Construction of a seismic model of the sun

A. G. Kosovichev and A. V. Fedorova

Crimean Astrophysical Observatory, USSR Academy of Sciences; Astronomy Institute, USSR Academy of Sciences
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New observational data on the frequencies of p modes of natural solar oscillations of intermediate degrees $l = 4-140$ are used to determine the radial structure of the sun, assuming spherical symmetry and hydrostatic equilibrium. On the basis of the solutions of the inverse problem of solar seismology obtained, an improved solar model is constructed, the oscillation frequencies of which agree considerably better with observations than the standard evolutionary model. The remaining discrepancy is associated mainly with the uncertain structure of the subphotospheric turbulent convection zone. A likely reason for this departure of the internal solar structure from the standard model may be mixing of matter in the central core. We calculate a class of theoretical models with partial mixing in the core. The degree of mixing is estimated by comparing them with the seismic model of the sun. A more accurate determination of the parameters of the solar core is possible from the frequencies of modes of low degree, $l = 0-3$. The available observations are not consistent with our seismic model, however, as manifested particularly in a higher density at the center of the sun reconstructed from these data. This may be related to systematic frequency shifts that were neglected in the observations.

1. INTRODUCTION

It is well known that the frequencies of natural solar oscillations observed in the 1.5-5 mHz range differ from the corresponding theoretical frequencies calculated for the standard model of internal solar structure (see, e.g., Ref. 1). Despite the fact that the frequency differences are small, about 1%, they are systematic and indicate departures of solar structure from the standard model. These departures may occur for two main reasons: first, because of an inaccurate description of microscopic physical processes inside the star, such as nuclear reactions and the transfer of radiant and convective energy, and an inaccurate equation of state of the solar plasma; second, due to evolutionary processes ignored in the standard scheme, such as total or partial mixing of matter in the zone of nuclear reactions and a heterogeneous distribution of chemical composition in the initial stage of evolution on the main sequence. The investigation of these phenomena is of interest for a wide range of physical and astrophysical problems, one of the most interesting of which is the solar neutrino deficit.3

By varying different input parameters and initial conditions and specifying some mixing of chemical composition, one can, in principle, attempt to construct a solar model for which the calculated and observed frequencies agree. Such a model will be far from unique, however, since assumptions that are not obvious and are difficult to test, especially involving the structure of turbulent convection and heat balance inside the sun, are used in constructing the theoretical models. The uncertainty of these and other assumptions may considerably distort the solar parameters being determined.

A different approach is to attempt to eliminate the influence of various uncertainties by solving the inverse problem.4 Under the minimal additional assumptions of spherical symmetry and hydrostatic equilibrium inside the sun, in particular, one can construct a seismic model specified by the radial distribution of the main thermodynamic functions: pressure $P$, density $\rho$, and ratio $\gamma$ of specific heats, or combinations of them: $u = p/\rho$ (Ref. 5), $W = (\gamma - 1)GM_{\odot}/4\pi r^3$ (Ref. 6), $c = \sqrt{\gamma P/\rho}$. If the equation of state of the solar plasma, i.e., the function $\gamma = \gamma(\rho, P, Y)$, which is now known to high accuracy,7 is included in the analysis, one can find the helium abundance $Y$, but only in the convective envelope, since $\gamma$ depends on $Y$ only in regions of hydrogen and helium ionization.5,11

Determinations of the profiles of temperature and helium abundance in the solar core are of utmost importance for solar physics, especially for the solution of the neutrino problem. To determine them from oscillation frequencies, one needs an additional condition on thermal balance between energy release in nuclear reactions and heat removal, which also depend on the nuclear reaction rates and the opacity coefficient of stellar matter, known with a certain error. The values of these physical parameters may be improved using solar seismological data, in principle, if additional assumptions are made about the internal structure of the sun. By

![FIG. 1. Optimum averaging kernels for density at $r_0 = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$, and 0.9; $\alpha = 10^3$ and $\beta = 1$.](image-url)
assumming chemical composition to be constant outside the region of energy release, for example, one can attempt to improve the opacity coefficient. The condition of thermal balance, widely used in model building, may be violated, however, due to instabilities associated with thermonuclear burning, so it is not as necessary to satisfy it inside the sun as to satisfy the condition of hydrostatic equilibrium. Thus, despite the fact that the chemical composition and temperature in the central regions may be found relatively easily by postulating thermal balance there, the accuracy in determining these parameters remains an open question.

Here we consider the construction of a seismic model of the sun based on a solution of the inverse problem obtained from the frequencies of $p$ modes of oscillations of intermediate degree $l = 4 - 140$ (Ref. 14). The advantage of these observations is the high relative precision in determining frequencies, achieved by analyzing rotational splitting of frequencies in power spectra. Unfortunately, the frequencies of modes with low and high angular degrees ($l = 0 - 3$ and $l > 140$, respectively) are presently known to low precision. This means that the structure of the central and surface layers of the sun is determined less reliably.

Modes with $l = 0 - 3$ carry the most important information about the solar core. As an analysis shows, however, the measured frequencies of these modes do not agree with the frequencies of modes of intermediate degree, and they are evidently shifted systematically relative to the true frequencies of natural oscillations. The difficulties in measuring these frequencies are partly due to the fact that the fine structure arising from rotation (for $l > 0$) cannot be resolved reliably.

In constructing a seismic model of the sun in the present paper, we assume spherical symmetry and hydrostatic equilibrium, and we use the equation of state from Ref. 15. The solution of the inverse problem obtained under these assumptions from the observations of Ref. 14 are given in Sec. 2. Evidence for the mixing of matter in the central core is noted. In Sec. 3 we calculate a standard evolutionary model of the sun, as well as a class of models with mixing in the core. Mixing has been modeled by a local perturbation in hydrogen and helium abundances relative to their distributions in the standard model at the present age. This has revealed the characteristic changes in other parameters as a result of mixing. In Sec. 4 these results are used to choose an extrapolation of the solution of the inverse problem near the center and surface of the sun, where, as noted, the seismic characteristics of the sun cannot yet be determined from the available data. A complete seismic model of the sun is then constructed by solving the hydrostatic equations. The frequencies of $p$ modes calculated for this model agree considerably better with the observed frequencies in the range $l = 4 - 140$ under consideration. The remaining difference is related mainly to uncertainty in the outer layers. We also discuss the problem of the frequency spectrum of modes of low degree $l$, which cannot be solved in the constructed seismic model. The conclusion that the solar core has a high density, obtained earlier from ground-based observations, has been confirmed by observations from the Phobos spacecraft.

2. INVERSE PROBLEM OF SOLAR SEISMOLOGY

An efficacious method of determining the seismic parameters of the sun from observed frequencies can be based on a variational principle, which is expressed in the form of nonlinear integral equations relating the oscillation frequencies to the radial distributions of density $\rho$, pressure $P$, and adiabatic index $\gamma$. Because of the extremal property of eigenvalues, small changes $\delta P/\rho$ in eigenfrequencies can be associated with perturbations $\delta P/\rho$, $\delta P/P$, and $\delta \gamma/\gamma$ as a function of radius inside the sun. The inverse problem entails determining these perturbations from the known frequency corrections.

Since the oscillation frequencies in the standard evolutionary model 1 constructed in Ref. 20 are fairly close to the observed frequencies, it is taken as the initial model. The general scheme for solving the inverse problem of solar seismology consists in the successive improvement of the initial model on the basis of the linearized variational principle and is an iterative method of solution of a nonlinear inverse problem. As shown in Sec. 4, even the first correction to the solar model 1 removes the main discrepancies between the observed and theoretical frequencies. Let us briefly describe the procedure for solving the inverse problem. A more detailed presentation, together with test examples, may be found in another paper.

Applying the equations of hydrostatics and state to eliminate $\delta P/P$ and $\delta \gamma/\gamma$ from the linearized variational principle, we obtain

$$\frac{\delta \omega_i}{\omega_i} = \int_0^R K_{(p, r)}^{(p, r)} \frac{\delta P}{\rho} \, dr + \int_0^R K_{(Y, r)}^{(Y, r)} \delta Y \, dr,$$

where $K_{(p, r)}^{(p, r)}(r)$ and $K_{(Y, r)}^{(Y, r)}(r)$ are the sensitivity functions of the oscillation frequencies to density variation at a constant helium abundance and to variation in helium abundance at a constant density, respectively. Similar expressions may be obtained for other parameters of solar structure, such as $u(r)$ and $W(r)$, instead of $\rho$. As noted, in this formulation the oscillation frequencies are sensitive only to the helium abundance in surface zones of ionization, among which the most interesting is the zone of second helium ionization, located in the adiabatic region of the convective zone, the structure of which does not depend on the detailed description of the convective processes. This important property enables us to determine the helium abundance without resorting to a particular convection model.

From a finite number of measured frequencies, the unknown functions may be determined from (1) only with finite radial resolution; in other words, only certain mean values can be determined. The method of optimal averages is used to find the latter. It is based on the Backus—Gilbert theorem, according to which the estimate of the mean values of the unknown functions in (1) at certain radii $r = r_0$ is a linear combination of observational data $\delta \omega_i/\omega_i$, i.e.,

$$\overline{\delta P/\rho} = \sum_{i=1}^N A_{(r_0, r)}^{(\omega_i)} \delta \omega_i,$$

and similarly for $Y$. Here $(\delta P/\rho)_{r_0}$ are the overall means of $\delta P/\rho(r)$:

$$\overline{\delta P/\rho} = \int_0^R A_{(r_0, r)}^{(\rho, r)} \delta P/\rho \, dr.$$

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The weighting functions $A^0(Y, r)$ are called optimal averaging kernels. They consist of linear combinations of sensitivity functions:

$$A^0(Y, r) = \sum_{i=1}^{N} a_i(Y, r) K_i(Y, r).$$

(4)

For the other unknown function $Y(r)$, Eqs. (2)-(4) have a similar form. The coefficients $a_i(Y, r)$ and $a_i(Y, r_0)$ are chosen from simultaneous observation of two conditions: closeness of the optimal averaging kernel to a $\delta$ function at the point $r = r_0$ for a certain unknown function and closeness of this kernel to zero for another function whose contribution we wish to eliminate. We may thus estimate $\delta \rho / \rho$ independently of $\delta Y$ and vice versa. The unknown coefficients are found by minimizing a quadratic function obtained by combining the aforementioned conditions. To determine $a_i(Y, r)$, for example, we seek to minimize

$$M (r, a, \alpha, \beta) = 12 \int_{0}^{R_{\odot}} (r - r_0)^2 [A^0(Y, r)]^2 dr + \beta \int_{0}^{R_{\odot}} [A^0(Y, r)]^2 dr + \alpha \sum_{i=1}^{N} E_{ij} a_i(Y, r) a_j(Y, r),$$

where $E_{ij}$ is the covariance matrix of the measurements errors. Here the first term expresses the condition of closeness of $A^0(Y, r)$ to a $\delta$ function, the second expresses the closeness of $A^0(Y, r_0)$ to zero, and the third term expresses minimization of the measurement errors. The parameter $\beta$ regulates the degree of influence of $\delta Y$ on the determination of $\delta \rho / \rho$ and is chosen such that this influence not exceed the measurement errors. The regularization parameter $\alpha$ is chosen from the condition of "balance" between the spatial resolution and the errors in estimating the unknowns, since striving to obtain the best resolution leads to large absolute values of the coefficients $a_i(Y, r)$, which means large errors in determining the corresponding optimum mean values from (2).

In solving inverse problems of solar seismology, one must bear in mind that high-frequency $p$ modes that penetrate into surface layers of the sun are more subject than others to the influence of nonadiabatic effects such as interaction with convective motion and radiative damping, which are ignored in our model. To avoid distortions, we have considered only the low-frequency range of the spectrum from 1.5 to 3 mHz. We chose a total of 598 frequencies.

In Fig. 1 we show the optimal averaging kernels for density at individual points in radius. The halfwidth of these kernels can serve to characterize the spatial resolution. The results of the solution for $\delta \rho / \rho$ and $\delta u / u$ are shown in Figs. 2 and 3. The parameters $\alpha$ and $\beta$ were $10^3$ and 1, respectively.

We see that the central region of the sun at $r < 0.1 R_{\odot}$ cannot be resolved using this set of frequencies, which does not contain modes with low $l$. The profile of $\delta u / u$ in the radiative zone is consistent, on the whole, with those found earlier from other observations by asymptotic and variational methods. In our solution for $\delta u / u$, however, the zone of an abrupt change in the speed of sound near the lower boundary of the convective zone can be resolved more clearly, which yields a more reliable determination of its depth.

The most interesting feature in our solutions in the solar core is the relatively abrupt change in density at $r = 0.2 R_{\odot}$. This feature can be traced more clearly if the spatial resolution is increased somewhat, at the cost of an increase in the error, by decreasing $\alpha$ to $10^2$ (Fig. 4). On the basis of our results, we believe that there is a small density jump near the boundary of the nuclear reaction zone. This feature may also show up, although less clearly, in the corresponding dependence of $\delta u / u$. The density jump may correspond to the boundary of the region of partial mixing of matter in the solar core. It is clear that if the mixing region is localized in the solar core, it will be separated from the surrounding matter by a certain density jump, since the molecular weight increases within that region as a result of nuclear reactions.

At present, as we noted, the structure of the core cannot be resolved well enough to determine the nature of the mixing. In this connection, it is of great interest to compare our solutions with theoretical solar models calculated with the introduction of partial local mixing in the core. Moreover, knowing the typical behavior of density in such models, we can attempt to extrapolate the solution of the inverse problem to construct a complete seismic model of the sun.

3. THEORETICAL SOLAR MODELS WITH MIXING IN THE CORE

In the course of evolution, a region with a lower hydrogen abundance $X$ and a higher helium abundance $Y$ than the surrounding matter is formed in the core as a result of nuclear reactions. Mixing decreases the radial gradients of hydrogen and helium abundance. We have simulated mixing by specifying a certain local perturbation in the functions $X(r)$ and $Y(r)$ obtained for the standard evolutionary model of the sun with the present age. A solar model with the given perturbed $X(r)$ and $Y(r)$ profiles was then calculated, assuming hydrostatic equilibrium and thermal balance. An equilibrium $^3$He abundance was also assumed.
We considered relative perturbations of hydrogen abundance

$$\frac{\delta X}{X} = C \cos \left( \frac{3}{2} \pi \frac{r}{r_0} \right),$$

where $C$ takes the values $C_1$ at $r \leq r_0/3$, $C_2$ at $r_0/3 < r < r_0$, and 0 at $r > r_0$; $C_1$ and $C_2$ are constants characterizing the amplitude of the perturbation in the core. This form of perturbation is qualitatively consistent with a determination of $\delta X/X(r)$ in the core directly from observed oscillation frequencies. The perturbed profiles of hydrogen and helium abundance were taken in the form

$$X'(r) = a \cdot X(r) \cdot (1 + \delta X/X), \quad Y'(r) = 1 - X'(r) - Z,$$

where $a$ is a numerical factor of order unity, the exact value of which was found together with the parameter $a$ of mixing length theory by requiring the equality of the luminosity and radius of the model with the present solar values.

The model building procedure and the input values of the physical parameters have been described in detail in Ref. 30. Our purpose is to study the relative perturbations of the radial structure in mixing, so possible uncertainties in the input data are not important in this case. The small relative perturbations depend more on the functional form of the aforementioned parameters than on their absolute values.

The perturbation parameters were varied within the following limits: $C_1$ and $C_2$ from 0.01 to 0.08, $r_0$ from 0.1 $R_\odot$ to 0.3 $R_\odot$. In Fig. 5 we show $X(r)$ profiles in the standard evolutionary model and in the perturbed model with $C_1 = 0.01$, $C_2 = 0.04$, and $r_0 = 0.2 R_\odot$. The relative perturbations $\delta X/X$, $\delta \rho/\rho$, and $\delta u/u$ are given in Fig. 6. The qualitative agreement between these functions and the solution of the inverse problem (Figs. 2 and 3) is worthy of note. This agreement is found to be closest for the indicated parameters $C_1$, $C_2$, and $r_0$, which gives some idea of the degree of possible mixing in the solar core.

The stronger perturbation $\delta u/u$ in the radiative zone, as suggested in Ref. 25, may be due to somewhat higher opacity.
4. SEISMIC MODEL OF THE SUN

A typical feature of the density perturbation in models with mixing in comparison with the standard model is the nonmonotonic variation as the center of the sun is approached: at a certain distance from the center, the density is higher, and near the center it is lower, than in the standard model. In constructing the seismic model, this qualitative behavior was taken into account in extrapolating the solution of the inverse problem in the vicinity of the center, which was chosen to have the form

\[ \frac{\delta \rho}{\rho} = \begin{cases} D_1 \cos \left( \frac{\pi}{2} \frac{r}{r_c} \right) & \text{at } r \leq r_c, \\ \left( \frac{\delta \rho}{\rho} \right)_{r_c} \sin \left( \frac{\pi}{2} \frac{r-r_c}{r_c} \right) & \text{at } r_c < r < r_e, \end{cases} \]

where \( r_c \) is the radius at which the density \( \left( \frac{\delta \rho}{\rho} \right)_{r_c} \) closest to the center in our solution of the inverse problem is determined (Fig. 2); \( r_f \) and \( D_1 \) are regularization parameters. The parameter \( D_1 \) was varied from \(-0.04\) to \(-0.01\), and \( r_f \) was found by requiring conservation of solar mass in the seismic model:

\[ \delta M_\odot = \int_0^{R_\odot} \delta \rho \, r^2 \, dr = 0. \]

The outer layers are not studied in detail here, so we have used the simplest extrapolation,

\[ \frac{\delta \rho}{\rho} = \left( \frac{\delta \rho}{\rho} \right)_{r_e} = \text{const}, \]

where \( r_e = 0.99 \).

From the known density distribution in the improved model,

\[ \rho'(r) = \rho(r) \left( 1 + \frac{\delta \rho}{\rho} \right) \]

the other hydrostatic parameters were calculated:

mass

\[ m'(r) = \int_0^r \rho'(r') r'' \, dr', \]

pressure

\[ P'(r) = P(R_\odot) + \int_r^{R_\odot} \frac{G \rho}{r^2} m'(r') \, dr', \]

and \( u'(r) = P'(r) / \rho' \) and \( W'(r) = (r^2 / GM_\odot) (du'/dr) \). In Fig. 7 we give the relative departures of the parameters \( \rho \) and \( u \) of the seismic model, obtained for \( D_1 = -0.015 \), from the initial standard model. The results of the solution of the inverse problem are also shown here. The perturbation of \( u \) in the calculated seismic model clearly agrees with its direct determination from oscillation frequencies.

FIG. 5. Distribution of hydrogen abundance in the solar core. Solid curve: standard evolutionary model; dashed curve: model with mixing for \( C_1 = 0.01, C_2 = 0.04, \) and \( r_0 = 0.2 \, R_\odot \).

FIG. 6. Relative departure from the standard model for the distributions of density (solid curve), the parameter \( u = P/\rho \) (short-dash line), and hydrogen abundance (long-dash line) in a model with mixing (\( C_1 = 0.01, C_2 = 0.04, \) \( r_0 = 0.2 \, R_\odot \)).

FIG. 7. Relative departure of the parameters \( \rho \) (curve 1) and \( u \) (curve 2) in the seismic model from the initial standard model. The same parameters determined from a solution of the inverse problem (curves with vertical error bars) are given for comparison.
FIG. 8. Relative difference between observed and theoretical frequencies of oscillation modes with $n = 7-16$ for the standard model (a) and the seismic model (b). Points corresponding to the same $n$ are joined by solid curves, to the right of which the respective $n$ is given.

FIG. 9. Parameter $W$ as a function of radius for the seismic model (solid curve) and the standard model (dashed curve).
In Fig. 8 we show the relative differences between the observed and theoretical frequencies for modes with radial order $n = 7-16$, calculated for the standard model 1 (a) and our seismic model (b). It would appear that even the first correction obtained by the variational method considerably improves the agreement between the observed and theoretical frequencies. The remaining difference is mainly a function of frequency and is related to the properties of the surface layer of turbulent convection on the sun, which can be investigated in detail based on the frequencies of modes with high $l$. We shall not consider that problem here, however.

The function $W(r)$ (Fig. 9) can be used to find the depth of the convective zone. This parameter reaches a constant value of $-2/5$, corresponding to adiabatic structure in deep layers of the convective zone, at $R_\infty = 0.713 \pm 0.003 R_\odot$. This corresponds to a depth $0.287 R_\odot$ for the convective zone, somewhat less than the estimate in Ref. 27 but in complete agreement with the value obtained in Ref. 29 using an asymptotic method to solve the inverse problem.

It is evident that the frequencies of $p$ modes with $l \geq 4$ are virtually insensitive to the structure of the innermost region of the sun at $r \leq 0.1 R_\odot$, so it is of great interest to compare the frequencies of $p$ modes of oscillations with $l = 0-3$ for the standard and seismic models with those obtained from observations, from which it follows that the relative frequency difference remains almost constant. This means that the constructed seismic model does not agree with the data of Ref. 31.

It is well known that an estimate of the parameters of the solar core from the frequencies of modes with $l = 0-5$ using data of Ref. 31 for $l = 0-3$ yields unusually high central densities: 10-20% higher than in the standard model. According to new data, in which the data of Ref. 31 were supplemented by more accurately measured frequencies with $l = 4$ and 5 from Ref. 14, the density perturbation at the center is somewhat smaller, $\approx 7\%$. A considerable discrepancy with the seismic model constructed from modes with intermediate $l$ (Fig. 10) is also obtained in this case in the range from 0.1 to 0.2 $R_\odot$, however. This contradiction indicates a discrepancy between observations of the frequencies of oscillations with low$^{31}$ and intermediate$^{14}$ values of $l$. The reason is not yet clear; it may lie in the less precise measurement of oscillation frequencies for low $l$, since here it is difficult to resolve the fine structure associated with rotation (for $l > 0$), or in systematic frequency shifts due to the solar activity cycle. It is obvious, however, that it will be impossible to draw reliable conclusions about the structure of the solar core unless this contradiction is resolved.

In Fig. 10 we also show a solution of the inverse problem of solar seismology by the method given in Sec. 2 using the frequencies of $p$ modes with $l = 0$, 1, and 2 based on observations from the Phobos spacecraft$^{16}$ (a total of 18 frequencies), supplemented by the frequencies of modes with $l = 4$ and 5 from Ref. 14. It is interesting that these new data confirm the high density in the core, so that the indicated contradiction remains.

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