III. Effect of Equilibrium Flow

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Abstract. The resonances that appear in the linear compressible MHD formulation of waves are studied for equilibrium states with flow. The conservation laws and the jump conditions across the resonance point are determined for 1D cylindrical plasmas. For equilibrium states with straight magnetic field lines and flow along the field lines the conserved quantity is the Eulerian perturbation of total pressure. Curvature of the magnetic field lines and/or velocity field lines leads to more complicated conservation laws. Rewritten in terms of the displacement components in the magnetic surfaces parallel and perpendicular to the magnetic field lines, the conservation laws simply state that the waves are dominated by the parallel motions for the modified slow resonance and by the perpendicular motions for the modified Alfvén resonance.

The conservation laws and the jump conditions are then used for studying surface waves in cylindrical plasmas. These waves are characterized by resonances and have complex eigenfrequencies when the classic true discontinuity is replaced by a nonuniform layer. A thin non-uniform layer is considered here in an attempt to obtain analytical results. An important result related to earlier work by Hollweg et al. (1990) for incompressible planar plasmas is found for equilibrium states with straight magnetic field lines and straight velocity field lines. For these equilibrium states the incompressible and compressible surface waves have the same frequencies at least in the long wavelength limit and there is an exact correspondence with the planar case. As a consequence, the conclusions formulated by Hollweg et al. still hold for the straight cylindrical case. The effects of curvature are subsequently considered.

1. Introduction

Hollweg et al. (1990; hereafter referred to as HYCG 90), studied resonant absorption of MHD surface waves in the presence of an equilibrium flow. As a preliminary illustration of the effects of an equilibrium flow they considered incompressible waves in a planar geometry. In an attempt to obtain analytical results they assumed that the surface waves are supported by a thin non-uniform plasma layer. The surface wave on a true discontinuity which has a real frequency (exp(−iot) dependency) is replaced by a surface wave which then has a complex frequency of which the imaginary part determines the damping or overstability of the wave. HYCG 90 found that an equilibrium flow can either increase or decrease the resonant absorption rate and that there
are values of the velocity shear for which the absorption rate is zero. In addition they found that there can be resonances leading to overstability even at velocity shears which are below the threshold for the Kelvin–Helmholtz instability.

A key assumption for obtaining the dispersion relation is that the Eulerian perturbation of total pressure across the non-uniform layer is constant. The imaginary part of the dispersion relation and as a consequence the imaginary part of the frequency is found by using the Landau prescription for going around the pole on the real axis.

The present paper takes the analysis by HYG 90 of surface waves in a stationary equilibrium further and puts the assumption that the Eulerian perturbation is constant across the non-uniform layer and the prescription for going around the pole on a firm basis. In Section 2 the conservation laws and the jump conditions at a resonance are obtained for a stationary equilibrium. This Section closely follows a recent paper by Sakurai, Goossens, and Hollweg (1991, Paper I) and generalizes the results found there for a static equilibrium to stationary equilibrium states. The jump conditions are then used in Section 3 to obtain the complex eigenfrequencies of MHD surface waves on thin non-uniform layers in cylindrical stationary equilibrium states. Section 4 summarizes the conclusions.

2. Jump Conditions across a Resonant Surface

This section provides a mathematical analysis of the jump conditions across a resonant surface in 1D equilibrium states that have both an equilibrium flow and magnetic field. It generalizes the jump conditions obtained for 1D magnetostatic (no equilibrium flow) states in Paper I. Details of the present analysis are given for a cylindrical equilibrium. In a cylindrical equilibrium the magnetic field lines can be curved in contrast to a planar equilibrium which has always straight field lines. We shall see that the curvature of the magnetic field lines plays an important role in the jump conditions. The jump conditions for a planar equilibrium are identical to those for a cylindrical equilibrium with straight field lines.

2.1. Differential Equations for \( \xi_r \) and \( P' \)

A system of cylindrical coordinates \((r, \varphi, z)\) is adopted in the subsequent analysis. The equilibrium physical variables are functions of the radial coordinate \( r \) only. The equilibrium magnetic field and flow velocity have longitudinal and azimuthal components

\[
B = (0, B_\varphi(r), B_z(r)), \quad \mathbf{v} = (0, v_\varphi(r), v_z(r)),
\]

and satisfy the equation of stationary equilibrium

\[
\frac{d}{dr} \left( p + \frac{B^2}{2\mu} \right) = \frac{\rho v^2_\varphi}{r} - \frac{B^2_\varphi}{\mu r}.
\]

Here \( \rho \) is density and \( p \) pressure. Since the equilibrium quantities depend on \( r \) only, the perturbed quantities can be Fourier-analyzed with respect to the ignorable coordinates.
\( \varphi \) and \( z \) and time \( t \) and put proportional to

\[
\exp[i(m\varphi + kz - \omega t)].
\]

We can eliminate all but two of the perturbed variables from the linearized MHD equations by algebraic means. One step is the definition of the perturbed total pressure \( P' \):

\[
P' = p' + \mathbf{B} \cdot \mathbf{B}'/\mu,
\]

where an accent denotes an Eulerian perturbation. The other key is the derivation of the compression term

\[
\text{div} \, \mathbf{v}' = \frac{i\Omega^2 P' + Qv'_r}{\rho(c^2 + v_A^2)(\Omega^2 - \omega_c^2)},
\]

where \( \Omega \) is the Doppler-shifted frequency

\[
\Omega = \omega - \omega_f,
\]

and

\[
Q = -(\Omega^2 - \omega_A^2) \rho v_r^2/r + 2\Omega^2 B^2/\mu r + 2\Omega f_B B \varphi v_\varphi/\mu r,
\]

with \( \omega_f \) the flow frequency defined as

\[
\omega_f = \frac{m}{r} v_\varphi + kv_z,
\]

and

\[
f_B = k \cdot \mathbf{B} = \frac{m}{r} B_\varphi + kB_z.
\]

\( c, v_A, \omega_A, \) and \( \omega_c \) stand for the adiabatic velocity of sound, the Alfvén velocity, the Alfvén frequency, and the cusp frequency. The squares of these quantities are defined as

\[
c^2 = \gamma p/\rho, \quad v_A^2 = B^2/\mu \rho, \quad \omega_A^2 = f_B^2/\mu \rho,
\]

\[
\omega_c^2 = \omega_A^2 c^2/(c^2 + v_A^2), \quad \omega_A^2 c^2/(c^2 + v_A^2).
\]

Note that \( \Omega, \omega_A, \omega_c, \) and \( \omega_f \) depend on the spatial coordinate \( r \) for a non-uniform equilibrium.

After algebraic reduction it is found that the perturbed variables \( v'_r \) and \( P' \) satisfy a pair of ordinary linear differential equations of first order. A simpler pair of differential equations can be obtained when the radial component of the Lagrangian displacement \( \xi \) is used instead of \( v'_r \). An invariant representation of the relation between \( \xi \) and \( v' \) is

\[
v' = \frac{\partial \xi}{\partial t} + \text{rot}(\xi \times v) - \xi \text{ div } v + v \text{ div } \xi.
\]

Since the present equilibrium flow is incompressible, the third term in the right-hand member of (9) vanishes. The components of the Eulerian perturbation of velocity are
then related to the components of the Lagrangian displacement as

\begin{align}
v'_r &= -i\Omega\xi'_r , \quad (10a) \\
v'_\varphi &= -i\Omega \xi'_{\varphi} - \zeta_r r \frac{d}{dr} \left( \frac{v_{\varphi}}{r} \right) , \quad (10b) \\
v'_z &= -i\Omega \xi'_{zz} - \xi'_r \frac{dv_z}{dr} . \quad (10c)
\end{align}

The differential equations for \( \xi_r \) and \( P' \) are

\begin{align}
D \frac{d}{dr} (r\xi'_r) &= C_1 r\xi'_r - C_2 rP' , \quad (11) \\
D \frac{dP'}{dr} &= C_3 \xi'_r - C_1 P' , \quad (12)
\end{align}

where

\begin{align}
D &= \rho(c^2 + v^2)(\Omega^2 - \omega^2)(\Omega^2 - \omega^2) , \quad (13a) \\
C_1 &= Q\Omega^2 - 2m(c^2 + v^2)(\Omega^2 - \omega^2)T/r^2 , \quad (13b) \\
C_2 &= \Omega^2 - (c^2 + v^2)(m^2/r^2 + k^2)(\Omega^2 - \omega^2) , \quad (13c) \\
C_3 &= D \left\{ \rho(\Omega^2 - \omega^2) + r \frac{d}{dr} \left[ \frac{1}{\mu} \left( \frac{B_\varphi}{r} \right)^2 - \rho \left( \frac{v_{\varphi}}{r} \right)^2 \right] \right\} + \\
&+ Q^2 - 4(c^2 + v^2)(\Omega^2 - \omega^2)T^2/r^2 , \quad (13d) \\
T &= \frac{f_B B_\varphi}{\mu} + \rho \Omega v_{\varphi} . \quad (13e)
\end{align}

Equations (11) and (12) govern the linear displacements of a compressible 1D cylinder in magnetostationary equilibrium. They agree with the equations obtained by Bondeson, Iacono, and Bhattacharjee (1987). In the absence of an equilibrium flow they reduce to the equations obtained by Appert, Gruber, and Vaclavik (1974) and by Hain and Lust (1958) for a static straight cylinder. It is convenient here to note that Equations (11)–(12) are formally identical to the corresponding equations for a static equilibrium. The coefficient functions \( D, C_1, C_2 \) and \( C_3 \) are now more complicated because of the equilibrium flow contributions, but the form of the equations is identical. This enables us to follow the analysis of Paper I. The other perturbed quantities \( (P', \rho' \text{ and so on}) \) can be computed once \( \xi_r \) and \( P' \) are known. For later use we note that the components of the displacement vector in the magnetic surfaces parallel to the magnetic field lines, \( \xi_{||} = \xi \cdot B/B \), and perpendicular to the magnetic field lines \( \xi_{\perp} = (\xi_{\varphi} B_z - \xi_z B_{\varphi})/B \), are related to \( \xi_r \) and \( P' \) as

\begin{align}
(\Omega^2 - \omega^2)\xi_{\perp} &= \frac{i}{\rho B} (g_B P' - 2B_z T \xi_r/r) , \quad (14a)
\end{align}
\[(\Omega^2 - \omega_c^2)\xi_r = \frac{i f_B}{\rho B} \frac{e^2}{c^2 + v_A^2} (\Omega^2 P' - Q \omega_r)/\Omega^2 - i(\Omega^2 - \omega_c^2) (2\Omega B_\phi v_\phi + f_B v_\phi^2)\xi_r/B \Omega^2 r . \tag{14b}\]

Here \(B = (B_\phi^2 + B_z^2)^{1/2}\) and
\[g_B = (k \times B)_r = \frac{m}{r} B_z - k B_\phi\tag{15}\]
as in Paper I.

In the absence of an equilibrium flow, expressions (14) reduce to the corresponding expressions found in Paper I. The most obvious effect of an equilibrium flow is the additional term in the right-hand member of Equation (14b) which introduces a displacement parallel to the field lines even when \(c^2 = 0\), provided that the field lines are curved (\(B_\phi \neq 0\)).

We can combine Equations (11)–(12) into one ordinary linear differential equation for \(\xi_r\) as
\[
\frac{d}{dr} \left[ f(r; \omega^2) \frac{d}{dr} (r \xi_r) \right] - g(r; \omega^2) r \xi_r = 0 , \tag{16}\]
where
\[f(r; \omega^2) = \frac{D}{r C_2} , \quad g(r; \omega^2) = \frac{d}{dr} \left( \frac{C_1}{r C_2} \right) - \frac{1}{r D} \left( C_3 - \frac{C_1^2}{C_2} \right) , \tag{17}\]
or for \(P'\):
\[
\frac{d}{dr} \left[ \tilde{f} (r; \omega^2) \frac{d}{dr} P' \right] - \tilde{g} (r; \omega^2) P' = 0 , \tag{18}\]
where
\[\tilde{f} (r; \omega^2) = \frac{r D}{C_3} , \quad \tilde{g} (r; \omega^2) = -\frac{d}{dr} \left( \frac{r C_1}{C_3} \right) - \frac{r}{D} \left( C_2 - \frac{C_1^2}{C_3} \right) . \tag{19}\]

The set of differential equations (11)–(12) supplemented with boundary conditions define an eigenvalue problem with \(\omega\) as eigenvalue parameter. Since \(\Omega\) and the characteristic frequencies \(\omega_A\) and \(\omega_c\) depend on \(r\) for a nonuniform equilibrium, the set of Equations (11)–(12) has mobile regular singularities for values of \(r\) so that
\[\Omega^2(r) = \omega_A^2(r) , \quad \Omega^2(r) = \omega_c^2(r) , \tag{20a}\]
or using the definition of \(\Omega\) at positions \(r\), where
\[\omega = \omega_f(r) \pm \omega_A(r) , \tag{20a}\]
\[\omega = \omega_f(r) \pm \omega_c(r) . \tag{20b}\]

In absence of an equilibrium magnetic field, Equations (20) define the flow resonance point where \(\omega = \omega_f(r)\). In the absence of an equilibrium flow, Equation (20a) defines...
the Alfvén resonance point where $\omega^2 = \omega_A^2(r)$, and Equation (20b) defines the slow resonance point where $\omega^2 = \omega_s^2(r)$. From the view point of the spectrum of linear ideal MHD of a stationary equilibrium, Equations (20) define four continuous ranges in the spectrum, which are not symmetric with respect to $\omega = 0$. It is easy to envisage cases where $\omega_j(r)$ is in absolute value comparable to or larger than $\omega_A(r)$ so that the ranges defined by Equations (20) differ largely from the classic Alfvén and slow continua of a static equilibrium. Nevertheless we shall in what follows refer to these four ranges as the Alfvén continuum and the slow continuum modified by the equilibrium flow.

For incompressible plasmas ($c^2 \rightarrow \infty$, div $v \rightarrow 0$) we have equations for $\xi$ and $P'$ which are formally the same as Equations (11)–(12) but the coefficient functions $D$, $C_1$, $C_2$, and $C_3$ now take the following simpler expressions:

\begin{align}
D &= \rho(\Omega^2 - \omega_A^2) , \\
C_1 &= -2mT/r^2 , \\
C_2 &= -(m^2/r^2 + k^2) , \\
C_3 &= D \left[ D + \frac{2B_\varphi}{\mu} \frac{d}{dr} \left( \frac{B_\varphi}{r} \right) - 2v_\varphi \frac{d}{dr} \left( \frac{v_\varphi}{r} \right) - \frac{v_\varphi^2}{r} \frac{d\rho}{dr} \right] - 4T^2/r^2 .
\end{align}

(21a) \quad (21b) \quad (21c) \quad (21d)

For incompressible plasmas the Alfvén continuum and the slow continuum coincide. Equations (21) result after cancelling one factor $(\Omega^2 - \omega_A^2)$ from Equations (11)–(12).

2.2. JUMP CONDITIONS FOR ALFVÉN MODES

We now focus on a frequency in the ‘modified’ Alfvén continuum and determine the spatial behaviour of the corresponding solution in the vicinity of the resonant or singular surface $r = r_A$ where

$$\Omega(r_A) = \omega_A(r_A)$$

and specify the jump conditions.

The main part of this subsection deals with equilibrium states that have non-zero azimuthal components of magnetic field and velocity. The special case $B_\varphi = 0$, $v_\varphi = 0$ is considered at the end of this subsection.

We follow the analysis in Paper I. We introduce the new radial variable $s$ defined as

$$s = r - r_A ,$$

and determine the series expansions of the coefficient functions $D$, $C_1$, $C_2$, and $C_3$. As for the case of a static equilibrium the expansion for $D$ starts with a term in $s$, $D = d_1s + \cdots$, while the series expansions of $C_1$, $C_2$, and $C_3$ all start with constant terms (at least when $B_\varphi \neq 0$). We have

$$d_1 = \rho v_\varphi^2 \omega_A^2 \Delta ,$$

$$C_{10} = -2\omega_A^2 B_z g_B T/\mu \rho ,$$

(22)
\[ C_{20} = -\omega^2_{\Lambda} g_B^2 / \mu \rho , \]
\[ C_{30} = -4 \omega^2_{\Lambda} B_z^2 T^2 / \mu \rho r^2 , \]
where
\[ \Delta = \frac{d}{dr} \left( \Omega^2 - \omega^2_{\Lambda}(r) \right) . \] (24)

All the quantities in the right-hand members of (23) have to be evaluated in \( r = r_{\Lambda} \) (\( s = 0 \)). Notice also that the expressions (23) are reduced to the corresponding expressions in Paper I in the absence of an equilibrium flow. As in the static case we have
\[ C_{10}^2 - C_{20} C_{30} = 0 \] (25)
and this implies that the right hand members of Equations (11)–(12) are linearly dependent in leading order. We shall come back to that in what follows. Let us first note that the second-order differential equations (16) and (18) reduce to (in case \( B_\omega \neq 0 \))
\[ \alpha \frac{d}{ds} \left( s \frac{d}{ds} \xi_r \right) + \beta \xi_r = 0 , \] (26)
\[ \tilde{\beta} \frac{d}{ds} \left( s \frac{d}{ds} P' \right) + \tilde{\beta} P' = 0 , \]
where \( \alpha, \beta, \bar{\alpha}, \) and \( \tilde{\beta} \) denote constants arising from the series expansions. Equations (26) both have an indicial equation with the double root \( \nu_{1,2} = 0 \). This is again the same as for the static case, so that we can borrow the form of the solutions from Paper I and write
\[ \xi_r(s) = R u(s) \ln |s| + \begin{cases} \xi^-_A(s), & s < 0 , \\ \xi^+_A(s), & s > 0 , \end{cases} \] (27)
\[ P'(s) = \tilde{R} \tilde{u}(s) \ln |s| + \begin{cases} P^-_A(s), & s < 0 , \\ P^+_A(s), & s > 0 . \end{cases} \] (28)

In these equations \( R \) and \( \tilde{R} \) are constant, and \( u(s), \tilde{u}(s), \xi_A(s), \xi^+_A(s), P^-_A(s), \) and \( P^+_A(s) \), are analytic functions of \( s \), all starting with a constant term. For convenience we use the normalization \( u(0) = \tilde{u}(0) = 1 \). We are concerned with the jumps in \( \xi_r(s) \) and \( P'(s) \) that are defined as
\[ [\xi_r] = \lim_{s \to 0^+} \xi_r(s) - \lim_{s \to 0^+} \xi_r(s) , \]
\[ [P'] = \lim_{s \to 0^+} P'(s) - \lim_{s \to 0^+} P'(s) , \] (29)
and in that context it suffices to take the constant terms in the series expansions of \( \xi^-_A(s), \xi^+_A(s), P^-_A(s), \) and \( P^+_A(s) \).
The behaviour of the solutions near the resonant point is different from that given by Equations (27) and (28) in the special case that the equilibrium magnetic field and the equilibrium flow have no \( \phi \)-components. In that case

\[
g_B = \frac{m B_z}{r}, \quad T \equiv 0, \quad C_1 \equiv 0, \tag{30}
\]

and as in the static case with straight magnetic field lines \((B_\phi = 0)\) the series expansions now start with a constant term for \( C_2 \), a term in \( s^2 \) for \( C_3 \), a term in \( s \) for \( f \) and \( g \), and a term in \( s^{-1} \) for \( \tilde{f} \) and \( \tilde{g} \). In the vicinity of \( s = 0 \) the second-order differential equations (16) and (18) reduce to

\[
\alpha \frac{d}{ds} \left( s \frac{d \xi_r}{ds} \right) + \beta s \xi_r = 0, \tag{31a}
\]

\[
\tilde{\alpha} \frac{d}{ds} \left( \frac{1}{s} \frac{d P'}{ds} \right) + \tilde{\beta} s P' = 0, \tag{31b}
\]

where \( \alpha, \beta, \tilde{\alpha}, \) and \( \tilde{\beta} \) denote constants arising from the series expansions. The indicial equation of (31a) has the double root \( \nu_{1,2} = 0 \), and the solution for \( \xi_r \) has a \( \ln |s| \) and a Heaviside contribution. The indicial equation of (31b) however has the roots \( \nu_1 = 0 \) and \( \nu_2 = 2 \). The root \( \nu_2 = 2 \) leads to a \( s^2 \ln |s| \) contribution which for \( |s| \ll 1 \) can be neglected compared to the constant term. For an equilibrium with straight magnetic field lines and an equilibrium flow along these straight field lines the Eulerian perturbation of total pressure, \( P' \), has no logarithmic singularity. We shall see below that in that case \( P' \) does not jump either.

Let us now return to (25) and the general case. This relation implies that the right-hand members of Equations (11) and (12) to leading order are linearly dependent. To make this point clear, we rewrite Equations (11)–(12) keeping only the dominant terms near \( r = r_A \) as

\[
\Delta s \frac{d \xi_r}{ds} = \frac{g_B}{\mu \rho^2 v_A^2} (g_B P' - 2 B_z T \xi_r / r), \tag{32}
\]

\[
\Delta s \frac{d P'}{ds} = \frac{2 T B_z}{\mu \rho^2 v_A^2 r} (g_B P' - 2 B_z T \xi_r / r). \tag{33}
\]

These equations imply that near the resonance point to leading order

\[
s \frac{d}{ds} (g_B P' - 2 B_z T \xi_r / r) = 0
\]

so that

\[
g_B P' - 2 B_z T \xi_r / r = C_A + C_B H(s) \tag{34}
\]

in the thin resonant layer, with \( C_A \) and \( C_B \) constants and \( H(s) \) the Heaviside function. The continuity of the 'large' solutions for \( \xi_r \) and \( P' \) implies that there cannot be a
Heaviside contribution to the right-hand member of (34) so that \( C_R = 0 \). The fundamental jump relation across the resonant surface for Alfvén continuum modes is
\[
g_B P' - 2B_z T \xi_r/r = C_A .
\]

The solutions for \( \xi_r(s) \) and \( P'(s) \) then take the form
\[
\xi_r(s) = \frac{g_B}{\mu \rho^2 v_A^2 \Delta} C_A \ln |s| + \begin{cases} 
\xi_A^-(s), & s < 0 , \\
\xi_A^+(s), & s > 0 , 
\end{cases} 
\]
\[
P'(s) = \frac{2TB_z}{\mu \rho^2 v_A^2 \Delta r} C_A \ln |s| + \begin{cases} 
P_A^-(s), & s < 0 , \\
P_A^+(s), & s > 0 , 
\end{cases} 
\]
where now the constant terms in \( R u(s) \) and \( \tilde{R}u(s) \) of Equations (27) and (28) have been identified. As for the static case we can use (14a) to rewrite the fundamental conservation law at the resonant surface as
\[
s \xi^\perp = \frac{iC_A}{\rho B \Delta} .
\]

Conservation law (38) tells us that \( \xi^\perp \) has a \( s^{-1} \) singularity and a \( \delta(s) \) contribution which dominate the \( \ln |s| \) singularity and \( H(s) \) contribution found for \( \xi_r, P' \), and \( \xi^\parallel \). The perturbations are still polarized in the magnetic surfaces perpendicular to the magnetic field lines as in the static case. This comes about because Equation (38) applies locally at the resonant layer, where \( v(r) \) is essentially uniform and provides mainly a Doppler shift without affecting the physics.

In the special case that the magnetic field is straight (\( B_\varphi = 0 \) and the flow along the magnetic field lines the conserved quantity reduces to
\[
\frac{m}{r_A} B_z P' = C_A ,
\]
that is, the total pressure perturbation \( P' \) is conserved across the singularity, a result already found in Paper I for a static equilibrium with straight field lines. Note, however, that in a stationary equilibrium we must have \( v_\varphi \equiv 0 \) in addition to \( B_\varphi \equiv 0 \) for (39) to hold.

Let us now return to the general case and consider the jumps \([ \xi_r ] \) and \([ P' ] \). Sakurai, Goossens, and Hollweg (1991) explained that these jumps are due to dissipative effects and have to be obtained from dissipative MHD equations containing terms due to electric diffusivity, and/or viscosity and/or other dissipative effects. They went through a rather lengthy computation (see Section 4 and the Appendix of Paper I) in visco-resistive MHD to find the jump conditions.

The visco-resistive equations close to the Alfvén singularity reduce to
\[
[s \Delta - i \Omega(v + \eta) \frac{d^2}{ds^2}] \frac{d\xi_r}{ds} = \frac{g_B}{\mu \rho^2 v_A^2} (g_B P' - 2B_z T \xi_r/r) ,
\]

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\[
\left[ sA - i\Omega(v + \eta) \frac{d^2}{ds^2} \right] \frac{dP'}{ds} = \frac{2TB_z}{\mu \rho^2 v_2^2 r} \left( g_B P' - 2B_z T\xi_r/r \right). \tag{40b}
\]

As a consequence conservation law (35) still holds when dissipation is included. Following the analysis in Paper I we find that the thickness of the dissipation layer can be measured by \( \delta_A \) defined as
\[
\delta_A = \left[ \frac{\Omega}{A} (v + n) \right]^{1/3}, \tag{41}
\]
and that the jump relations for \( \xi_r \) and \( P' \) are
\[
[\xi_r] = -i\pi \frac{\text{sign } \Omega}{|A|} g_B \frac{mB_z}{\mu \rho^2 v_2^2 A} C_A, \tag{42}
\]
\[
[P'] = -i\pi \frac{\text{sign } \Omega}{|A|} \frac{2TB_z}{\mu \rho^2 v_2^2 r} C_A. \tag{43}
\]

Sakurai, Goossens, and Hollweg (1990) noted that these jumps relations are not unexpected because \( \ln s = \ln(-s) \pm i\pi \) for the complex \( \ln \) function but the \( \pm \) sign is not arbitrary but uniquely determined by dissipation. At this point we should note that in the Appendix of Paper I \( \omega \) is assumed to be real which corresponds to external excitation at a given frequency. In this paper we shall also deal with the eigenvalue problem and try to determine frequencies of eigenmodes. The frequencies shall be complex and the jump conditions (42)–(43) will only be valid conditions if \( |\text{Im } \omega| \ll |\text{Re } \omega| \).

For the special case that the magnetic field lines are straight \( (B_\varphi = 0) \) and the equilibrium flow is along the magnetic field lines \( (v_\varphi = 0) \), the jump conditions then reduce to
\[
[P'] = 0, \quad C_A = \frac{mB_z}{r_A} P', \tag{44a}
\]
\[
[\xi_r] = -i\pi \frac{\text{sign } \Omega}{|A|} \frac{m^2}{\rho r^2} P'. \tag{44b}
\]

In an equilibrium with \( B_\varphi = 0, v_\varphi = 0 \), the Eulerian perturbation of total pressure, \( P' \), is continuous across the resonant point. Similar considerations apply to the planar geometry, which is formally obtained from the above formulation by letting \( r \to \infty \) and keeping \( m/r \) finite. In the vicinity of the resonance point equations of the form (31) are obtained. The equation for \( P' \) contains an \( s^2 \ln|s| \) which can be neglected compared to the constant term. This is consistent with assuming that \( P' = \text{constant} \) in the thin resonant layer, as was done by Hollweg (1987) and Hollweg and Yang (1988) for the planar geometry. In addition we have in the planar case \( [P'] = 0 \) according to (44a). Note from Equation (44c) that \( \xi_r \) does not jump for \( m = 0 \) when \( B_\varphi = 0, v_\varphi = 0 \). Modes
with an azimuthal wave number \( m = 0 \) are not resonantly absorbed in an equilibrium with \( B_\phi = 0 \) and \( v_\phi = 0 \), they require \( B_\phi \neq 0 \) and/or \( v_\phi \neq 0 \) for resonant absorption at the Alfvén resonance point.

2.3. Jump Conditions for Slow Modes

We now consider a frequency in the ‘modified’ slow continuum and focus on the spatial behaviour of the solution in the vicinity of the resonant surface \( r = r_s \), where

\[
\Omega(r_s) = \omega_c(r_s).
\]

The radial variable \( s \) is now defined as

\[
s = r - r_s. \tag{45}
\]

The first terms in the series expansions for \( D, C_1, C_2, \) and \( C_3 \) have the coefficients \((B_\phi \neq 0, v_\phi \neq 0)\)

\[
d_1 = -\rho v_\phi^2 \omega_\lambda^2 \Delta, \quad C_{10} = \omega_c^2 Q, \quad C_{20} = \omega_c^4, \quad C_{30} = Q^2, \tag{46}
\]

where

\[
\Delta = \frac{d}{dr} (\Omega^2(r) - \omega_c^2(r)) \tag{47}
\]

and \( Q \) is to be evaluated at \( r = r_s \) where \( \Omega^2 = \omega_c^2 \) so that

\[
Q = 2\omega_c B_\phi (\omega_c B_\phi + f_B v_\phi)/\mu r + \rho v_\phi^2 \omega_c^2 v_\phi^2/r c^2. \tag{48}
\]

Again

\[
C_{10}^2 - C_{20} C_{30} = 0. \tag{49}
\]

In the vicinity of the singularity the second-order differential equations for \( \xi_r \) and \( P' \) reduce to equations of the form (26), and the solutions for \( \xi_r \) and \( P' \) are of the same form as given in Equations (27)–(28), at least when \( B_\phi \neq 0 \) and/or \( v_\phi \neq 0 \).

When the magnetic field lines are straight \((B_\phi = 0)\) and the flow is along the magnetic field lines \((v_\phi = 0)\) things are different for \( P' \), but for \( \xi_r \) they remain unchanged. As in the static case with straight field lines, the differential equation for \( P' \) reduces to

\[
\frac{d^2 P'}{ds^2} - \frac{\beta}{s} P' = 0 \tag{50}
\]

leading to an indicial equation with roots \( v_1 = 0 \) and \( v_2 = 1 \). The root \( v_2 = 1 \) gives rise to a \( s \ln|s| \) contribution to the solution which for small \( |s| \) values can be neglected compared to the constant term. This \( s \ln|s| \) behaviour is also found in the static equilibrium with straight field lines (Hollweg and Yang, 1988; Sakurai, Goossens, and Hollweg, 1991).

For the general case, condition (49) implies that close to the singularity the right-hand members of Equations (11)–(12) are linearly dependent. These equations can now be
written as

\[ \Delta s \frac{d \xi_r}{ds} = \frac{\omega_c^2}{\rho v_A^2 \omega_A^2}(\omega_c^2 P' - Q \xi_r), \]  

(51)

\[ \Delta s \frac{d \xi}{ds} = \frac{Q}{\rho v_A^2 \omega_A^2}(\omega_c^2 P' - Q \xi_r). \]  

(52)

A similar analysis as given for the Alfvén resonance gives the fundamental conservation law for slow continuum modes

\[ \omega_c^2 P' - Q \xi_r = C_S, \]  

(53)

where \( C_S \) is a constant.

Using (3), (10a), and (14b) we can rewrite the conservation law (53) as

\[ s \text{ div } \nu' = -\frac{i \omega_c}{\rho (c^2 + v^2_A)} \frac{C_S}{\Delta}, \]  

(54)

\[ s \xi_{\parallel} = \frac{if_B}{\rho B \Delta \left( \frac{c^2}{c^2 + v^2_A} \right)} \frac{C_S}{\omega_c^2} - i s \xi_{(s)}(2 \Omega B \varphi v_{\varphi} + f_B v^2_{\varphi})/B \omega_c^2 r_S. \]  

(55)

The second term of the right-hand side of (55) contains \( \xi_{(s)} \) and introduces a \( s \ln |s| \) term, but this \( s \ln |s| \) can be neglected in comparison to the first term which is constant. So conservation law (53) then expresses that \( s \text{ div } \nu' \) and \( s \xi_{\parallel} \) are conserved across the resonant point, or that \( \text{ div } \nu' \) and \( \xi_{\parallel} \) and have a \( s^{-1} \) singularity and a \( \delta(s) \) contribution which dominate the \( s \ln |s| \) singularity and the \( H(s) \) contribution found for \( \xi_{\parallel}^r, P_{\parallel} \), and \( \xi_{\perp} \). The solution is polarized in the magnetic surfaces parallel to the magnetic field lines. Again as for the Alfvén resonance the equilibrium flow does not affect the polarization properties of the solutions. Equations (51)–(52) can now be integrated once to

\[ \xi_r(s) = \frac{\omega_c^2}{\rho v_A^2 \omega_A^2 \Delta} C_S \ln |s| + \begin{cases} \xi_{\parallel}^S, & s < 0, \\ \xi_{\parallel}^S, & s > 0 \end{cases}, \]  

(56)

\[ P'(s) = \frac{Q}{\rho v_A^2 \omega_A^2 \Delta} C_S \ln |s| + \begin{cases} P_{\parallel}^S, & s < 0, \\ P_{\parallel}^S, & s > 0 \end{cases}. \]  

(57)

The jumps in \( \xi_r \) and \( P' \) are due to dissipative effects. Following the analysis in Paper I we obtain

\[ [\xi_r] = -i \pi \frac{\text{sign } \Omega}{|\Delta|} \frac{\omega_c^2}{\rho v_A^2 \omega_A^2} C_S, \]  

(58)

\[ [P'] = -i \pi \frac{\text{sign } \Omega}{|\Delta|} \frac{Q}{\rho v_A^2 \omega_A^2} C_S. \]  

(59)
In an equilibrium with straight magnetic field lines \((B_\phi = 0)\) and flow along the magnetic field lines \((v_\phi = 0)\) \(P'\) is a conserved quantity and the jump relations reduce to

\[
[P'] = 0, \quad C_S = \omega_c^2 P', \quad [\xi_r] = -i\pi \frac{\text{sign } \Omega}{\rho |A|} \left( \frac{c^2}{c^2 + v_A^2} \right)^2 k^2 P'.
\]

(60)

For slow modes the jump in \(\xi_r\) is independent of the azimuthal wave number \(m\), and for given \(P'\) proportional to \(k^2\). As a consequence axisymmetric perturbations are resonantly absorbed at the slow resonance point in an equilibrium with \(B_\phi = 0\) and \(v_\phi = 0\). This is in contrast with the Alfvén resonance point where axisymmetric perturbations require \(B_\phi \neq 0\) and/or \(v_\phi \neq 0\) to be resonantly absorbed.

### 2.4. Incompressible Plasmas

We consider incompressible plasmas in a separate subsection because the behaviour of the solutions in the resonant layer differs from the behaviour found there so far when \(B_\phi \neq 0\).

When \(B_\phi = 0\) and \(v_\phi = 0\), we find the classic roots \(\nu_1, \nu_2 = 0\) and \(\nu_1 = 0, \nu_2 = 2\) for the indicial equation for \(\xi_r\) and \(P'\), respectively, so that

\[
[P'] = 0, \quad [\xi_r] = -i\pi \frac{\text{sign } \Omega}{\rho |A|} \left( \frac{m^2}{r^2} + k^2 \right) P',
\]

(61)

where now

\[A = \frac{d}{ds} (\Omega^2 - \omega_A^2).\]

Equation (61b) implies that axisymmetric perturbations \((m = 0)\) have a jump in \(\xi_r\) at the Alfvén resonance point when \(B_\phi = 0\) and \(v_\phi = 0\). This seems to contradict the result obtained for compressible plasmas at the end of Section 2.2. However, in an incompressible plasma the Alfvén and slow resonance points coincide, and as pointed out at the end of the preceding Section 2.3 axisymmetric perturbations have a jump in \(\xi_r\) at the slow resonance point in a compressible plasma when \(B_\phi = 0, v_\phi = 0\). The jump in \(\xi_r\) for \(m = 0\), present in Equation (61b), is a consequence of the slow nature of the resonance.

When \(B_\phi \neq 0\) and/or \(v_\phi \neq 0\) we obtain in contrast to earlier results (25) that

\[C_{10}^2 - C_{20} C_{30} = -4k^2 T^2/r_A^2 < 0,\]

(62)

where \(T\) has to be evaluated at \(r_A\). This differs from the earlier result (25) because the Alfvén and the slow continua coincide in the incompressible limit, but in deriving Equations (21) one factor \((\Omega^2 - \omega_A^2)\) could be cancelled; thus the subscript ‘0’ has a different meaning in (62) than in (25). The indicial equation has now two purely imaginary roots both for \(\xi_r\) and \(P'\):

\[\nu_{1,2} = \pm iM, \quad M = 2|kT/\rho |A| r_A > 0,\]

(63)
leading to solutions for $\xi_r$ and $P'$ that are linear combinations of
\begin{equation}
\exp(-\text{Marg}.s) \exp(iM \ln |s|), \quad \exp(\text{Marg}.s) \exp(-iM \ln |s|) .
\end{equation}

Similar solutions are found in the study of critical layers of internal gravity waves (see Leblond and Mysak, 1978). Leblond and Mysak discuss how arg $s$ has to be chosen and explain how the solutions are attenuated across the singular layer giving rise to absorption of momentum and energy.

3. Surface Waves

In this section we investigate what happens to the classic surface waves studied in the literature (see, e.g., Edwin and Roberts, 1983) when the discontinuity is replaced by a thin transition layer. We shall use the jump conditions found in Section 2. The jump conditions are strictly speaking only valid for real $\omega$, while the eigenvalues that we will find are complex. This way of doing is reasonable when $|\text{Im}\.\omega| \ll |\text{Re}\.\omega|$ which can be expected to be the case when the discontinuity is replaced by a ‘thin’ nonuniform transition layer.

3.1. $B_\varphi = 0$, $v_\varphi = 0$

The choice $B_\varphi = v_\varphi = 0$ implies $Q = 0$ and $T = 0$ and $\omega^2 = k^2 B^2 / \mu \rho$.

Let us first consider incompressible plasmas. The relevant equations are
\begin{equation}
\rho(\Omega^2 - \omega^2) \frac{d}{dr} (r \xi_r) = (m^2 / r^2 + k^2) r P',
\end{equation}
\begin{equation}
\frac{dP'}{dr} = \rho(\Omega^2 - \omega^2) \xi_r .
\end{equation}

When the plasma is uniform, the second-order differential equation for $P'$ simplifies to
\begin{equation}
\frac{d^2 P'}{dr^2} + \frac{1}{r} \frac{dP'}{dr} - \left( k^2 + \frac{m^2}{r^2} \right) P' = 0 .
\end{equation}

The solutions to this equation are linear combinations of the modified Bessel functions $I_{|m|}(|k| r)$ and $K_{|m|}(|k| r)$.

Consider first the case of a true discontinuity at the position $r = R$ which separates two regions denoted by index 1 ($r \leq R$) and 2 ($r \geq R$), respectively. In these two regions the equilibrium quantities take constant but different values. We then have
\begin{equation}
P'(r) = A_1 I_{|m|}(|k| r), \quad \xi_r(r) = \frac{|k|}{\rho_1 (\Omega^2_1 - \omega^2_A)} A_1 I_{|m|}(|k| r), \quad r \leq R ,
\end{equation}
\begin{equation}
P'(r) = A_2 K_{|m|}(|k| r), \quad \xi_r(r) = \frac{|k|}{\rho_2 (\Omega^2_2 - \omega^2_A)} A_2 K_{|m|}(|k| r), \quad r \geq R .
\end{equation}
An accent on a Bessel function denotes the derivative with respect to the argument. Continuity of $\xi_r$ and $P'$ at $r = R$ leads to the dispersion relation

$$\rho_1 (\Omega_1^2 - \omega_{A_1}^2) - \rho_2 (\Omega_2^2 - \omega_{A_2}^2) \frac{I_{m_i}(|k| R) K_{m_i}(|k| R)}{I_{m_i}(|k| R) K'_{m_i}(|k| R)} = 0 \quad \text{(68)}$$

Using the first term in the expansions of the modified Bessel functions for small argument $|k| R \ll 1$ for $m \geq 1$ we obtain

$$\rho_1 (\Omega_1^2 - \omega_{A_1}^2) + \rho_2 (\Omega_2^2 - \omega_{A_2}^2) = 0 \quad \text{for} \quad |k| R \ll 1 \leq m \quad \text{(69)}$$

with solutions

$$\omega = k \frac{\rho_1 v_1 + \rho_2 v_2}{\rho_1 + \rho_2} + \omega_{cm} \quad \text{(70)}$$

Here $\omega_{cm}$ is the centre-of-mass frequency as in Hollweg et al. (1990):

$$\omega_{cm} = \pm \left\{ \omega_k^2 - \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2} k^2 (v_1 - v_2)^2 \right\}^{1/2} \quad \text{(71)}$$

and $\omega_k$ is the kink frequency:

$$\omega_k^2 = (\rho_1 \omega_{A_1}^2 + \rho_2 \omega_{A_2}^2)/(\rho_1 + \rho_2) = k^2 (B_{z_1}^2 + B_{z_2}^2)/\mu(\rho_1 + \rho_2) \quad \text{(72)}$$

The frequency of the surface wave is degenerate with respect to the azimuthal mode number $m$ at least in the limit considered here. This degeneracy will be removed when either a $B_\phi$ and/or $v_\phi$ component is included in the equilibrium state, or when higher order terms in $|k| R$ are included. Equation (71) contains the Kelvin–Helmholtz instability which occurs for any velocity shear in the absence of a magnetic field.

Let us now replace the discontinuity by a thin nonuniform transition region $[R - a, R + a]$ of width $2a$. In this transition region the equilibrium variables vary in a continuous manner from their constant values in region 1 to their constant values in region 2, in particular $\omega_k^2(r)$ varies there from $\omega_{A_1}^2$ to $\omega_{A_2}^2$. Because of Equation (69) the frequency $\omega$ defined by Equation (70) is in the Alfvén continuum. Replacing the discontinuity by a continuous variation leads to a resonance.

We can now proceed along two lines. First we can follow HYCG 90 and use the condition $P' = $ constant to solve Equation (65a) in the transition zone $[R_1 = R - a, R_2 = R + a]$ for $\xi_r$, evaluate the integral with the Landau prescription for going around the pole and impose the conditions for continuity at $r = R_1 = R - a$ and at $r = R_2 = R + a$. This would give us the required dispersion relation. On the other hand we know the jumps experienced by $P'$ and $\xi$, when moving from region 1 to region 2. So there is no need to solve Equation (65a) in the transition layer, we can just write

$$[P'] = A_2 K_{m_i}(|k| R_2) - A_1 I_{m_i}(|k| R_1) = 0$$
\[
\frac{\xi_r}{\rho_2(\Omega_2^2 - \omega_{A_2}^2)} = \frac{|k| A_2}{\rho_2(\Omega_2^2 - \omega_{A_2}^2)} K_{|m|}(|k| R_2) - \frac{|k| A_1}{\rho_1(\Omega_1^2 - \omega_{A_1}^2)} I_{|m|}'(|k| R_1) =
\]
\[
= -i \pi \left\{ \frac{\text{sign } \Omega}{\rho |\Delta|} \left( \frac{m^2}{r^2} + k^2 \right) \right\}_{r_A} A_1 I_{|m|}(|k| R_1) .
\]

Elimination of \(A_1\) and \(A_2\) leads to the dispersion relation
\[
\rho_1(\Omega_1^2 - \omega_{A_1}^2) - \rho_2(\Omega_2^2 - \omega_{A_2}^2) \frac{I_{|m|}'(|k| R_1) K_{|m|}(|k| R_2) +}{I_{|m|}(|k| R_1) K_{|m|}'(|k| R_2)}
\]
\[
+ i \pi \frac{1}{|k|} \rho_1(\Omega_1^2 - \omega_{A_1}^2) \rho_2(\Omega_2^2 - \omega_{A_2}^2) \left\{ \frac{\text{sign } \Omega}{\rho |\Delta|} \left( \frac{m^2}{r^2} + k^2 \right) \right\}_{r_A} \times
\]
\[
\times \frac{K_{|m|}(|k| R_2)}{K_{|m|}'(|k| R_2)} = 0 .
\] (73)

Comparison of the dispersion relations (68) and (73) shows that the resonance at \(r_A\) produces an imaginary contribution to the dispersion relation and makes the frequency \(\omega\) a complex quantity \(\omega = \omega_r + i \gamma\). Our main concern is the determination of \(\gamma\), since \(\gamma\) determines the damping or the instability of the wave.

Again we can use the expansions of the modified Bessel functions for small arguments \((k^2 r^2 \ll 1 \leq m^2)\) to obtain
\[
\rho_1(\Omega_1^2 - \omega_{A_1}^2) + \rho_2(\Omega_2^2 - \omega_{A_2}^2) -
\]
\[
- i \pi \rho_1 \rho_2(\Omega_1^2 - \omega_{A_1}^2)(\Omega_2^2 - \omega_{A_2}^2) \left\{ \frac{\text{sign } \Omega}{\rho |\Delta|} \frac{|m|}{r^2} \right\}_{r_A} R = 0 .
\] (74)

Here we have dropped the distinction between \(R, R_1,\) and \(R_2\) since \(a/R \ll 1\) by assumption.

The frequency of the wave is now complex and its imaginary part \(\gamma\) is given by (see, e.g., Krall and Trivelpiece, 1973)
\[
\gamma = -\frac{D_r(\omega_r, k)}{\partial D_r / \partial \omega_r} ,
\] (75)

where \(D_r\) and \(D_i\) denote the real and imaginary parts of the dispersion relation. The real part of the frequency \(\omega_r\) is approximated by its value obtained for a true discontinuity. The derivative \(\partial D_r / \partial \omega_r\) is then to be evaluated for \(\omega_r\) given by Equation (70) and equals \(2(\rho_1 + \rho_2) \omega_{cm}\). We obtain for \(\gamma\)
\[
\gamma = -\frac{\rho_1^2(\Omega_1^2 - \omega_{A_1}^2)^2}{2(\rho_1 + \rho_2) \omega_{cm}} \frac{\text{sign } \Omega}{\rho(r_A)|\Delta|} \frac{|m| \pi}{r_A} ,
\] (76)

where \(\Omega\) is evaluated at \(r_A\).
The degeneracy of the frequency of the surface wave with respect to \( m \) that we found for the true discontinuity is now partly removed as the damping rate is proportional to \( |m| \) for \( |kR| \ll 1 \leq |m| \).

In the absence of flow \( \gamma \) takes the form

\[
\gamma = -\frac{|m| \pi}{2R} \frac{\rho_1^2 \rho_2^2}{\rho(r_\Delta)(\rho_1 + \rho_2)^3} \frac{|\omega_{A2}^2 - \omega_{A1}^2|}{|\omega_k| |\Delta|} < 0 .
\] (77)

In the special case that \( \omega_\Delta^2(r) \) varies linearly across the nonuniform transition layer \( |\Delta| \) equals

\[
|\Delta| = |\omega_{A2}^2 - \omega_{A1}^2|/2a
\]

and Equation (77) for \( \gamma \) reduces to

\[
\gamma = -\frac{|m| \pi}{R} \frac{a}{\rho(r_\Delta)(\rho_1 + \rho_2)^3} \frac{|\omega_{A2}^2 - \omega_{A1}^2|}{|\omega_k|} < 0 .
\] (78)

Equation (78) shows that \( \gamma \) is proportional to \( a/R \) and the assumption that the nonuniform transition region is thin implies that \( a/R \ll 1 \). The thin region approximation then naturally implies that \( |\gamma| \ll |\omega_k| \). If the density is constant \( (\rho_1 = \rho_2) \) so that the linear variation of \( \omega_\Delta^2 \) is completely due to the (linear) variation of \( B^2 \), we can simplify Equation (78) further to

\[
\gamma = -\frac{|m| \pi}{8} \frac{a}{R} \frac{\omega_{A2}^2 - \omega_{A1}^2}{|\omega_k|} = -\frac{|m| \pi}{4} \frac{a}{R} \frac{B_2^2 - B_1^2}{B_1^2 + B_2^2} |\omega_k| .
\] (79a)

In the special case that \( B_1 = B_2 \) and that the density \( \rho \) varies linearly across the nonuniform region, so that \( \omega_\Delta^2 \) varies like \( r^{-1} \) there, Equation (77) can be simplified to

\[
\gamma = -\frac{|m| \pi}{4} \frac{a}{R} \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} |\omega_k| .
\] (79b)

Let us now return to the special result found in Equation (79a). This result should be compared with the result obtained by Goedbloed (1983) for the damping of the quasi-mode in planar geometry (Chapter X, C). Goedbloed determines the damping of this mode by considering the poles of the Green’s function. He finds no poles on the principal Riemann sheet, corresponding to the fact that complex eigenvalues do not exist in ideal MHD. On the other Riemann sheets, however, he does find poles and he shows that for \( n = \pm 1 \) these poles lead to a damped ‘quasi-mode’. The approximations made by Goedbloed are the long wavelength limit (cf. our assumption \( |k| R \ll 1 \)) so that he can use series expansions of the Bessel functions and the assumption that \( \text{Re} \omega \approx \omega_{\text{kin}} \) and \( |\text{Im} \omega| \ll |\text{Re} \omega| \) which are again the assumptions that we have adopted. It is interesting to note that when in our result \( |m| \pi/R \) is replaced by Goedbloed’s \( k \) the results are identical. An analysis of the planar geometry using the jump conditions actually recovers Goedbloed’s result. It is comforting that these two rather different
methods give the same result. This suggests that it is justified to use the jump conditions in a normal mode analysis as long as $|\text{Im } \omega| \ll |\text{Re } \omega|$.

For $m = 0$, a similar analysis for $|kR| \ll 1$ which is not given here, leads to a damping rate proportional to $(kR)^2 \ln |kR|$

Let us now turn to compressible plasmas. The relevant set of equations is

$$D \frac{d}{dr} (r \xi_r) = - C_2 r P',$$

$$\frac{dP'}{dr} = \rho (\Omega^2 - \omega^2_A(r)) \xi_r.$$  

For a uniform plasma the second-order differential equation for $P'$ can be written as

$$\frac{d^2 P'}{dr^2} + \frac{1}{r} \frac{dP'}{dr} - \left( K^2 + \frac{m^2}{r^2} \right) P' = 0,$$

where

$$K^2 = - \frac{(\Omega^2 - k^2 c^2)(\Omega^2 - k^2 v_A^2)}{(c^2 + v_A^2)(\Omega^2 - \omega^2)} > 0.$$ 

Here $K^2$ is positive as we are interested in surface waves. The solutions for $P'$ can again be written in terms of the modified Bessel functions as in the incompressible case, but now $|k|$ is to be replaced by $|K_1|$ and $|K_2|$, respectively. For a true discontinuity the continuity conditions for $\xi_r$ and $P'$ lead to the dispersion relation

$$\rho_1 (\Omega_1^2 - \omega_{A_1}^2) - \rho_2 (\Omega_2^2 - \omega_{A_2}^2) \left[ \frac{K_1}{|m_1|} \frac{I_{m_1} (|K_1| R)}{K_{m_1} (|K_1| R)} \frac{K_{m_1} (|K_2| R)}{I_{m_1} (|K_2| R)} \right] = 0.$$ 

Again we can use the series expansions of the modified Bessel functions to obtain a simpler version of the dispersion relation. When we keep the first term in each of the expansions ($|K_1| R_1 \ll 1, |K_2| R_2 \ll 1$) we recover Equation (69), i.e., the dispersion relation obtained for incompressible plasmas in the limit $|k| R \ll 1$. Replace now the discontinuity by a thin transition layer and use the jump conditions (44) to write

$$[P'] = A_2 K_{m_1} (|K_2| R_2) - A_1 I_{m_1} (|K_1| R_1) = 0,$$

$$[\xi_r] = \frac{|K_2| A_2}{\rho_2 (\Omega_2^2 - \omega_{A_2}^2)} K_{m_1} (|K_2| R_2) - \frac{|K_1| A_1}{\rho_1 (\Omega_1^2 - \omega_{A_1}^2)} I_{m_1} (|K_1| R_1) =$$

$$= -i \pi \left\{ \frac{\text{sign } \Omega}{\rho |A|} \right\}_{r_\Lambda} A_1 I_{m_1} (|K_1| R_1).$$
Elimination of $A_1$ and $A_2$ leads to the required dispersion relation:

$$
\rho_1 (\Omega_1^2 - \omega_{A_1}^2) - \rho_2 (\Omega_2^2 - \omega_{A_2}^2) \frac{K_1 \mid I_{m1}(\mid K_1 \mid R_1) K_{m1}(\mid K_2 \mid R_2)}{K_2 \mid I_{m1}(\mid K_1 \mid R_1) K_{m1}(\mid K_2 \mid R_2)} + 
$$

$$
+ i \pi \left\{ \frac{\text{sign } \Omega}{\rho} \frac{m^2}{|A| r^2} \right\} \rho_1 \rho_2 (\Omega_1^2 - \omega_{A_1}^2)(\Omega_2^2 - \omega_{A_2}^2) \times
$$

$$
\frac{K_{m1}(\mid K_2 \mid R_2)}{K_2 \mid K_{m1}(\mid K_2 \mid R_2)} = 0 .
$$

(84)

Again we can use the first term in each of the series expansions of the Bessel functions to simplify the dispersion relation. In the limit $|K_1| R_1 \ll 1$, $|K_2| R_2 \ll 1$ we again obtain the dispersion relation (74) found in the limit $|k| R \ll 1$ for incompressible plasmas. Thus in this limit there is no distinction between the compressible and incompressible dispersion relations; Equation (74) applies to both cases. Moreover, it is easily shown that (74) has an exact correspondence with the incompressible dispersion relation in slab geometry obtained by Hollweg et al. (1990); if $k$ in their dispersion relation (19) is replaced with $|m|/r_{\lambda}$, we obtain (74). Thus the results of Hollweg et al. (1990) on resonant instability due to velocity shear can be applied directly to the cylindrical case considered here.

3.2. $B_{\varphi} \neq 0$, $v_{\varphi} = 0$

Let us now consider the case that the equilibrium magnetic field has a non-zero $\varphi$ component. Obtaining analytic expressions for the growth rates is severely complicated now because the differential equations no longer have constant coefficients in general. Thus we present only a special case as an illustration of the technique.

A finite electric current on the axis requires that $B_{\varphi} \to 0$ when $r \to 0$ at least as fast as $r$, and this variation of $B_{\varphi}$ produces non-constant coefficients in the differential equations in general. In order to make analytic progress we make simplifying assumptions about the wave numbers $m$ and $k$, and $B_{\varphi}$. We take a low-$\beta$ plasma ($c^2 \ll v_{\lambda}^2$), assume that $B_{\varphi}$ is small compared to $B_z$ and focus on the long wave length limit in the $z$-direction, so that

$$
B_{\varphi}/B_z = o(\varepsilon), \quad kr/m = o(\varepsilon) ,
$$

where $\varepsilon \ll 1$. In addition we choose a linear profile for $B_{\varphi}$ in region 1 and take $B_{\varphi}$ to be zero in region 2:

$$
B_{\varphi} = \begin{cases} 
B_0 r/R , & r \leq R , \\
0 , & r > R .
\end{cases}
$$

(85)

$B_0$ is a constant. This choice of $B_{\varphi}$ implies that there is a surface current in the equilibrium. For simplicity we take a flow in the $z$-direction ($v_z \neq 0$, $v_{\varphi} = 0$) and assume that $\Omega^2 \approx \omega_{\lambda}^2 \gg \omega_c^2$ where the last inequality is due to our low $\beta$ approximation. The
differential equations for $\xi_r$ and $P'$ can then be simplified to

$$\rho(\Omega^2 - \omega_A^2) \frac{d}{dr} (r \xi_r) = -2mf_B \frac{B_0}{\mu R} \xi_r + \frac{m^2}{r} P',$$

$$\rho(\Omega^2 - \omega_A^2) \frac{dP'}{dr} = 2 \frac{m}{\mu r} f_B \frac{B_0}{R} P' + \left\{ \rho^2 (\Omega^2 - \omega_A^2)^2 - \frac{4f_B^2}{\mu^2} \frac{B_0^2}{R^2} \right\} \xi_r . \quad (86)$$

For a plasma with uniform distributions of $\rho$, $B_z$, and $v_z$, we can obtain a simple differential equation for $\xi_r$:

$$\frac{d}{dr} \left[ r \frac{d}{dr} (r \xi_r) \right] = m^2 \xi_r . \quad (87)$$

The solutions for $\xi_r$ and $P'$ that satisfy the regularity conditions at $r = 0$ and $r = \infty$ are

$$\xi_r = A_1 r^{\lambda_1} ,$$

$$P' = \frac{A_1}{m} \left[ \rho_1 (\Omega_1^2 - \omega_{A_1}^2) \right] \text{sign} \ m + 2f_B B_0/\mu R \right] r^{\lambda_1 + 1}$$

for $0 \leq r \leq R , \quad (88)$

and

$$\xi_r = A_2 r^{\lambda_2} ,$$

$$P' = -\frac{A_2}{m} \text{sign} \ (m) \rho_2 (\Omega_2^2 - \omega_{A_2}^2) r^{\lambda_2 + 1} \quad \text{for} \quad r \geq R . \quad (89)$$

The exponents $\lambda_1$ and $\lambda_2$ are

$$\lambda_1 = |m| - 1 , \quad \lambda_2 = -|m| - 1 . \quad (90)$$

Continuity at $r = R$ of $\xi_r$ and the Lagrangian perturbation of total pressure $\delta P$ will lead to the required dispersion relation. We have to use $\delta P$,

$$\delta P = P' + \frac{dP}{dr} \xi_r = P' - \xi_r B_{\phi}^2 / r , \quad (91)$$

because $B_{\phi}$ and as a consequence the derivative of $P$ are discontinuous in the equilibrium. The dispersion relation is

$$\rho_1 (\Omega_1^2 - \omega_{A_1}^2) + \rho_2 (\Omega_2^2 - \omega_{A_2}^2) + 2 \text{sign} \ (m) f_B B_0 / \mu R - |m| B_{\phi}^2 / \mu R^2 = 0 . \quad (92)$$

In an equilibrium with a continuous $\phi$-component of the magnetic field, continuity of $\delta P$ is equivalent to the continuity of $P'$. The discontinuity of $B_{\phi}$ in the present equilibrium has as a consequence that the continuity condition on $\delta P$ introduces an extra
term \( A_1 B_0 R_1^{\lambda_1 - 1} \) in addition to the terms that one would obtain when imposing continuity of \( P' \). This extra terms leads to the extra (last) term \( -|m| B_0^2 \mu R^2 \) in the left-hand side member of the dispersion relation (92). The solution of dispersion relation (92) is of the same form as the one given in Equation (70) but \( \omega_k^2 \) now takes the form

\[
\omega_k^2 = \frac{\omega_{A_1}^2 + \rho_2 \omega_{A_2}^2}{\rho_1 + \rho_2} - \frac{\text{sign} \ m}{\mu (\rho_1 + \rho_2)} \left( \frac{m B_0^2}{R^2} + 2 k B_{z1} \frac{B_0}{R} \right).
\]

In this expression for \( \omega_k^2 \) modifications to the classic expression for \( \omega_k^2 \) are introduced by the azimuthal magnetic field \( B_\phi \) and the azimuthal wave number \( m \) through \( \omega_{A_1}^2 \), in the first term and through the second term. To make the dependence of \( \omega_k^2 \) and of \( \omega^2 \) on \( B_\phi(B_0) \) and \( m \) more explicit we rewrite the expression for \( \omega_k^2 \) as

\[
\mu (\rho_1 + \rho_2) \omega_k^2 = k^2 (B_{z1}^2 + B_{z2}^2) + \frac{B_0^2}{R^2} (m^2 - |m|) + 2 k B_{z1} \frac{B_0}{R} (m - \text{sign} \ m).
\]

(93)

The last two terms in the right-hand side are the terms that depend on the azimuthal wave number \( m \). These two terms remove the degeneracy of the frequencies of surface waves with respect to the azimuthal wave number \( m \), at least when \( |m| > 1 \). For \( |m| = 1 \) the two terms containing the azimuthal wave number vanish and we get the same frequencies as in the case that \( B_\phi = 0 \). For \( |m| > 1 \) the degeneracy is completely removed. The square of the frequencies of surface waves with a given wave number \( k \) and opposite azimuthal wave numbers \( |m| \) and \( -|m| \) are split by the amount \( 4 k B_{z1} B_0 (|m| - 1)/\mu (\rho_1 + \rho_2) R \) in a static equilibrium.

We now replace the discontinuity in the equilibrium at \( r = R \) by a thin transition region \( R_1 = R - a \leq r \leq R + a = R_2 \). We assume that all the equilibrium quantities with the one exception of \( B_\phi \), vary there continuously from their constant values in region 1 to their constant values in region 2. We assume that \( B_\phi \) varies linearly in the transition region also, \( B_\phi = B_\phi (r)/R_2 \), \( a \leq r \leq R_2 \), but that it vanishes identically for \( r > R_2 \). Use now the jump conditions (42)–(43) combined with Equation (35) for \( C_A \) to write

\[
C_A = - A_2 B_2 (\Omega_2^2 - \omega_{A_2}^2) \text{sign} \ (m) R_2^{\lambda_2} + A_2 R_2^{\lambda_2} - A_1 R_1^{\lambda_1} = - i \pi \Omega \left[ \frac{\text{sign} \ (\Omega) g_B}{\rho^2 v_A^2 |A|} \right] \frac{C_A}{r_A} - \frac{A_2}{m} \text{sign} \ (m) \rho_2 (\Omega_2^2 - \omega_{A_2}^2) R_2^{\lambda_2 + 1} - A_1 \text{sign} \ (m) \rho_1 (\Omega_1^2 - \omega_{A_1}^2) R_1^{\lambda_1 + 1} -
\]

\[
- 2 \frac{A_1}{\mu m} f_B B_0 R_1^{\lambda_1} = - i \pi \Omega \left[ \frac{\text{sign} \ (\Omega) 2 T B_z}{\rho^2 v_A^2 |A|} \right] \frac{C_A}{r_A}.
\]

(94)

Expression (43) for the jump in \( P' \) has been derived for an equilibrium with continuously varying equilibrium quantities. The jump in \( P' \) given in the second of Equations (94)
is due to the resonance at \( r_\alpha \) only. The equilibrium that we are considering here has a discontinuous \( B_\varphi \) profile and this discontinuity causes an additional jump in \( P' \) and adds an extra term \( A_1 B_0 R_1^{\varphi - 1} \) to the left hand member of the second of Equation (94). Elimination of \( A_1 \) and \( A_2 \) leads to the required complex dispersion relation. The real part of the dispersion relation is the left-hand member of Equation (92). The imaginary part is

\[
D_i(\Omega, k, m) = i \pi \text{sign}(\Omega) \rho_2 (\Omega_2^2 - \omega_{A_2}^2) B_{z2} \times
\]

\[
\times \left\{ \frac{1}{\rho B^2 f} \left[ \frac{2B_z T/r}{m/R_2} + \rho_2 \text{sign}(m) (\Omega_2^2 - \omega_{A_2}^2) g_B \right] \right\}_{r_\alpha}.
\]

(95)

When we take \( B_\varphi \equiv 0 \), expression (95) for \( D_i \) reduces to the limit for \( k |r| \ll |m| \) of the imaginary part of the dispersion relation (94). The contributions of \( B_\varphi \) to \( D_i \) are contained in \( \omega_{A_2}^2, T \), and \( g_B \). The derivative of \( D_r \) with respect to \( \omega_{cm} \) is to be computed with the aid of the dispersion relation (92) for the true discontinuity and is again \( 2(\rho_1 + \rho_2) \omega_{cm} \), with \( \omega_{cm} \) given by Equations (71) and (93). The damping (or growth) rate is obtained by dividing Equation (95) by \( 2(\rho_1 + \rho_2) \omega_{cm} \). In the case of a static equilibrium the factor \( \Omega_2^2 - \omega_{A_2}^2 \) takes the form

\[
\omega_{k - \omega_{A_2}} = \frac{-\rho_1}{\rho_1 + \rho_2} (\omega_{A_2}^2 - \omega_{A_1}^2) - \frac{\text{sign} m}{\rho_1 + \rho_2} \left( \frac{B_0^2}{R^2} + \frac{2k B_{z1} B_0}{R} \right) \frac{1}{\mu}
\]

and the computation of \( \gamma \) is straightforward when the equilibrium quantities \( \rho_1, B_{z1}, B_0, \rho_2, \) and \( B_{z2} \) are specified.

4. Conclusions

The analysis by Hollweg et al. (1990) of surface waves in stationary equilibrium states has been taken further. The conservation laws and jump conditions at Alfvén and slow resonance points obtained by Sakurai, Goossens, and Hollweg (1990) have been generalized to include an equilibrium flow. The assumption that the Eulerian perturbation of total pressure is constant has been recovered as the special case of the conservation law for an equilibrium with straight magnetic field lines and flow along the magnetic field lines. In addition, the prescription for going round the pole has been put on a firm basis. The jump conditions are then used to find the dispersion relations and the eigenfrequencies. For equilibrium states with straight magnetic field lines the important result is that in the long wavelength limit compressible and incompressible surface MHD waves have the same eigenfrequencies and that there is an exact correspondence with the planar case. The results obtained for the incompressible planar case, therefore, also hold for the compressible cylindrical case with straight field lines. For equilibrium states with curved magnetic field lines the dispersion relation becomes rather complicated and numerical computation for specified equilibrium profiles is required in order to formulate statements on the eigenfrequencies.
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