances (depending on the ratio of the densities). If the gas density within the field exceeds that in the field-free medium, then the $f$-mode may propagate. The $f$-mode exists in a restricted band of horizontal wavenumber and only when the field-free medium is warmer than the magnetic atmosphere. An analytical form for the wave speed of the $f$-mode is obtained for small values of the Alfvén speed. It is shown that the $f$-mode is related to the fast magnetoacoustic surface wave, merging into that mode at short wavelengths.

1. Introduction

A theoretical understanding of magnetoacoustic surface waves in a gravitationally stratified atmosphere permeated by a magnetic field is of natural interest. We shall refer to such waves as magnetoacoustic-gravity surface waves. In the solar atmosphere an interest in such surface waves occurs in studies of coronal heating (Ionson, 1978, 1985; Wentzel, 1979; Rae and Roberts, 1981; Lee and Roberts, 1986; Hollweg, 1986, 1987a, b), in oscillations in sunspot penumbrae (Nye and Thomas, 1976; Small and Roberts, 1984), and in helioseismology (Deubner and Gough, 1984; Christensen-Dalsgaard, Gough, and Toomre, 1985; Leibacher et al., 1985).

There is thus a widespread interest in the behaviour of surface waves (see the review by Roberts, 1991) and their possible application to observations of waves in the solar atmosphere. The ‘running penumbral wave’ sunspot phenomenon and chromospheric canopy modes are two possible applications of surface wave theory. In any case, an understanding of surface waves in a stratified atmosphere is an essential prerequisite to any application to the photospheric and chromospheric plasmas. We present a detailed systematic investigation of magnetic surface waves on a single horizontal magnetic interface in a stratified atmosphere (see also Miles and Roberts, 1991).

In general, any disturbance in the solar atmosphere is subject to the three restoring forces of buoyancy, compressibility, and magnetism. The inclusion of gravity not only
introduces a preferred direction additional to that determined by the magnetic field, but also imposes length and time scales in the system. Consequently, the nature of the propagation of magnetoacoustic-gravity surface waves is necessarily complicated, a reflection of the highly anisotropic character of a magnetically structured and stratified atmosphere. The length scales introduced by the gravitational field are defined by the equilibrium density and pressure profiles, while the imposed time scales arise from the acoustic cutoff frequency and the buoyancy (Brunt–Väisälä) frequency.

We consider the propagation of magnetoacoustic-gravity surface waves parallel to an applied horizontal magnetic field at an isothermal magnetic interface one side of which is field-free. The field strength of the horizontal magnetic field is assumed to decrease exponentially with height in such a manner that the Alfvén speed is constant. Such an equilibrium profile is amenable to an analytical investigation yielding a relatively simple, though transcendental, dispersion relation.

An alternative model for solar applications is that of an isothermal atmosphere permeated by a uniform magnetic field. In this case the Alfvén speed increases exponentially with height. We examine this case in Miles, Allen, and Roberts (1992; Paper II); see also Miles and Roberts (1990) and Evans and Roberts (1990).

The dispersion relation discussed here was first obtained in Miles and Roberts (1991). An extension to include the effects of non-parallel propagation is given briefly in Jain and Roberts (1991a). A recent paper by González and Gratton (1991) takes up the problem, considering the general case of non-parallel propagation. González and Gratton investigate in detail the stability constraints on the equilibrium and go on to explore the spectrum of permitted surface modes in the non-parallel case. Here, for the case of parallel propagation, we focus attention on the inter-relationships that exist for the various modes and their behaviours in a number of limiting cases (including the extreme of small wavelength, the non-magnetic case, and the incompressible case). We thus make clear, for example, the connection between our modes and the more familiar fast and slow surface modes in the standard frequency–wavenumber diagram of helioseismology.

To begin our investigation, recall that in the absence of gravity a magnetic interface supports two surface waves (Roberts, 1981a; Miles and Roberts, 1989; Jain and Roberts, 1991a, b), depending on the relative temperatures of the media either side of the interface. These are the fast and slow magnetoacoustic surface waves. Their counterparts arise when gravity is included. However, the inclusion of gravity permits a third surface mode to propagate, although only for a limited range of the horizontal wavenumber. This surface wave is the $f$-mode, modified by the presence of the magnetic field. Like the fast magnetoacoustic surface wave, its existence depends upon the relative temperatures of the media either side of the interface; both the $f$-mode and the fast magnetoacoustic surface wave can propagate only when the field-free region is warmer than the magnetic atmosphere.

The $f$-mode is an important oscillation of the Sun, of use in determining the structure and dynamics of the solar interior. Together with $p$-modes, the $f$-mode with period around 5 min has been observed with great accuracy (see, for example, Duvall et al.,
1988; Libbrecht and Kaufman, 1988; Libbrecht, Woodard, and Kaufman, 1990). The chromosphere is dominated by magnetic field and therefore magnetism will have an influence on the nature of \( p \)- and \( f \)-modes. It is thus important to consider the effect of a magnetic field on these oscillations (see Campbell and Roberts, 1989; Evans and Roberts, 1990). We examine the influence of a horizontal magnetic field on the \( f \)-mode.

Finally, we note that surface waves are also important in laboratory plasmas, especially with regard to stability considerations (e.g., Gratton, Gratton, and González, 1988; González and Gratton, 1990).

To motivate our study, then, recall that on a magnetic interface between two uniform and incompressible fluids, a surface wave exists which propagates with a speed \( \omega/k_x \) given by

\[
\frac{\omega^2}{k_x^2} = \frac{\rho_0}{(\rho_0 + \rho_e)} v_A^2.
\]  

(1)

Here \( \rho_0 \) is the density in the magnetic field region, where the Alfvén speed is \( v_A \), and \( \rho_e \) is the density in the field-free fluid. This is the well-known result (see Kruskal and Schwarzschild, 1954; Chandrasekhar, 1961) that a surface wave propagates at a speed that is intermediate between the Alfvén speeds of the two media, with one of those speeds here taken to be zero. However, the assumptions of incompressibility and zero gravity that permit the result (1) are too drastic to allow it to be of more than a rough guide to the actual behaviour of surface waves in the solar atmosphere, which of course is far from incompressible and is also stratified.

Compressibility modifies (1) to the result (Wentzel, 1979; Roberts, 1980, 1981a, b) that a surface wave propagates according to

\[
\frac{\omega^2}{k_x^2} = \frac{\rho_0}{(\rho_0 + \rho_e)} \left(\frac{m_0}{m_e}\right) v_A^2,
\]

(2)

where \( m_0 \) and \( m_e \), both positive, are functions of \( \omega^2 \) and \( k_x^2 \). In the incompressible limit \( m_0/m_e \) approaches unity and we recover the relation (1). Generally, however, the dispersion relation (2) is transcendental and may admit two modes (Roberts, 1981a), which have speeds that are sub-Alfvénic.

How does stratification modify the above results? The presence of gravity modifies the equilibrium state of a medium and so the behaviour of a surface wave is changed in two ways: first, because the wave samples a non-uniform medium, and secondly because additional forces – buoyancy forces – arise to modify the propagation of the wave. These two effects seriously complicate any description of the waves. However, when the medium is both incompressible and uniform in density, we have the result (e.g., Chandrasekhar, 1961) that

\[
\frac{\omega^2}{k_x^2} = \frac{\rho_0}{(\rho_0 + \rho_e)} v_A^2 - \frac{g}{k_x} \left(\frac{\rho_0 - \rho_e}{\rho_0 + \rho_e}\right),
\]

(3)
revealing that in the presence of gravity the surface mode is rendered dispersive and is also subject to instability (the Rayleigh–Taylor instability) at long wavelengths if a dense fluid rests on top of a light fluid (i.e., if $\rho_0 > \rho_e$). Equation (3) is a familiar result, often used in astrophysical applications. But we should remember the restrictions under which it is derived, namely an incompressible fluid, uniform (unstratified) in density (though stratified in pressure). Neither of these assumptions is likely to be met under solar conditions.

So what replaces the result (3) when a stratified gas is considered? This is the subject of our paper. We show, in fact, that for the case of two isothermal gases in contact at a magnetic–non-magnetic interface, with constant Alfvén speed within the magnetic region, the dispersion relation (3) is replaced by the highly transcendental form

$$\frac{\omega^2}{k_x^2} = \frac{\rho_0}{\left(\rho_0 + \rho_e \frac{(M_e + 1/2H_e)m_0^2}{(M_0 - 1/2H_B)m_e^2}\right)} v_A^2 - \frac{\rho_0 c_0^2}{(k_x^2 c_0^2 - \omega^2)} - \frac{\rho_e c_e^2}{(k_x^2 c_e^2 - \omega^2)} - g \frac{\rho_0}{(M_0 - 1/2H_B)m_0^2} + \rho_e \frac{(M_e + 1/2H_e)}{m_e^2},$$

(4)

where $M_0, M_e$ are functions of $\omega^2$ and $k_x$ and $H_B, H_e$ are constants. It is the derivation of this relation and an unfolding of its properties that provides our topic of investigation.

### 2. Magnetoacoustic-Gravity Waves

#### 2.1. The Governing Differential Equations

Consider a plane-parallel stratified atmosphere with gas pressure $p(z)$ and density $\rho(z)$ within which is embedded a horizontal magnetic field $B(z)\hat{x}$. Since the field is straight there are no tension effects and the equilibrium is one of magnetohydrostatic balance determined by

$$\frac{d}{dz} \left( p(z) + \frac{B^2(z)}{2\mu} \right) = -\rho(z)g,$$

(5)

demonstrating that the magnetic pressure may provide support against gravity $-g\hat{z}$. Then two-dimensional, linear, isentropic disturbances of the form

$$v(x, z, t) = (v_x(z), 0, v_z(z)) e^{i(\omega t - k_x x)},$$

(6)

for frequency $\omega$ and horizontal wavenumber $k_x$, satisfy the equations

$$\left(1 - \frac{k_x^2 c_x^2}{\omega^2}\right) \Delta = \frac{dv_x}{dz} - \frac{gk_x^2}{\omega^2} v_z,$$

(7)


\[ c_s^2 \frac{dA}{dz} - (\gamma - 1)gA - \frac{\gamma B}{\mu \rho} \frac{dB}{dz} A = g \frac{dv_z}{dz} + (k_x^2 v_A^2 - \omega^2) v_z - \frac{1}{\rho} \frac{d}{dz} \left( \frac{B^2}{\mu} \frac{dv_z}{dz} \right), \]

(8)

where \( A = \text{div} \mathbf{v}. \) Here

\[ c_s(z) = (\gamma p(z)/\rho(z))^{1/2} \]

and

\[ v_A(z) = \left( \frac{B^2(z)}{\mu \rho(z)} \right)^{1/2} \]

are the sound and Alfvén speeds, respectively; \( \gamma \left( = \frac{5}{3} \right) \) is the adiabatic index and \( \mu \) the permeability of a vacuum.

The ordinary differential Equations (7) and (8) describe magnetoacoustic waves in a stratified gas. They also describe gravity waves and their unstable counterpart, convection. Eliminating the compression \( A \) leads to the second-order ordinary differential equation (Adam, 1977; Small and Roberts, 1984; Roberts, 1985; cf. Goedbloed, 1971)

\[
\frac{d}{dz} \left\{ \frac{\rho(z) (c_s^2(z) + v_A^2(z)) (\omega^2 - k_x^2 c_s^2(z))}{(k_x^2 c_s^2(z) - \omega^2)} \frac{dv_z}{dz} \right\} - \\
- \left\{ \rho(z) (\omega^2 - k_x^2 c_s^2(z)) + \frac{g^2 k_x^2 \rho(z)}{(k_x^2 c_s^2(z) - \omega^2)} \right\} v_z + \\
g k_x^2 \frac{d}{dz} \left( \frac{\rho(z) c_s^2(z)}{k_x^2 c_s^2(z) - \omega^2} \right) = 0,
\]

(9)

where \( c_T = c_s v_A / (c_s^2 + v_A^2)^{1/2} \) is the cusp speed. Equation (9) possesses a complex structure reflecting the fact that the medium is compressible, magnetic and stratified in temperature and density.

2.2. THE GENERAL DISPERSION RELATION

We consider a magnetic interface located at \( z = 0 \). The region \((z > 0)\) above the interface is taken to be isothermal and permeated by a non-uniform horizontal magnetic field \( \mathbf{B}(z) = (B(z), 0, 0) \), the strength of which declines exponentially at the same rate as the square root of the gas density, thus producing an Alfvén speed that is constant (see also Yu, 1965; Thomas, 1983). The field-free medium \((z < 0)\) below the interface is also isothermal but of a possibly different temperature to that above the interface. Quantities above \((z > 0)\) the interface are denote by a subscript '0', quantities below \((z < 0)\) by a subscript 'e'. See Figure 1.

Pressure balance at the interface \( z = 0 \) dictates that the total (gas plus magnetic)
pressure (as $z \to 0_+$) equals that in the field-free region below (as $z \to 0_-$):

$$p_0(0_+) + \frac{B_0^2}{2\mu} = p_e(0_-),$$  \hspace{1cm} (10)

where $B_0$ is the magnetic field strength at $z = 0$. Equation (10) coupled with the ideal gas law gives a relationship between the densities on either side of the interface: namely,

$$\frac{c_e^2}{c_0^2} \frac{\rho_e(0_-)}{\rho_0(0_+)} = 1 + \frac{1}{2} \gamma \frac{\beta}{\beta},$$  \hspace{1cm} (11)

where $\beta = c_0^2/v_A^2$ is the squared ratio of the sound speed $c_0 (= \gamma p_0/\rho_0)^{1/2}$ to the Alfvén speed $v_A (= (B_0^2/\mu \rho_0(0_+))^{1/2})$ at the base of the magnetic atmosphere, and $c_e (= (\gamma p_e/\rho_e)^{1/2})$ is the sound speed in the field-free region.

The assumptions of constant sound and Alfvén speeds imply a density profile of the form

$$\rho(z) = \begin{cases} 
\rho_0 e^{-z/H_B}, & z > 0, \\
\rho_e e^{-z/H_e}, & z < 0,
\end{cases}$$  \hspace{1cm} (12)

where $H_B (= c_0^2/\Gamma g)$ and $H_e (= c_e^2/\gamma g)$ are the density scale-heights above and below the interface, respectively, and $\Gamma$, defined by

$$\Gamma = \frac{2\beta \gamma}{2\beta + \gamma},$$  \hspace{1cm} (13)
is the magnetically-modified adiabatic exponent. (In the absence of a magnetic field \( \Gamma = \gamma \) and \( H_B = H_0 \) \( \equiv c_0^2/\gamma g \).) We have written \( \rho_e \equiv \rho_e(0_-) \) and \( \rho_0 \equiv \rho_0(0_+) \).

We should note that the equilibrium considered here is subject to instability. There is the possibility of a Rayleigh–Taylor instability when the gas in the upper region is heavier than that in the lower region and furthermore there is field interchange instability arising from the support that the field imparts to the gas in the upper region (Yu, 1965; Thomas, 1983). González and Gratton (1991) discuss these instabilities and their relationships to the surface modes in some detail.

For the stratification defined by Equation (12), the governing ordinary differential Equation (9) for the magnetic field region reduces to (Yu, 1965; Campbell and Roberts, 1989)

\[
\frac{d^2 v_z}{dz^2} - \frac{1}{H_B} \frac{dv_z}{dz} + A_B v_z = 0, \quad z > 0, \tag{14}
\]

where

\[
A_B = \frac{(\Gamma - 1)g^2 k_x^2 + (\omega^2 - k_x^2 c_0^2)(\omega^2 - k_x^2 c_T^2)}{(c_0^2 + v_A^2)(\omega^2 - k_x^2 c_T^2)}. \tag{15}
\]

Equation (14) possesses the general solution

\[
v_z(z) = \left( d_1 \exp\left( \frac{z(1 - 4A_B H_B^2)^{1/2}}{2H_B} \right) + d_2 \exp\left( -z(1 - 4A_B H_B^2)^{1/2} \right) \right) \times
\]

\[
\exp\left( \frac{z}{2H_B} \right), \quad z > 0, \tag{16}
\]

where \( d_1 \) and \( d_2 \) are arbitrary constants.

In the non-magnetic region Equation (9) reduces to

\[
\frac{d^2 v_z}{dz^2} - \frac{1}{H_e} \frac{dv_z}{dz} + A_e v_z = 0, \quad z < 0, \tag{17}
\]

where

\[
A_e = \frac{(\gamma - 1)g^2 k_x^2 + \omega^2(\omega^2 - k_x^2 c_e^2)}{\omega^2 c_e^2}. \tag{18}
\]

Equation (17) possesses the general solution

\[
v_z(z) = \left( d_3 \exp\left( \frac{z(1 - 4A_e H_e^2)^{1/2}}{2H_e} \right) + d_4 \exp\left( -z(1 - 4A_e H_e^2)^{1/2} \right) \right) \times
\]

\[
\exp\left( \frac{z}{2H_e} \right), \quad z < 0, \tag{19}
\]

where \( d_3 \) and \( d_4 \) are arbitrary constants.
The requirement that the total (kinetic plus magnetic) energy density remains finite as $|z| \to \infty$, together with the assumptions (see Section 3 for a discussion) that $4A_B H_B^2 \leq 1$ and $4A_e H_e^2 \leq 1$, implies that $d_1 = d_4 = 0$. The vertical velocity component $v_z(z)$ is therefore of the form

$$v_z(z) = \begin{cases} 
  d_2 \exp \left( \frac{1}{2H_B} - M_0 \right) z, & z > 0, \\
  d_3 \exp \left( \frac{1}{2H_e} + M_e \right) z, & z < 0,
\end{cases}$$

(20)

where

$$M_0 = \frac{(1 - 4A_B H_B^2)^{1/2}}{2H_B}$$

(21)

and

$$M_e = \frac{(1 - 4A_e H_e^2)^{1/2}}{2H_e}.$$ 

(22)

We note that $M_0, M_e > 0$, since we take the positive square roots of $(1 - 4A_B H_B^2)$ and $(1 - 4A_e H_e^2)$. Notice that $v_z(z)$ declines in the dense region ($z < 0$) but may or may not decline in the more tenuous region ($z > 0$). We explore this aspect more fully in Section 4.2.

We require that the normal component of velocity across the interface $z = 0$ be continuous; so $d_2 = d_3$. Additionally, integration of Equation (9) about a small neighbourhood of the interface shows that

$$\frac{\rho(z) (c_s^2(z) + v_\Lambda^2(z)) (\omega^2 - k^2 c_s^2(z))}{(k^2 c_s^2(z) - \omega^2)} \frac{dv_z(z)}{dz} - \frac{k^2 g \rho(z)c_s^2(z)v_z(z)}{(k^2 c_s^2(z) - \omega^2)}$$

is continuous across $z = 0$. Application of these matching conditions to the solution (20) then yields the transcendental dispersion relation (see also Miles and Roberts, 1991)

$$\frac{\rho_0(c_0^2 + v_\Lambda^2)(k^2 c_0^2 - \omega^2)}{(k^2 c_0^2 - \omega^2)} \left( \frac{1}{2H_B} - M_0 \right) + \frac{k^2 g \rho_0 c_0^2}{(k^2 c_0^2 - \omega^2)} =$$

$$= \frac{\rho_e c_e^2}{(k^2 c_e^2 - \omega^2)} \left\{ k^2 g - \left( \frac{1}{2H_e} + M_e \right) \omega^2 \right\}.$$ 

(24)

It may be noted that the Lamb modes $\omega = k_x c_0$ and $\omega = k_x c_e$ (Lamb, 1932) do not satisfy this dispersion relation but act as separatrices in the usual diagnostic diagram (see, for example, Roberts, 1985; see also Section 4.2).
The dispersion relation (24) describes the parallel propagation of surface waves at a single magnetic interface in a gravitationally stratified atmosphere under the assumption of a constant Alfvén speed in the magnetic region. The case of non-parallel propagation is discussed briefly in Jain and Roberts (1991a) and in detail by González and Gratton (1991).

Finally, we note that Equation (24) may be written as

\[
\frac{\omega^2}{k_x^2} = \frac{\rho_0}{\rho_0 + \rho_e} \left( \frac{M_e + 1/2H_e}{M_0 - 1/2H_B} \right) \left( \frac{v_A^2}{m_0^2} - \frac{\rho_0c_0^2}{(k_x^2c_0^2 - \omega^2)} - \frac{\rho_ec_e^2}{(k_x^2c_e^2 - \omega^2)} \right) \left( \frac{g}{\rho_0} \frac{(M_0 - 1/2H_B)}{m_0^2} + \frac{\rho_e}{m_e^2} \frac{(M_e + 1/2H_e)}{m_e^2} \right),
\]

where

\[
m_0^2 = \frac{(k_x^2v_A^2 - \omega^2)(k_x^2c_0^2 - \omega^2)}{(c_0^2 + v_A^2)(k_x^2c_T^2 - \omega^2)}
\]

and

\[
m_e^2 = \frac{(k_x^2c_e^2 - \omega^2)}{c_e^2}.
\]

The form (25) of the general dispersion relation proves useful for comparing with certain special cases (e.g., the non-magnetic case).

3. Cutoff Curves

Before considering a numerical solution of the general dispersion relation (25) it is necessary to examine the constraints under which the equation is derived. We require that \(1 - 4A_BH_B^2 > 0\) and \(1 - 4A_eH_e^2 > 0\). These constraints represent the cutoff frequencies for the modes. If either \(1 - 4A_BH_B^2\) or \(1 - 4A_eH_e^2\) is negative, then solutions with oscillatory structure arise; these are internal (or body) modes (Lighthill, 1978), modified by the presence of a magnetic field. They will not be discussed further here (but see González and Gratton, 1991).

The condition \(4A_eH_e^2 < 1\) generates curves \(R_1\) and \(R_2\) in the \(\omega - k_x\) space which contain the allowed modes given by Equation (25). Similarly, the condition \(4A_BH_B^2 < 1\) generates confining curves \(R_3\) and \(R_4\). The form of these constraints depends upon whether \(\omega > k_xc_T\) or \(\omega < k_xc_T\). Thus, if \(\omega > k_xc_T\) then

\[
\max\left(\frac{c_T}{c_e}, R_1, R_3\right) \leq \frac{\omega}{k_xc_e} \leq \min\left(R_2, R_4\right).
\]
However, if \( \omega < k_x c_T \) then either

\[
R_1 \leq \frac{\omega}{k_x c_e} \leq \min \left( \frac{c_T}{c_e}, R_2, R_3 \right),
\]

(29)

or

\[
\max(R_1, R_4) \leq \frac{\omega}{k_x c_e} \leq \min \left( \frac{c_T}{c_e}, R_2 \right).
\]

(30)

Here \( R_1 \) and \( R_2 \) are given by

\[
R_{1,2}^2 = \frac{(1 + 4k_x^2 H_e^2) \pm \sqrt{(1 + 4k_x^2 H_e^2)^2 - 64k_x^2 H_e^2 \left( \frac{\gamma - 1}{\gamma^2} \right)}}{8k_x^2 H_e^2},
\]

(31)

and \( R_3^2 \) and \( R_4^2 \) satisfy

\[
1 - \frac{4c_0^2 \Gamma^2}{c_e^4} \left( \frac{c_0^2}{c_e^2} + \frac{v_A^2}{c_e^2} \right) \left( \frac{R_{3,4}^2}{c_e^2} - \frac{c_T^2}{c_e^2} \right) \left( \frac{R_{3,4}^2}{c_e^2} - \frac{c_T^2}{c_e^2} \right) = 0.
\]

(32)

In the limit as \( k_x H_e \to \infty \), which includes the case \( \gamma \to 0 \), we obtain

\[
R_1 \to 0, \quad R_2 \to 1
\]

(33)

and

\[
R_3 \to \min \left( \frac{c_0^2}{c_e^2}, \frac{v_A^2}{c_e^2} \right), \quad R_4 \to \max \left( \frac{c_0^2}{c_e^2}, \frac{v_A^2}{c_e^2} \right).
\]

(34)

In the limit as \( k_x H_e \to 0 \) we find that

\[
R_1^2 \to \frac{4(\gamma - 1)}{\gamma^2}, \quad R_2^2 \to \infty
\]

(35)

and

\[
R_3^2 \to \frac{c_0^2 \Gamma^2}{c_e^2} \left( \frac{v_A^2}{c_e^2} + 4 \frac{c_0^2}{c_e^2} (\Gamma - 1) \right), \quad R_4^2 \to \infty.
\]

(36)

Thus, for example, with \( c_0/c_e = 0.9 \) and \( v_A/c_e = 0.5 \) (as used later in the numerical
solutions) we have
\[ R_1 \to 0, \quad R_2 \to 1, \quad R_3 \to 0.9, \quad R_4 \to 0.5 \quad \text{as} \quad k_x H_e \to \infty \]
and
\[ R_1 \to 0.98, \quad R_2 \to \infty, \quad R_3 \to 0.81, \quad R_4 \to \infty \quad \text{as} \quad k_x H_e \to 0. \]
In the above, and in all the subsequent illustrations, we have taken \( \gamma = \frac{5}{3}. \)
The cutoff curves are illustrated in the subsequent figures.

4. Special Cases

4.1. Magnetoacoustic Surface Waves

It is of interest to examine the general dispersion relation (25) in some limiting cases. For example, in the limit of zero gravity we have \( H_B^{-1} = H_0^{-1} = 0 \) and
\[ A_B \to -m_0^2, \quad A_e \to -m_e^2 \quad \text{as} \quad g \to 0, \quad (37) \]
and then Equation (25) reduces to
\[ \frac{\omega^2}{k_x^2} = \frac{\rho_0}{\left( \rho_0 + \frac{m_0}{m_e} \right)} \frac{v_A^2}{\rho_0}, \quad (38) \]
where \( m_0, m_e > 0. \) Thus, we recover the dispersion relation describing the parallel propagation of surface waves on a magnetic interface one side of which is field-free (Wentzel, 1979; Roberts, 1980, 1981a, b; Somasundaram and Uberoi, 1982; Miles and Roberts, 1989; Jain and Roberts, 1991b).

The dispersion relation (38) for surface waves in an unstratified medium implies that the longitudinal phase-speed \( \omega/k_x \) of a surface wave must lie below the Alfvén speed. In fact, there may be two surface waves (Roberts, 1981a). The interface always supports a slow surface wave, satisfying \( \omega/k_x < \min(c_T, c_e). \) But if \( v_A > c_0 \) and \( c_e > c_0, \) then a second mode may propagate with a longitudinal phase-speed satisfying \( c_0 < \omega/k_x < \min(c_e, v_A); \) this is the fast surface wave. The properties of magnetoacoustic surface waves (in the absence of gravity) are described in detail in Miles and Roberts (1989) and extended to non-parallel propagation in Jain and Roberts (1991b); see also Uberoi (1982) for the cold plasma approximation and Uberoi (1972) for the incompressible case.

4.2. Non-Magnetic Case

It is also interesting to examine the dispersion relation (25) in the limit of zero magnetic field. With \( B_0 = 0, \) \( H_B = H_0 \) and Equation (25) reduces to the dispersion relation
\[ \frac{\rho_0 c_0^2}{(k_x^2 c_0^2 - \omega^2)} \left( k_x^2 g - \left( M_0 - \frac{1}{2H_0} \right) \omega^2 \right) = \]
\[ = \frac{\rho_e c_e^2}{(k_x^2 c_e^2 - \omega^2)} \left( k_x^2 g - \left( M_e + \frac{1}{2H_e} \right) \omega^2 \right), \quad (39) \]
where now
\[ M_0 = \frac{(1 - 4A_0H_0^2)^{1/2}}{2H_0} \]

is the non-magnetic version of \( M_0 \) given by Equation (21).

We note immediately that since there is no magnetic field there can be no magnetoacoustic surface mode solutions to this dispersion relation. In the notation of Equation (25), we may rewrite Equation (39) as
\[ \frac{\omega^2}{k_x^2} = -\frac{gk_x^2(\rho_0 - \rho_e)}{\rho_0(M_0 - 1/2H_0)m_e^2 + \rho_e(M_e + 1/2H_e)m_0^2}. \]

It is interesting to observe that we can remove the transcendentality of Equation (41). This was first noted by Bernstein and Book (1983) in their investigation of the effect of compressibility on the Rayleigh–Taylor instability. After much algebra, Equation (41) may be rewritten as a polynomial in \( \Omega^2 \equiv \omega^2/k_x g^2 \):
\[ (\Omega^4 - 1)(\Omega^2g - k_xc_0^2)^2(\Omega^2g - k_xc_e^2)^2[\Omega^8 - 2S\Omega^6 + \gamma] \]
\[ + (S^2 + 2\gamma - 1)\Omega^4 - 2(\gamma - 1)S\Omega^2 - D^2 = 0, \]
(42)
where \( S = k_x(c_0^2 + c_e^2)/g \) and \( D = k_x(c_0^2 - c_e^2)/g \). Bernstein and Book (1983) considered only the quartic (in \( \Omega^2 \)) factor of Equation (42); we consider the full expression as given by Equation (42).

Clearly, the process of removing the transcendentality nature of Equation (41), through squaring several times, may introduce spurious roots which satisfy the polynomial (42) but not the original dispersion relation (41). For example, Equation (42) possesses the solutions \( \Omega^2 = k_xc_0^2/g \) and \( \Omega^2 = k_xc_e^2/g \), roots of which are the Lamb modes \( \omega = k_xc_0 \) and \( \omega = k_xc_e \). However, although these are solutions to the polynomial Equation (42) they do not satisfy the transcendentally dispersion relation given by Equation (41).

The other possible solutions to the polynomial (42) are \( \Omega^2 = \pm 1 \) and those arising from the quartic (in \( \Omega^2 \)), namely
\[ \Omega^8 - 2S\Omega^6 + (S^2 + 2\gamma - 1)\Omega^4 - 2(\gamma - 1)S\Omega^2 - D^2 = 0. \]
(43)

The solution \( \Omega^2 = 1 \) (i.e., \( \omega^2 = gk_x \)) of Equation (42) is termed the \( f \)-mode; the \( f \)-mode is observed on the Sun. The solution \( \Omega^2 = 1 \) is in fact permitted only when \( \rho_0 > \rho_e \). This is the unstable case. We return to the \( f \)-mode in Section 5.1. There are no stable solutions to the \( \Omega^2 \)-quartic (43), which are also stable solutions to the dispersion relation (41), when \( \rho_0 > \rho_e \).

If \( \rho_0 < \rho_e \), then the opposite is true; that is, the solution \( \omega^2 = gk_x \) of Equation (42) does not satisfy the dispersion relation (41) and the only stable solutions to the dispersion relation (41) are those resulting from the quartic (43). In fact, when \( \rho_0 < \rho_e \) there is again only one stable solution of relation (41). This mode is shown in Figure 2(i) and is the gravity surface wave. The mode asymptotes to distinct limits at short and long wavelengths. We discuss these limits in turn.
Fig. 2. (i) The dimensionless phase-speed versus dimensionless horizontal wavenumber for the stable solution to the non-magnetic dispersion relation (41) when \( \rho_0 < \rho_1 \) (i.e., \( c_0 > c_e \)). Here \( c_0/c_e = 1.4 \), giving \( \omega/k_x c_e \rightarrow 1.38 \) as \( k_x H_e \rightarrow 0 \) and \( \omega/k_x c_e \rightarrow 0 \) as \( k_x H_e \rightarrow \infty \). The dashed curves \( R_1, R_2, R_3, \) and \( R_4 \) are determined from the requirements that \( 4A_0 H_0^2 \leq 1 \) and \( 4A_2 H_2^2 \leq 1 \). (ii) As in (i) with the inclusion of the curve \( A_0 = 0 \), shown as dot-dashed lines. The curve \( A_0 = 0 \) divides the region of evanescence in the upper medium into a domain wherein the vertical velocity component is growing with height \( z \) from a domain where it is declining with height. Together with the conditions imposed by the cutoff curves, this means that the mode has a growing \( v_z \) only in the region shown shaded. Elsewhere on the dispersion curve \( v_z \) declines with height.
Consider, first, the short wavelength limit. As \( k_x \to \infty, S \to \infty, \) and \( D \to \infty; \) then the dominant terms in the quartic (43) reduce it to

\[
S^2 \Omega^4 - D^2 = 0,
\]

the stable solution of which, for \( \rho_0 < \rho_e, \) is the familiar Rayleigh–Taylor dispersion relation

\[
\frac{\omega^2}{k_x^2} = -\frac{g}{k_x} \frac{\rho_0 - \rho_e}{\rho_0 + \rho_e}.
\]

This result in terms of Figure 2 means that the dimensionless phase-speed \( \omega/k_x c_e \to 0 \) as \( k_x H_e \to \infty. \) This is to be expected since as \( k_x H_e \to \infty \) then \( g \to 0 \) and so from Equation (45) \( \omega/k_x \to 0. \)

In the long wavelength limit, as \( k_x \to 0, \) the quartic (43) reduces to

\[
(2\gamma - 1)\Omega^4 - 2(\gamma - 1)S\Omega^2 - D^2 = 0,
\]

which possesses the solution

\[
\frac{\omega^2}{k_x^2 c_e^2} = \frac{(\gamma - 1)(\rho_0 + \rho_e) + ((\gamma - 1)^2(\rho_0 + \rho_e)^2 + (2\gamma - 1)(\rho_0 - \rho_e)^2)^{1/2}}{\rho_0(2\gamma - 1)}.
\]

Figure 2(i) is drawn for the parameter value \( c_0/c_e = 1.4 \) for which Equation (47) gives \( \omega^2/k_x^2 c_e^2 \approx 1.9 \) (i.e., \( \omega/k_x c_e \to 1.38 \) as \( k_x \to 0 \)). The mode has the property that its vertical velocity component in the upper atmosphere \( (z > 0) \) changes in character from exponentially growing for small values of \( k_x H_e \to \) exponentially declining for large values of \( k_x H_e. \) The mode always declines in the lower atmosphere \( (z < 0) \). This is illustrated in Figure 3 where the eigenfunctions for the mode are plotted for three values of \( k_x H_e. \)

This behaviour of the mode may be easily understood by an examination of the vertical velocity component:

\[
v_z(z) \sim \begin{cases} 
\exp \left\{ \frac{1}{2H_0} \left[ 1 - (1 - 4A_0 H_0^2)^{1/2} \right] z \right\}, & z > 0, \\
\exp \left\{ \frac{1}{2H_e} \left[ 1 + (1 - 4A_e H_e^2)^{1/2} \right] z \right\}, & z < 0,
\end{cases}
\]

where we take the positive square roots of \( (1 - 4A_0 H_0^2)^{1/2} \) and \( (1 - 4A_e H_e^2)^{1/2}. \) Evidently, for \( z < 0, v_z \) is always exponentially declining in nature. However, for \( z > 0, v_z \) declines only if \( A_0 < 0. \) Thus \( A_0 = 0 \) determines where the mode changes in character from exponentially growing to exponentially declining. The curve \( A_0 = 0 \) is drawn in Figure 2(ii), superimposed on the mode given in Figure 2(i). The curve is contained within the cutoff curves \( R_3 \) (with \( B_0 = 0 \)) and \( R_4 \) (with \( B_0 = 0 \)), asymptoting to these curves as \( k_x H_e \to \infty. \)

Now for evanescent solutions in the lower atmosphere the mode must lie within the region defined by \( R_1 \) and \( R_2, \) while for evanescent solutions in the upper atmosphere
Fig. 3. The eigenfunction $v_e(z)/v_e(0)$ for the gravity surface mode shown in Figure 2 at three values of $k_xH_e$. As the value of $k_xH_e$ is increased (i.e., as we move from left to right in Figure 2), so the eigenfunction of the mode switches from spatial growth to decay. There is a critical value of $k_xH_e$: if $k_xH_e$ exceeds $2 ((\gamma - 1)/\gamma^2)^{1/2} (\rho_0/\rho_e)^2$, then the mode will have a decaying character. For $c_0 = 1.4c_e$, this critical value of $k_xH_e$ is approximately 0.5.

The mode must lie within the region defined by $R_3 (B_0 = 0)$ and $R_4 (B_0 = 0)$. For the mode to have a declining $v_z$ it must lie within the region defined by the curve $A_0 = 0$. Thus, in Figure 2(ii) the unshaded region defined by $A_0 < 0$ and satisfying

$$\max (R_3(B_0 = 0), R_1) \leq \frac{\omega}{k_xc_e} \leq \min (R_2, R_4(B_0 = 0))$$

defines the region where the mode has a declining vertical velocity component. The shaded region represents the region where the mode has a spatially growing vertical velocity component.

The turning point of the curve $A_0 = 0$ is given by

$$\frac{\omega}{k_xc_e} = \frac{\rho_e}{\rho_0 \sqrt{2}} ; \quad k_xH_e = 2 \left(\frac{\gamma - 1}{\gamma^2}\right)^{1/2} \left(\frac{\rho_0}{\rho_e}\right)^2 ;$$

for the parameter values of Figure 2(ii) this gives

$$\frac{\omega}{k_xc_e} \approx 0.99 ; \quad k_xH_e \approx 0.5 .$$
Fig. 4. The familiar diagnostic diagram for an isothermal atmosphere, with the inclusion of the curve $A = 0$ (shown as a dot-dash line). Evanescent (non-propagating) modes exist only within the region defined by the cutoff curves (the solid lines). The unshaded region defined by $A < 0$ is where an evanescent mode will have a vertical velocity component that declines with height (cf. Figure 2(ii)).

Thus, the mode has a growing vertical velocity component in the region $k_x H_e < 0.5$ of Figure 2.

Figure 2 is in fact equivalent to the more familiar $\omega - k_x$ diagnostic diagram. In Figure 4 we present the usual diagnostic diagram for an isothermal atmosphere with sound speed $c_s$. Included in Figure 4 is the curve $A = 0$ defined by

$$\frac{k_x^2 \omega_g^2}{\omega^2} + \frac{(\omega^2 - k_x^2 c_s^2)}{c_s^2} = 0,$$

where $\omega_g = (\gamma - 1)^{1/2} c_s/\gamma H$ is the buoyancy (Brunt–Väisälä) frequency of the atmosphere. As in Figure 2, cutoff curves divide the $\omega - k_x$ space into regions of evanescence and regions of propagation. However, in contrast to Figure 2, there is only a single pair of cutoff curves in Figure 4. This is because the standard diagnostic diagram is usually drawn for one medium only (and not for two media as in Figure 2). Application of Figure 4 to the upper atmosphere gives the curves $R_4 (B_0 = 0)$ and $R_3 (B_0 = 0)$ of Figure 2(i); the curve $R_4$ corresponds to the upper (acoustic branch) cutoff curve of Figure 4, and the curve $R_3$ corresponds to the lower (gravity branch) cutoff curve of Figure 4.

As $k_x \to 0$, the upper cutoff curve tends to $\omega_a (= c_s/2H)$, the acoustic cutoff frequency for the atmosphere, while the lower cutoff curve tends to zero. As $k_x \to \infty$, the upper
cutoff curve tends to $\omega = k_x c_s$, the Lamb frequency for the atmosphere, while the lower
cutoff curve tends to the buoyancy frequency $\omega_g$. Note that $\omega_a > \omega_g$. The time scales
$\omega_a^{-1}$ and $\omega_g^{-1}$ are imposed in the atmosphere due to the presence of the gravitational
field. The region above the upper cutoff curve is one in which sound waves may
propagate. The region below the lower cutoff curve is where gravity or $g$-modes may
exist. As in Figure 2, the region bounded by these cutoff curves is where surface modes
may occur, corresponding to vertical evanescence (non-propagation). The curve $A = 0$
divides the region of evanescence still further and asymptotes to $\omega = k_x c_s$ and to $\omega = \omega_g$
as $k_x \to \infty$. In the region enclosed by the cutoff curves and where $A < 0$ a mode
possesses a decaying vertical velocity component; in the shaded region a mode has a
growing vertical velocity component.

The turning point of the curve $A = 0$ is given by

$$\omega = \omega_g \sqrt{2}, \quad k_x = \frac{2\omega_g}{c_s} .$$

(52)

Thus, in Figure 4, if $k_x < 2\omega_g/c_s$ then the mode will have a growing vertical velocity component.

Thus we conclude that in the absence of a magnetic field, the dispersion relation (41)
possess two distinct stable modes, the $f$-mode and the surface gravity mode, only one
of which may propagate in any given circumstances depending on the ratio of the
densities. If $\rho_0 > \rho_e$, then the $f$-mode exists (though the equilibrium is here unstable);
while if $\rho_0 < \rho_e$, it is the surface gravity mode that exists.

4.3. INCOMPRESSIBLE FLUID

Another case of the dispersion relation (25) that is of interest is that of an incompressible
medium, corresponding to $c_0$, $c_e \to \infty$. Consider the special case of a fluid with uniform
distributions of density, so that $\rho_0(z) = \rho_0$ and $\rho_e(z) = \rho_e$ (see Miles, 1991 for the case
of an incompressible fluid with non-uniform distributions of density). Then, for $c_0$ and
c_e large we obtain

$$m_0, m_e, M_0, M_e \to k_x$$

and thus Equation (25) reduces to the well-known result (e.g., Chandrasekhar, 1961)

$$\frac{\omega^2}{k_x^2} = \frac{\rho_0}{(\rho_0 + \rho_e)} \frac{v_A^2}{k_x} - \frac{g}{k_x} \frac{(\rho_0 - \rho_e)}{(\rho_0 + \rho_e)} .$$

(54)

In the absence of gravity Equation (54) exhibits a surface wave with speed $c_k$
($=(\rho_0/(\rho_0 + \rho_e))^{1/2} v_A$), given by Kruskal and Schwarzschild (1954) (see also Dungey
and Loughhead, 1954). In the absence of a magnetic field Equation (54) illustrates that
the interface is unstable ($\omega^2 < 0$) if $\rho_0 > \rho_e$, i.e., if the upper medium is denser than the
lower one. This is the familiar Rayleigh–Taylor instability (Chandrasekhar, 1961). The
effect of including a magnetic field is to stabilize the interface against short wavelength
oscillations.
5. Surface Waves

5.1. The f-mode

In the absence of a magnetic field Equations (7) and (8) possess the solution $A = 0, v_z \sim e^{k_x z}$, $\omega^2 = g k_x$ (i.e., $\Omega^2 = 1$). The solution $\Omega^2 = 1$ is the f-mode. The f-mode is incompressible ($A = 0$) and its frequency and spatial dependence are independent of the form of the thermal stratification. In the presence of a horizontal magnetic field the f-mode is no longer incompressible and its frequency is affected by the magnetic forces and the compressibility of the gas (Campbell and Roberts, 1989; Evans and Roberts, 1990).

The f-mode will be a solution to the dispersion relation (25) in the limit of zero field provided the mode satisfies the conditions on $v_z(z)$ under which the dispersion relation was derived. In the case of zero magnetic field, Equation (25) with $\Omega^2 = 1$ reduces to

$$\frac{1}{(\gamma k_x H_e c_0^2/c_e^2 - 1)} \left\{ \frac{1}{2k_x H_e c_0^2/c_e^2} [1 - |1 - 2k_x H_e c_0^2/c_e^2| - 1] \right\} =$$

$$= \frac{1}{(\gamma k_x H_e - 1)} \left\{ \frac{1}{2k_x H_e} [1 + |2k_x H_e - 1| - 1] - 1 \right\}. \quad (55)$$

A scrutiny of Equation (55) reveals that it is satisfied only if $1 < 2k_x H_e < c_0^2/c_e^2$; that is, Equation (55) is satisfied only if the horizontal wavenumber lies in the interval

$$\frac{1}{2H_e} < k_x < \frac{1}{2H_0}. \quad (56)$$

With $k_x$ lying in this interval there is a mode with $\omega^2 = g k_x$ and $v_z \sim e^{k_x z}$; this satisfies boundness of the kinetic energy density but the velocity itself is unbounded in the upper atmosphere ($z > 0$). It follows immediately from the inequality (56) that the f-mode can propagate only if $H_e > H_0$, i.e., if $c_0 < c_e$ or (equivalently) $\rho_0 > \rho_e$. Thus the upper region must be warmer and more tenuous than the underlying atmosphere for the f-mode to exist in an isothermal non-magnetic media. This is the same condition that pertains for the existence of fast magnetoacoustic surface waves on an unstratified field-free interface (Roberts, 1981a; Miles and Roberts, 1989; Jain and Roberts, 1991b). Of course, with $\rho_0 > \rho_e$, the equilibrium is unstable to the Rayleigh–Taylor mode and so one might argue that the f-mode is consequently of no interest. However, against this is the fact that the Sun supports an f-mode (see, for example, the observations by Libbrecht and Kaufman, 1988). Added to this, we show below that with the inclusion of a magnetic field the magnetically-modified f-mode evolves with increasing field strength into the fast magnetoacoustic surface wave. In this respect, then, the f-mode provides a ‘seed-bed’ for the fast magnetoacoustic surface mode.

We consider now the influence of the magnetic field on the f-mode. For a weak field, we seek an approximate solution to the dispersion relation (25), with the property that
\( \Omega^2 \rightarrow 1 \) as \( v_\Lambda/c_e \rightarrow 0 \), by writing
\[
\Omega^2 = 1 + \alpha v_\Lambda^2/c_e^2, \quad v_\Lambda/c_e \rightarrow 0, \tag{57}
\]
for constant \( \alpha \). Expanding the dispersion relation (25) for small \( v_\Lambda/c_e \) allows us to determine \( \alpha \). After some detailed algebra, we find that
\[
\alpha = \frac{\gamma(2k_xH_e - 1)}{2(1 - c_0^2/c_e^2)}. \tag{58}
\]
Since the conditions \( 2k_xH_e > 1 \) and \( c_0^2/c_e^2 < 1 \) are already imposed in deriving the dispersion relation (55), we see that \( \alpha \) is positive. Thus, in the presence of a magnetic field the frequency of the \( f \)-mode is increased. Other investigations of the effect of a magnetic field on the \( f \)-mode frequency have similarly found an increase (Campbell and Roberts, 1989; Evans and Roberts, 1990).

In terms of the original variables, the first-order correction to the \( f \)-mode in the presence of a magnetic field gives the result
\[
\frac{\omega^2}{gk_x} \approx 1 + \frac{\gamma}{2} \frac{(2k_xH_e - 1)}{(c_e^2 - c_0^2)} v_\Lambda^2, \tag{59}
\]
valid for \( c_e > c_0 \) and \( 2k_xH_e > 1 \). We note that Equation (59) is valid only within the domain given by inequality (56), since otherwise the expansion (57) does not reduce to the \( f \)-mode as the magnetic field tends to zero. The analytic result (59) compares favourably with a full numerical solution (see Miles, 1991, for a detailed discussion). The numerical solution shows that the effect of including the magnetic field is two-fold: the restriction on the horizontal wavenumber is lessened and shifted to the right, thereby resulting in a smaller maximum longitudinal phase-speed.

The vertical velocity of the disturbance in the \( f \)-mode, given by Equation (48), can be calculated as a function of height for any point on the dispersion curve. Figure 5(i) compared the eigenfunctions for the \( f \)-mode for two different values of the horizontal wavenumber, namely for \( k_xH_e = 0.5135 \) (just as the \( f \)-mode begins to propagate) and for \( k_xH_e = 1.8287 \) (at the other extreme of its range of propagation). Evidently the eigenfunction of the \( f \)-mode is depressed as the horizontal wavenumber is increased. We note that \( v_z \rightarrow 0 \) as \( z \rightarrow \infty \). This is because the \( f \)-mode lies entirely within the region \( A_0 > 0 \). However, the conditions imposed on the solutions (48) ensure that the energy density (kinetic plus magnetic) remains finite as \( |z| \rightarrow \infty \). The profiles of kinetic energy density for the eigenfunctions in Figure 5(i) are shown in Figure 5(ii).

5.2. MAGNETOACOUSTIC-GRAVITY SURFACE WAVES

We turn now to an examination of all of the modes given by the numerical solution of the dispersion relation (25). This includes the \( f \)-mode modified by magnetism, as discussed above, and makes clear the complex interconnections that exist between the various surface waves. The behaviour of the modes is explored over the range of the various parameters.
Fig. 5. (i) The eigenfunction $v_z(z)/v_z(0)$ for the $f$-mode with $c_0/c_\infty = 0.5$ and cases (a) $k_x = 0.5135$ and (b) $k_x H_\infty = 1.8287$, chosen close to the limits of allowable wavenumber. (ii) The profile of kinetic energy density, $\frac{1}{2} \rho(z) v_z^2$, of the $f$-mode for the eigenfunctions in (i), with $c_0/c_\infty = 0.5$. (a) $k_x H_\infty = 0.5135$, (b) $k_x H_\infty = 1.8287$. The kinetic energy density is plotted in units of $\frac{1}{2} \rho_0 v_z^2(0)$. 
Fig. 6a–d. The phase-speeds of magnetoacoustic-gravity surface modes at an isothermal magnetic interface with a constant Alfvén speed. Cases (a) $c_0/c_e = 0.9$, $v_A/c_e = 0.5$; (b) $c_0/c_e = 0.9$, $v_A/c_e = 0.75$; (c) $c_0/c_e = 0.9$, $v_A/c_e = 1.0$, and (d) $c_0/c_e = 1.4$, $v_A/c_e = 0.75$. The horizontal dashed lines are $\omega = k_x c_T$ and the asymptotes to which the surface modes tend as $k_x H_\parallel \to \infty$, as determined by the case $g = 0$ (see Equation (38)). The dashed curves $R_1$, $R_2$, $R_3$, and $R_4$ are determined from the requirements that $4A_p H_\parallel^2 \leq 1$ and $4A_p H_\parallel^2 \leq 1$ (see Section 3).
Figure 6 illustrates the variation of the phase-speed with the parameter $k_x H_e$, taking $c_0 < c_e$ (specifically $c_0/c_e = 0.9$) for three values of $v_A/c_e$ (Figures 6(a–c)) and taking $c_0 > c_e$ (specifically $c_0/c_e = 1.4$) for case (d) $v_A/c_e = 0.75$ (Figure 6(d)). The dashed horizontal lines correspond to $\omega = k_x c_T$ and the asymptotes to which the surface modes tend as $k_x H_e \to \infty$. These asymptotes correspond to the values which the fast and slow magnetoacoustic surface modes possess in the zero gravity case and are determined by the transcendental dispersion relation (38).
Recall that in the zero gravity case (discussed in detail in Roberts, 1981a; Miles and Roberts, 1989; Jain and Roberts, 1991b) that whereas the slow mode always occurs, the fast magnetoacoustic surface wave propagates only when both \( v_A > c_0 \) and \( c_e > c_0 \). This is evident here. For example, in Figures 6(a, b), for which \( v_A < c_0 \), one mode only (the slow surface wave) propagates for large \( k_x H_e \) (which includes \( g \to 0 \)). By contrast, in Figure 6(c), where \( v_A > c_0 \) and \( c_e > c_0 \), both surface waves propagate in the zero gravity limit and therefore for large \( k_x H_e \). We note also the absence of the fast surface mode as \( k_x H_e \to \infty \) when \( c_e < c_0 \) (see Figure 6(d)), which again is consistent with the case of zero gravity.

The lower phase-speed curves in Figure 6(d) are readily identified as the slow magnetoacoustic-gravity surface wave (that is, the slow magnetoacoustic surface wave modified by the presence of gravity). Similarly, the upper phase-speed curve in Figure 6(c) can be described as the fast magnetoacoustic-gravity mode. In Figure 6(a, b), however, the upper phase-speed curve is more akin to the \( f \)-mode modified by the magnetic field. Thus, we see that as the magnetic field strength is increased (through the parameter \( v_A/c_e \)) and \( k_x H_e \) is increased, so the \( f \)-mode develops into the fast magnetoacoustic-gravity surface mode.

The upper mode in Figure 6(d) is the magnetic equivalent to the gravity surface wave in Figure 2. In the non-magnetic case the gravity surface wave occurs only when \( c_0 > c_e \) and so the \( f \)-mode is not a solution to the dispersion relation (41). Similarly, here in the magnetic case \( c_0 > c_e \) and therefore the fast magnetoacoustic-gravity surface wave (having developed from the \( f \)-mode) is not a solution to the dispersion relation (25).

By comparing Figures 6(a–c), we may see the development of the \( f \)-mode from the non-magnetic case \((\omega^2 = g k_x)\) through a gradual increase of the magnetic field strength to its eventual merger with the fast magnetoacoustic surface wave (as \( k_x H_e \to \infty \)) in Figure 6(c).

It is of interest to see how the results we have obtained compare with the well-known dispersion relation for surface waves in an \textit{incompressible} medium with uniform distribution of densities (see Equation (54)). Unfortunately, such a comparison is difficult to make from Figure 6. Accordingly, in Figure 7, we have represented our dispersion curves, normalized in units appropriate for such a comparison. Of particular interest is the slow surface wave. Figure 7 illustrates the phase-speed (in units of the Alfvén speed \( v_A \)) as a function of wavenumber \( k_x \) (in units of \( g/v_A^2 \)) for increasing values of the sound speed \( c_0 \) within the field, holding fixed the sound speed \( c_e \) in the field-free region. Comparing the slow surface wave (shown as a solid curve in Figure 7) with its incompressible counterpart (determined by Equation (54) and shown in Figure 7 as a dot-dashed curve) we see a close similarity between the two, the two curves merging together as we approach (for large \( c_0/v_A \)) the incompressible extreme. This result is to be expected, since for fixed \( \gamma (= \frac{3}{2}) \) as we increase the sound speed, the scale-height is also increased and so the density has a weaker exponential dependence, and ultimately tends to a constant (as is the case described by Equation (54)).

It can be seen from Equation (54) that the incompressible mode asymptotes to \( c_e/v_A \) as \( k_x v_A^2/g \to \infty \). We note that as the value of \( c_0/v_A \) is increased the trend of the modes
Fig. 7a–d. The phase-speeds of magnetoacoustic-gravity surface modes normalized with respect to the Alfvén speed, with $c_s/v_A = 5.0$. Cases (a) $c_0/v_A = 2.5$; (b) $c_0/v_A = 3.75$; (c) $c_0/v_A = 4.5$; and (d) $c_0/v_A = 7.0$. The solid curve is the slow surface mode, the dot-dashed one its incompressible counterpart (determined by Equation (54)). For the sake of clarity, the fast mode has been omitted.
switches from monotonically increasing (Figures 7(a–c)) to monotonically decreasing (Figure 7(d)). This is most easily explained by considering the incompressible Equation (54). Clearly, if \( \rho_0 > \rho_e \) then the mode asymptotes from below to \( c_k/v_A \) as \( k_x v_A^2/g \to \infty \), whereas if \( \rho_0 < \rho_e \) then the mode asymptotes from above.

Finally, in Figure 8 we compare the eigenfunctions of the \( f \)-mode (in the presence of a magnetic field) with those of the fast and slow magnetoacoustic-gravity surface waves.
The eigenfunction of the $f$-mode (modified by the presence of the magnetic field) at $k_x H_e = 1.0$, displayed in Figure 6(c), is shown together with the eigenfunctions of the fast and slow magnetoacoustic-gravity surface modes at $k_x H_e = 5.0$ (Figure 6(c)). The curves are plotted for a fixed ratio of the sound speeds, namely $c_0/c_e = 0.9$, and field strength $v_A/c_e = 1.0$. We note that the $f$-mode (in the presence of a magnetic field) and the fast magnetoacoustic-gravity surface mode have similar profiles, demonstrating the connection which exists between them. By contrast, there is a marked difference between the eigenfunctions of the $f$-mode and the slow magnetoacoustic-gravity surface mode, the latter having a much more symmetric profile.

Observe that the $f$-mode (in the presence of a magnetic field) has a declining vertical velocity component. This is the case only when both $v_A \geq c_0$ and $c_e > c_0$. If we were to impose the curve $A_B = 0$ (see Equation (15)) on Figure 6(c) then the upper mode (which 'includes' the $f$-mode) is found to lie entirely within the region $A_B < 0$. However, if we were to impose the curve $A_B = 0$ on Figures 6(a) and 6(b) then the upper mode would lie in the region $A_B > 0$ and therefore have a growing vertical velocity component. The fast magnetoacoustic-gravity surface wave will always lie in the region $A_B < 0$, since it must satisfy the condition that both $v_A \geq c_0$ and $c_e > c_0$. In fact, both the slow and fast magnetoacoustic-gravity surface waves always have a bounded vertical velocity component.
6. Conclusions

The dispersion relation governing the parallel propagation of magnetoacoustic-gravity surface waves and the $f$-mode at a single magnetic interface between isothermal gases is highly transcendental and conceals a complex array of modal behaviour. By considering the behaviour of this dispersion relationship in the extremes of high horizontal wavenumber (short wavelength), zero magnetic field and for the limit of an incompressible fluid, we have attempted to shed light on this intricate behaviour. The treatment of these special cases is complemented by a full numerical investigation of the general case.

In the absence of a magnetic field there arises the possibility of a mode which has a phase-speed that is independent of the compressibility of the atmosphere; this is the $f$-mode, with a frequency (squared) given by $\omega^2 = gk_x$ for horizontal wavenumber $k_x$. For the non-magnetic configuration of two gases exponentially stratified in density, the $f$-mode exists only when $\rho_0 > \rho_e$ (i.e., when the density of the gases immediately above the interface is greater than that immediately below) and then its wavenumber $k_x$ is restricted in value. In the presence of a magnetic field, the restriction on the $f$-mode (now modified slightly in frequency) is weakened. If the magnetic field is so strong that its Alfvén speed equals the sound speed within the field, then the $f$-mode propagates for arbitrarily large wavenumber and merges with the fast surface wave of the unstratified ($g = 0$) medium. The restriction at low wavenumber (high wavelength) remains and signals the occurrence of wave leakage (at high wavelength) through the generation of internal waves.

If $\rho_0 < \rho_e$ and the field is absent, then the $f$-mode is replaced by a surface gravity wave. This mode too is restricted to wavenumbers greater than a critical value. The introduction of a magnetic field is able to remove this restriction at low wavenumber but brings in a break in the permitted wavenumber range for propagation (see Figure 6(d)). At small wavelengths (high wavenumber), the surface gravity mode in the presence of a field becomes the slow magnetoacoustic surface wave.

It has been shown that the domain of evanescence in the kinetic energy of a mode can be divided into two regions corresponding to whether a mode has a growing or decaying vertical velocity component. The vertical velocity component of the $f$-mode is always exponentially growing in nature. However, the fast magnetoacoustic-gravity surface wave (having developed from the $f$-mode) and the slow magnetoacoustic-gravity surface wave always have a decaying vertical velocity component. All the modes have decaying total (magnetic plus kinetic) energy.

Dispersion curves for the fast and slow magnetoacoustic-gravity surface waves show that both modes propagate with phase-speeds which decrease as the horizontal wave number is increased, gradually asymptoting to distinct limits as $k_x H_e \to \infty$, provided both $c_e > c_0$ and $v_A \geq c_0$. If either of these conditions is not met then the propagation of the fast magnetoacoustic-gravity surface mode is restricted in wavenumber and only the slow magnetoacoustic surface wave exists as $k_x H_e \to \infty$.

It is natural to ask how these modes fit into the well-known spectrum of $p$-modes observed on the Sun. In Figure 9 we present the fast and slow surface gravity waves
The cyclic frequency of magnetoacoustic-gravity surface modes non-dimensionalized with respect to the acoustic cutoff (cyclic) frequency \( v_{ac} = \frac{\omega_{ac}}{2\pi} \), where \( \omega_{ac} = \frac{c_e}{2H_e} \) is the acoustic cutoff frequency for the field-free region (with \( \frac{c_0}{c_e} = 0.9 \), \( \frac{v_A}{c_e} = 0.5 \) (cf. Figure 6(a))). The dot-dashed line \( n = 0 \) is the \( f \)-mode. The other dot-dash lines, labelled \( n = 1, 2, \ldots \) are the \( p \)-modes (given by Equation (60)). The dashed curves \( R_1, R_2, \) and \( R_4 \) are determined from the requirements that \( 4A_p H_p^2 \leq 1 \) and \( 4A_e H_e^2 \leq 1 \) (see Section 3).

The curve \( R_1 \) tends to \( \omega_{ac} \) as \( k_x H_e \to \infty \). For the sake of clarity, \( R_3 \) has been omitted.

for the case \( \frac{c_0}{c_e} = 0.9 \) and \( \frac{v_A}{c_e} = 0.5 \) (cf. Figure 6(a)), using a standard frequency-wavenumber plot. Added to this figure are the familiar \( p \)-modes of an adiabatically stratified field-free medium, taken here to have cyclic frequency \( v = \frac{\omega}{2\pi} = v_n \), given by (see, for example, Campbell and Roberts, 1989)

\[
v_n^2 = \left( 1 + \frac{2n}{m} \right) g k_x
\]

(60)

for mode number \( n = 1, 2, 3, \ldots \) and polytropic index \( m (= \frac{3}{2}) \). It can be seen that the surface modes are detectable only at a high degree \( l \). We may relate the degree \( l \) to our horizontal wavenumber \( k_x \) by setting \( k_x \approx l/R_0 \), for solar radius \( R_0 \approx 7 \times 10^5 \) km. Thus, with a scale height \( H_e = 10^2 \) km (typical of the temperature minimum) we see that the fast surface wave makes an appearance at degree \( l \geq 4000 \), and its frequency is above the acoustic cutoff frequency (of 5.38 MHz) in the temperature minimum.
Only very recently have observers begun to study systematically modes of such high degree and frequency (Fernandes et al., 1992). Accordingly, it will be interesting to see in the near future whether any observational (helioseismological) evidence emerges for magnetoacoustic surface modes.

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References