RESONANT BEHAVIOUR OF MHD WAVES ON MAGNETIC
FLUX TUBES

I. Connection Formulae at the Resonant Surfaces

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Abstract. A basic procedure is presented for dealing with the resonance problems that appear in MHD of which resonant absorption of waves at the Alfvén resonance point is the best known example in solar physics. The procedure avoids solving the full fourth-order differential equation of dissipative MHD by using connection formulae across the dissipation layer.

1. Introduction

Free oscillations of a body of finite size are, in most cases, characterized by a discrete eigenfrequency spectrum. If a body is shaken at an arbitrary frequency, the waves reflected at the border of the body destructively interfere with the original waves and the oscillation is not generally sustained. The eigenfrequencies are such particular frequencies that the reflected waves coherently make up a sustained oscillation. However for a certain class of waves, the propagation of wave energy is restricted to a particular direction, and there is no interaction between neighbouring oscillating elements. Under such circumstances, an infinitesimal portion of the medium can oscillate with its own eigenfrequency, without disturbing the rest of the medium. Then, as there are arbitrarily many oscillating elements, the spectrum of the eigenoscillations is continuous.

In magnetohydrodynamics (MHD), a continuous eigenfrequency spectrum arises in two cases. One is for Alfvén waves (Tataronis and Grossmann, 1983), since the group velocity of Alfvén waves is directed strictly along the direction of the magnetic field. The other case is for slow mode waves whose group velocity is equal to the cusp speed, \( V_c = cV_A / \sqrt{c^2 + V_A^2} \) (\( V_A \) and \( c \) stand for the Alfvén speed and the sound speed, respectively). As the group velocity diagram of slow mode waves makes a cusp at the cusp speed which is tangent to the magnetic field vector, the propagation of the energy
is restricted to the direction along the magnetic field. Therefore, MHD media support the Alfvén continuum and the cusp continuum (Goedbloed, 1983). If \( k \) is the wave number in the direction of the magnetic field, these continua are characterized by the frequencies \( \omega_A = kV_A \) and \( \omega_c = kV_c \). The cusp frequency \( \omega_c \) is smaller than the Alfvén frequency \( \omega_A \).

The continuous eigenspectrum in MHD leads to two important phenomena, namely resonant absorption and phase mixing. If a medium is driven externally at a frequency \( \omega \) that falls in the band of the continuous eigenfrequencies of the system, the amplitude of the oscillation excited in the system peaks at the resonance point where the frequency of a field line (either \( \omega_A \) or \( \omega_c \)) matches the frequency \( \omega \) (Chen and Hasegawa, 1974). The amplitude and the width of the peak of oscillation scale as \((v + \eta)^{-1/3}\), and \((v + \eta)^{1/3}\), respectively (Kappraff and Tataronis, 1977), where \( v \) is the kinematic viscosity and \( \eta \) is the magnetic diffusivity. If no dissipation mechanisms are included, the amplitude diverges at the resonance point. Dissipation takes place around the resonance point where the perturbations develop large gradients. A resonance also occurs when a system with a discrete eigenfrequency spectrum is excited at one of its eigenfrequencies. The amplitude of the oscillation in this case, however, diverges everywhere (as \( \sim (v + \eta)^{-1} \)), in contrast to the resonance in the continuous spectrum.

Resonant absorption is a process in an externally driven system (i.e., forced oscillation). Phase mixing on the other hand is a free oscillation of the system (Rae and Roberts, 1981; Lee and Roberts, 1986). If oscillations are excited initially in a system of field lines but no external driving is applied afterwards, the oscillations become gradually out of phase among neighbouring field lines because each field line oscillates with a different eigenfrequency. A large gradient will develop across the field lines, and the energy will be dissipated. The characteristic decay rate of this phase-mixing process, which we designate by \( \gamma_{\text{mix}} \), scales as \((v + \eta)^{1/3}\) (Heyvaerts and Priest, 1983).

If an external driving (with frequency \( \omega \)) is turned on at some instant, both free and forced oscillations are excited in the system. As time progresses, the free oscillations decay due to phase mixing, and only the forced oscillation remains, which shows a peak in amplitude at the resonance point. Resonant absorption therefore manifests itself most clearly after phase mixing of the free oscillations is over.

On the other hand, if a system is excited by an external driver which has a broad, continuous frequency spectrum, the excited oscillations will not show any peak. (In other words, peaks are distributed continuously and therefore are smeared.) The energy is supplied to the system by the resonance, and is transferred to free oscillations which later decay by phase mixing. The root-mean-square amplitude of the oscillation in this case is a smooth function in space, but a snapshot of the oscillation shows a highly phase-mixed behaviour. Whether the frequency spectrum of the external driver is narrow enough for the resonant behaviour to be seen in the system is determined by comparing the width of the power spectrum of the driver (\( \delta\omega \)) and the decay rate of free oscillation due to phase mixing, \( \gamma_{\text{mix}} \). If \( \delta\omega < \gamma_{\text{mix}} \), the resonant peak will show up in the system because the free oscillations die out within the characteristic life time of the external driver. On the other hand, if \( \delta\omega > \gamma_{\text{mix}} \), the external driver changes its character of
oscillation (frequency, phase, etc.) before the phase-mixing damping is completed. Therefore, phase-mixing oscillations exist all the time, whose energy is replenished from the external driver via resonance. The basic wave equation that describes resonant absorption and phase mixing is a second-order differential equation if dissipation is neglected. This dissipationless equation has a singularity at the resonance point, so that it cannot be solved as a boundary value problem in a conventional sense (Mok and Einaudi, 1985). In the absence of a singularity, a solution \( y(x) \) in an interval \( x_1 \leq x \leq x_2 \) which satisfies the boundary condition \( y(x_1) = y_1 \) and \( y(x_2) = y_2 \) can be found by a usual shooting method. Namely, one generates solutions from \( (x = x_1, y = y_1) \) with various values of the derivative \( y'_1 \) and the solution that hits \( y = y_2 \) at \( x = x_2 \) is the solution to the boundary value problem. The reality is, however, that the equation to be solved has a singularity at \( x_r \), in the interval, and most solutions diverge there. The solution which remains finite at the singularity will not generally satisfy the boundary condition at \( x = x_2 \).

If dissipative effects are included, the equation becomes a fourth-order differential equation and the singularity is removed. However, the highest derivatives have coefficients proportional to resistivity and viscosity which are small and have effects only near the resonance point \( x_r \). Therefore, we encounter a so-called singular perturbation problem. To solve the equation, we start the solution from \( x = x_1 \) without including dissipative effects. As the solution approaches the resonance point, we switch to the full fourth-order equation, say at \( x = x_r - \delta \). The fourth-order differential equation has two solutions that diverge toward \( |x - x_r| \gg \delta \). Therefore, at \( x = x_r - \delta \), \( y \) and \( y' \) are specified by the solution already obtained in \( x < x_r - \delta \), but \( y'' \) and \( y''' \) have to be adjusted so that the solution remains finite at \( x = x_r + \delta \). Beyond \( x = x_r + \delta \), we can again switch to the second-order equation without dissipation, and the solution will be found up to \( x = x_2 \).

The most cumbersome part of the process described above is to solve the full fourth-order differential equation near the resonance point. As far as the global behaviour of the solution is concerned, however, the full solution near the resonance simply gives a connection formula across the resonance point. Therefore, once such a connection formula is found, one does not have to solve the full fourth-order equation (except when the behaviour of the solution near the resonance point itself is the topic to be studied). To establish a connection formula, an important observation is that, although the equation without dissipative effects is singular at the resonance point, not all the physical quantities diverge here. Hollweg (1987a) noticed for the first time that, in a planar geometry, the perturbed total (gas plus magnetic) pressure is not divergent but continuous across the singularity. It is natural to assume that such a conserved quantity of the dissipationless equations would be conserved even if dissipative effects are included. By using this approach, Hollweg (1987a, b) and Hollweg and Yang (1988) were able to find solutions to the problem without explicitly solving the full dissipative equation. Hollweg et al. (1990) used this property to obtain complex eigenfrequencies of quasi-modes in ideal MHD. They considered incompressible perturbations in a planar geometry and included an equilibrium flow in their analysis.
In an equilibrium with curved magnetic field lines, like a straight cylinder with an azimuthal field $B_\varphi$, the Eulerian perturbation of total pressure is no longer a conserved quantity across the resonance point. In this paper new conservation formulae are presented and based on these conservation formulae a general method is formulated for determining the resonant absorption of driven waves in magnetic flux tubes. In a subsequent paper our formulation is applied to the interaction between sound waves and magnetic flux tubes (Sakurai, Goossens, and Hollweg, 1991, Paper II). This application is motivated by the observations that sound waves in the solar atmosphere are absorbed very efficiently by sunspots (Braun, Duvall, and LaBonte, 1988).

2. Basic Equations for Dissipationless Cases

We adopt a system of cylindrical coordinates $(r, \varphi, z)$ and assume a cylindrically-symmetric, static equilibrium configuration. The components of the equilibrium magnetic field $\mathbf{B}(0, B_\varphi, B_z)$ as well as the pressure $p$ and density $\rho$ are functions of the radial coordinate $r$ only. They satisfy the radial force balance equation

$$\frac{d}{dr} \left( p + \frac{B^2}{2\mu} \right) = -\frac{B_\varphi^2}{\mu r} . \quad (1)$$

Gravity is not included in the present analysis, and as a consequence $\rho$ does not appear in Equation (1).

The perturbed system is described by the linearized forms of the induction equation, the equation of motion, the continuity equation and the adiabatic condition:

$$\frac{\partial \mathbf{B}'}{\partial t} = \nabla \times (\mathbf{V}' \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}' , \quad (2)$$

$$\rho \frac{\partial \mathbf{V}'}{\partial t} = -\nabla p' + \frac{1}{\mu} \left[ (\nabla \times \mathbf{B}) \times \mathbf{B}' + (\nabla \times \mathbf{B}') \times \mathbf{B} \right] +$$

$$+ \rho \left[ \nu \nabla^2 \mathbf{V}' + \zeta \nabla (\nabla \cdot \mathbf{V}') \right] , \quad (3)$$

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho \mathbf{V}') = 0 , \quad (4)$$

$$\frac{\partial p'}{\partial t} + V_r \frac{dp}{dr} = \frac{\gamma p}{\rho} \left( \frac{\partial \rho'}{\partial t} + V_r \frac{d\rho}{dr} \right) . \quad (5)$$

In these equations $\eta$ is the electric diffusivity, $\nu$ and $\zeta' = \zeta - \nu/3$ are the kinematic shear and volume viscosities and $\gamma$ is the ratio of specific heats. All these quantities are assumed to be constant. Although dissipative terms are included in Equations (2)–(3), the energy equation is approximated by the adiabatic equation. The Eulerian perturbation of a quantity $f$ is denoted by $f'$. Since the equilibrium quantities depend on $r$ only,
the perturbed quantities can be Fourier-analyzed with respect to the ignorable coordinates $\varphi$ and $z$ and time $t$ and put proportional to
\[ \exp \left[ i(m\varphi + kz - \omega t) \right]. \]

Here $m$ (= an integer) and $k$ are the azimuthal and axial wave numbers.

Let us now turn to the equations that govern the perturbations in absence of dissipation. We can eliminate all but two of the perturbed variables from the linearized MHD equations by algebraic means. One step is the definition of the perturbed total pressure $P'$:
\[ P' = p' + B \cdot B'/\mu. \quad (6) \]

The other key is the derivation of the compression term
\[ \text{div} \xi = -\frac{\omega^2(P' - 2B^2_{\varphi}\xi_r/\mu r)}{\rho(c^2 + V_A^2)(\omega^2 - \omega_c^2)}. \quad (7) \]

Here $\xi_r$ is the radial component of the displacement defined by
\[ -i\omega \xi_r = V_r', \]
and $c$, $V_A$, $\omega_A$, and $\omega_c$ denote the adiabatic speed of sound, the Alfvén speed, the Alfvén frequency, and the cusp frequency. The squares of these quantities are defined as
\[ c^2 = \frac{\gamma p}{\rho}, \quad V_A^2 = \frac{B^2}{\mu \rho}, \]
\[ \omega_A^2 = f_B^2 \frac{\mu}{\mu \rho}, \quad \omega_c^2 = \frac{c^2}{c^2 + V_A^2} \omega_A^2, \quad f_B = \frac{m}{r} B_\varphi + kB_z. \quad (8) \]

After algebraic reduction it is found that the $r$-dependence of the perturbations in absence of dissipation is governed by two linear first-order differential equations for $\xi_r$ and $P'$:
\[ D \frac{d}{dr} (r\xi_r) = C_1 r\xi_r - C_2 rP', \quad (9) \]
\[ D \frac{dP'}{dr} = C_3 \xi_r - C_1 P', \quad (10) \]
where
\[ D = \rho(c^2 + V_A^2)(\omega^2 - \omega_A^2)(\omega^2 - \omega_c^2), \quad (11a) \]
\[ C_1 = 2\omega^4 \frac{B^2_\varphi}{\mu r} - 2m \frac{f_B B_\varphi}{\mu r^2} (c^2 + V_A^2)(\omega^2 - \omega_c^2), \quad (11b) \]
\[ C_2 = \omega^4 - \left( c^2 + V_A^2 \right) \left( \frac{m^2 + k^2}{r^2} \right) (\omega^2 - \omega_0^2), \quad (11c) \]

\[ C_3 = D \left[ \rho (\omega^2 - \omega_A^2) + 2 \frac{B_\varphi}{\mu} \frac{d}{dr} \left( \frac{B_\varphi}{r} \right) \right] + \]

\[ + 4 \left( \frac{B_\varphi}{\mu r} \right)^2 \omega^4 - 4 \rho (c^2 + V_A^2) (\omega^2 - \omega_0^2) \omega_A^2 \frac{B_\varphi^2}{\mu r^2}. \quad (11d) \]

Equations (9)–(10) govern the linear perturbations of a straight compressible cylinder. They were first obtained in this form by Appert, Gruber, and Vaclavik (1974). The other perturbed quantities \((p', \rho', \text{etc.})\) can be computed once \(\xi_r\) and \(P'\) are known. For later use we note that the components of the displacement vector in the magnetic surfaces parallel to the magnetic field lines, \(\xi_\parallel = \xi \cdot B / B\), and perpendicular to the magnetic field lines, \(\xi_\perp = (\xi_\varphi B_z - \xi_z B_\varphi) / B\), are related to \(\xi_r\) and \(P'\) as

\[ (\omega^2 - \omega_A^2) \xi_\perp = \frac{i}{\rho B} \left( g_B P' - 2 f_B B_\varphi B_z \xi_r / \mu r \right), \quad (12a) \]

\[ (\omega^2 - \omega_0^2) \xi_\parallel = \frac{i f_B}{\rho B} \frac{c^2}{c^2 + V_A^2} \left( P' - 2 B_\varphi^2 / \mu r \right). \quad (12b) \]

Here \(B = (B_\varphi^2 + B_z^2)^{1/2}\) and

\[ g_B = \frac{m}{r} B_z - k B_\varphi. \quad (13) \]

Equations (9)–(10) can be combined into one second-order differential equation for \(\xi_r\) (Hain and Lüst, 1958):

\[ \frac{d}{dr} \left[ f(r; \omega^2) \frac{d}{dr} (r \xi_r) \right] - g(r; \omega^2) r \xi_r = 0, \quad (14) \]

where

\[ f(r; \omega^2) = \frac{D}{r C_2}, \quad g(r; \omega^2) = \frac{d}{dr} \left( \frac{C_1}{r C_2} \right) - \frac{1}{r D} \left( C_3 - \frac{C_1^2}{C_2} \right), \quad (15) \]

or for \(P'\),

\[ \frac{d}{dr} \left[ \tilde{f}(r; \omega^2) \frac{dP'}{dr} \right] - \tilde{g}(r; \omega^2) P' = 0, \quad (16) \]
where
\[ \tilde{f}(r; \omega^2) = \frac{rD}{C_3}, \quad \tilde{g}(r; \omega^2) = -\frac{d}{dr} \left( \frac{rC_1}{C_3} \right) - \frac{r}{D} \left( C_2 - \frac{C_1^2}{C_3} \right). \] (17)

Equations (9)–(10) define an eigenvalue problem with \( \omega^2 \) as eigenvalue parameter when they are supplemented with proper boundary conditions. In a driven problem \( \omega \) is prescribed. Equations (9)–(10) have regular singular points at the zeros of the coefficient function \( D \). From Equation (11a) there are mobile regular singularities at positions \( r \) where
\[ \omega^2 = \omega_A^2(r), \] (18a)
\[ \omega^2 = \omega_c^2(r). \] (18b)

Equation (18a) defines the Alfvén resonance point, while Equation (18b) defines the slow or cusp resonance point. Since \( \omega_A^2(r) \) and \( \omega_c^2(r) \) are functions of position, Equations (18a, b) define, from the viewpoint of the spectrum of ideal MHD, two continuous ranges in the spectrum which are classically referred to as the Alfvén continuum and the slow continuum.

3. Conservation Laws at the Resonance Point

3.1. ALFVÉN RESONANCE POINT

We now focus on a frequency in the Alfvén continuum and determine the spatial behaviour of the corresponding perturbation close to the resonant or singular surface \( r = r_A \) where the condition \( \omega^2 = \omega_A^2(r_A) \) is satisfied. It is convenient to introduce the new radial variable \( s \) defined as
\[ s = r - r_A \] (19)
for determining the series expansions of the coefficient functions \( D, C_1, C_2, \) and \( C_3 \) in Equations (9)–(10). The series expansion of \( D \) starts with a term in \( s, d_1s, \) while the series expansions of \( C_1, C_2, \) and \( C_3 \) all start with constant terms \( C_{10}, C_{20}, \) and \( C_{30} \) (at least when \( B_\phi \neq 0 \)). We have
\[ d_1 = \rho V_A^2 \omega_A^2 d, \]
\[ C_{10} = -2B_\phi B_z \omega_A^2 f_B g_B / pr \mu^2, \]
\[ C_{20} = -\omega_A^2 g_B^2 / r \mu, \]
\[ C_{30} = -4B_\phi^2 B_z^2 \omega_A^2 / r^2 \mu^2, \] (20)

where
\[ \Delta = \frac{d}{dr} (\omega^2 - \omega_A^2(r)), \] (21)
and all quantities in the right-hand members of Equation (20) have to be evaluated at 
$r = r_\Lambda (s = 0)$. For later use it is important to note that

$$C_{10}^2 - C_{20} C_{30} = 0.$$  \hspace{1cm} (22)

In the same way we can obtain series expansions for the functions $f, f, \tilde{f},$ and $\tilde{g}$
in Equations (14)–(17). The important point here is to note that the series expansions
of $g$ and $\tilde{g}$ (for $B_\varphi \neq 0$) both start with non-zero constant terms. In the vicinity of $s = 0$,
the second-order differential equations (14) and (16) reduce to

$$\alpha \frac{d}{ds} \left( s \frac{d\xi_r}{ds} \right) + \beta \xi_r = 0 ,$$  \hspace{1cm} (23)

$$\tilde{\alpha} \frac{d}{ds} \left( s \frac{dP'}{ds} \right) + \tilde{\beta} P' = 0 ,$$

where $\alpha, \beta, \tilde{\alpha},$ and $\tilde{\beta}$ denote constants arising from the series expansions.
Equations (23) both have an indicial equation with the double root $v_{1,2} = 0$. This implies
that one of the two independent solutions both for $\xi_r$ and $P'$ has a logarithmic function.
Since the interval contains only one singular point, the solutions for $\xi_r(s)$ and $P'(s)$ may
be written as

$$\xi_r(s) = \begin{cases} S_1 u(s) + R_1 (u(s) \ln |s| + v(s)) , & s < 0 , \\ S_2 u(s) + R_2 (u(s) \ln |s| + v(s)) , & s > 0 , \end{cases}$$  \hspace{1cm} (23a)

$$P'(s) = \begin{cases} P_1 \tilde{u}(s) + Q_1 (\tilde{u}(s) \ln |s| + \tilde{v}(s)) , & s < 0 , \\ P_2 \tilde{u}(s) + Q_2 (\tilde{u}(s) \ln |s| + \tilde{v}(s)) , & s > 0 . \end{cases}$$

In these equations $S_1, S_2, R_1, R_2, P_1, P_2, Q_1,$ and $Q_2$ are constants, and $u(s), v(s), \tilde{u}(s),$ and $\tilde{v}(s)$ are analytic functions of $s,$ starting with a constant. Hence, we can use the
normalization $u(0) = 1, \tilde{u}(0) = 1$ for convenience. The regular solutions $u(s)$ and $\tilde{u}(s)$
are called the ‘small’ solutions while the solutions containing the $\ln |s|$ term are the ‘large’
solutions. It can be shown (see, e.g., Goedbloed, 1983) that the large solutions have to be
continuous, whereas the small solutions may jump: $R_1 = R_2 = R,$ $S_1 \neq S_2,$
$Q_1 = Q_2 = Q,$ $P_1 \neq P_2.$ The solutions for $\xi_r$ and $P'$ diverge logarithmically but the
differences in these logarithmic terms remain balanced. The solution then takes the form

$$\xi_r(s) = Ru(s) \ln |s| + \begin{cases} \xi^-_A (s) , & s < 0 , \\ \xi^+_A (s) , & s > 0 , \end{cases}$$  \hspace{1cm} (24)

$$P'(s) = Q\tilde{u}(s) \ln |s| + \begin{cases} P^-_A (s) , & s < 0 , \\ P^+_A (s) , & s > 0 . \end{cases}$$  \hspace{1cm} (25)
We are concerned with the jumps in $\xi_r(s)$ and $P'(s)$ that are defined as

\[
\begin{align*}
[\xi_r] &= \lim_{s \to 0^+} \xi_r(s) - \lim_{s \to 0^-} \xi_r(s), \\
[P'] &= \lim_{s \to 0^+} P'(s) - \lim_{s \to 0^-} P'(s),
\end{align*}
\]

(26)

and in that context it suffices to take the constant terms in the series expansions of $\xi_A^-(s)$, $\xi_A^+(s)$, $P_A^-(s)$, and $P_A^+(s)$.

Let us now see what happens when the magnetic field lines are straight ($B_\varphi = 0$). In that case $C_1 \equiv 0$ and the series expansions start with a constant term for $C_2$, a term in $s^2$ for $C_3$, a term in $s$ for $f$ and $g$ and a term in $s^{-1}$ for $\tilde{f}$ and $\tilde{g}$. In the vicinity of the resonant surface ($s = 0$) the second-order differential equations for $\xi_r$ and $P'$ reduce to

\[
\begin{align*}
\alpha \frac{d}{ds} \left( s \frac{d\xi_r}{ds} \right) + \beta s \xi_r &= 0 , \\
\tilde{\alpha} \frac{d}{ds} \left( \frac{1}{s} \frac{dP'}{ds} \right) + \tilde{\beta} \frac{P'}{s} &= 0 ,
\end{align*}
\]

(27a) (27b)

where again $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ denote constants arising from the series expansions. The indicial equation of Equation (27a) has the double root $v_{1,2} = 0$ leading to a log $|s|$ and a Heaviside function contribution to the solution for $\xi_r$. The indicial equation of Equation (27b) has the roots $v_1 = 0$ and $v_2 = 2$. The root $v_2 = 2$ leads to a $s^2 \log |s|$ contribution to the solution which for $|s| \ll 1$ can be neglected compared to the constant term. This $s^2 \ln |s|$ behaviour was also found by Hollweg (1987b) for the Alfvén resonance in incompressible plasmas. For an equilibrium with straight magnetic field lines $P'$ has no logarithmic singularity. We shall see below that in that case $P'$ does not jump either.

Let us now return to Equation (22). This equation implies that the right-hand members of Equations (9)–(10) are linearly dependent in leading order. To make this point clear, we rewrite Equations (9)–(10) close to the singularity as

\[
\begin{align*}
 sA \frac{d\xi_r}{ds} &= \frac{\alpha_B}{\rho B^2} \left( g_B P' - 2f_B B_\varphi B_z \xi_r / \mu r \right) , \\
 sA \frac{dP'}{ds} &= 2 \frac{f_B B_\varphi B_z}{\mu r \rho B^2} \left( g_B P' - 2f_B B_\varphi B_z \xi_r / \mu r \right).
\end{align*}
\]

(28) (29)

In these equations all equilibrium quantities are evaluated at $s = 0$ ($r = r_A$). From Equations (28)–(29) we obtain that

\[
 s \frac{d}{ds} \left( g_B P' - 2f_B B_\varphi B_z \xi_r / \mu r \right) = 0 .
\]

(30)
This equation indicates that the quantity in the parentheses might be conserved. However, because of the factor \( s \), we should generally write

\[
g_B P' - 2f_B B_\phi B_z \xi_r / \mu r = C_A + C_B H(s) ,
\]

where \( C_A \) and \( C_B \) are constants and \( H(s) \) stands for the Heaviside function. The continuity of the ‘large’ solutions for \( \xi_r \) and \( P' \) implies that there cannot be a Heaviside contribution to the right-hand member of (31). The fundamental conservation law at the Alfvén point resonance then is

\[
g_B P' - 2f_B B_\phi B_z \xi_r / \mu r = C_A .
\]

Equations (28) and (29) can then be integrated to give

\[
\xi_r(s) = \frac{g_B}{\rho AB^2} C_A \ln |s| + \begin{cases} 
\xi^-_A(s), & s < 0 , \\
\xi^+_A(s), & s > 0 ,
\end{cases}
\]

\[
P'(s) = \frac{2f_B B_\phi B_z}{\mu \rho AB^2 r_A} C_A \ln |s| + \begin{cases} 
P^-_A, & s < 0 , \\
P^+_A, & s > 0 ,
\end{cases}
\]

where now the constant terms in \( Ru(s) \) and \( Q\tilde{u}(s) \) of Equations (24) and (25) have been identified. The jumps in \( \xi_r \) and \( P' \) are due to dissipative effects and will be specified later. Using (12a) we can rewrite conservation law (32) as

\[
s \xi_\perp = iC_A / \rho B A .
\]

Conservation law (32) just expresses that \( s \xi_\perp \) is constant across the resonant layer, or that \( \xi_\perp \) has a \( s^{-1} \) singularity and a \( \delta(s) \) contribution which dominate the \( \ln |s| \) singularity and \( H(s) \) contribution found for \( \xi_r, \xi_\parallel, \) and \( P' \). The dominant singularities in the solutions reside in the components tangent to the magnetic surfaces and perpendicular to the magnetic field lines (see also Goedbloed, 1983). If \( B_\phi = 0 \), the conserved quantity reduces to

\[
\frac{m}{r_A} B_z P' = C_A ,
\]

that is, the total pressure perturbation \( P' \) is conserved across the singularity.

3.2. Slow resonance point

Next we consider a frequency in the slow continuum giving rise to a singularity at the position \( r = r_c \) where \( \omega^2 = \omega_c^2(r_c) \). The radial variable \( s \) is now defined as

\[
s = r - r_c .
\]

The series expansion of \( D \) again starts with a term in \( s \), while the series expansions of
the other coefficient functions all start with a constant term. We have

\[ d_1 = -\rho V^2_A \omega^2_A \Delta, \]

\[ C_{10} = 2\omega_c^4 B^2_\phi/\mu r, \]

\[ C_{20} = \omega_c^4, \]

\[ C_{30} = 4\omega_c^4 B^4_\phi/\mu^2 r^2, \]

where

\[ \Delta = \frac{d}{dr} (\omega^2 - \omega^2_c(r)), \]

and all quantities in the right-hand member of (37) have to be evaluated at \( r = r_c (s = 0) \). Note that again

\[ C_{10}^2 - C_{20} C_{30} = 0. \]

Close to the singularity the second-order differential equations for \( \xi \) and \( P' \) reduce to equations of the form (23), and the solutions for \( \xi_r \) and \( P' \) are of the same form as given in Equations (24)–(25), at least when \( B_\phi \neq 0 \).

When \( B_\phi = 0 \), things do not change for \( \xi_r \). We get an equation of the form (23a) and a corresponding solution for \( \xi_r \) as given in Equation (24). The differential equation for \( P' \) however reduces to

\[ \tilde{x} \frac{d^2 P'}{d s^2} + \tilde{\beta} \frac{P'}{s} = 0 \]

leading to an indicial equation with roots \( \nu_1 = 0, \nu_2 = 1 \). The root \( \nu_2 = 1 \) gives rise to a \( s \ln |s| \) contribution to the solution which for \( |s| \ll 1 \) can be neglected compared to the constant term. This \( s \ln |s| \) behaviour was also found by Hollweg and Yang (1988) for the slow resonance in planar geometry. Again for an equilibrium with straight magnetic field lines \( P' \) has no logarithmic singularity. We shall see below that \( P' \) does not jump either when \( B_\phi = 0 \).

Let us now return to condition (39). Close to the singularity, Equations (9)–(10) can be written as

\[ s \Delta \frac{d \xi_r}{ds} = \frac{\mu \omega_c^4}{B^2 \omega_A^2} (P' - 2B^2_\phi \xi_r/\mu r), \]

\[ s \Delta \frac{d P'}{ds} = 2 \frac{\omega_c^4}{r B^2 \omega_A^2} (P' - 2B^2_\phi \xi_r/\mu r). \]

A similar analysis as given for the Alfvén resonance leads to the fundamental con-
servation law for slow continuum modes:

\[ P' - 2B_\phi^2 \frac{\xi_r}{\mu r} = C_S, \tag{43} \]

where \( C_S \) is a constant.

Using (7) and (12b) we can rewrite the conservation law (43) as

\[ s \text{ div } \xi = -\frac{\omega_c^2}{\rho(c^2 + V_A^2)} \frac{C_S}{\Delta}, \]

\[ s \xi_\parallel = \frac{if_B}{\rho B A} \frac{c^2}{c^2 + V_A^2} C_S. \tag{44} \]

Conservation law (43) then expresses that \( s \text{ div } \xi \) and \( s \xi_\parallel \) are constant across the resonant point, or that \( \text{div } \xi \) and \( \xi_\parallel \) have a \( s^{-1} \) singularity and a \( \delta(s) \) contribution which dominate the \( \ln |s| \) singularity and the \( H(s) \) contribution found for \( \xi_r, \xi_\perp \), and \( P' \). The dominant singularities in the solutions now reside in div \( \xi \) and in the components tangent to the magnetic surfaces and parallel to the magnetic field lines. Equations (41) and (42) can then be integrated to give

\[ \xi_r(s) = \frac{\mu \omega_c^4}{B^2 \omega_A^2 A} C_S \ln |s| + \begin{cases} \xi_{s_{-}}, & s < 0, \\ \xi_{s_{+}}, & s > 0, \end{cases} \tag{45} \]

\[ P'(s) = \frac{2 \omega_c^4 B_\phi^2}{r B^2 \omega_A^2 A} C_S \ln |s| + \begin{cases} P_{s_{-}}, & s < 0, \\ P_{s_{+}}, & s > 0. \end{cases} \tag{46} \]

The jumps in \( \xi_r \) and \( P' \) are due to dissipative effects and will be specified in Section 4.2.

In an equilibrium with a straight magnetic field \( (B_\phi = 0) \) the conserved quantity reduces to

\[ P' = C_S, \]

that is the total pressure perturbation is conserved across the singularity.

4. Effects of Dissipation and Connection Formulae

The effects of dissipation are generally small and are localized in the vicinity of the resonance layer. We therefore introduce the following approximations: (a) terms involving \( v^2, \eta^2 \), etc., are neglected and only those linear in \( v, \eta, \) and \( \zeta \) are retained; (b) in the dissipative terms, only the \( r \)-derivatives of the perturbed quantities are retained while those of equilibrium quantities are neglected. Then it can be shown that the basic equations reduce to the form of Equations (9)–(10), if Equation (11a) for the definition
of $D$ is replaced by

$$D = \rho (c^2 + V_A^2) \left[ \omega^2 - \omega_A^2 - i \omega (v + \eta) \frac{d^2}{dr^2} \right] \times$$

$$\times \left[ \omega^2 - \omega_Z^2 - i \omega \left( v + \frac{\omega_Z^2}{\omega_A^2} \eta \right) \frac{d^2}{dr^2} \right].$$

(47)

The contribution from $\zeta$-terms is proportional to $d/dr$ and can be neglected.

4.1. CONNECTION FORMULAE AT THE ALFVÉN RESONANCE POINT

When dissipative effects are included, Equations (28)–(29) close to Alfvén singularity, $r = r_A$, take the form

$$s \Delta - i \omega (v + \eta) \frac{d^2}{ds^2} \frac{d \xi_r}{ds} = \frac{g_B}{\rho B^2} \left( g_B P' - 2 f_B B_\varphi B_z \xi_r / \mu r \right),$$

(48)

$$s \Delta - i \omega (v + \eta) \frac{d^2}{ds^2} \frac{d P'}{ds} = \frac{2 f_B B_\varphi B_z}{\mu r \rho B^2} \left( g_B P' - 2 f_B B_\varphi B_z \xi_r / \mu r \right).$$

(49)

As a consequence conservation law (32) still holds when dissipative effects are included. Therefore the two dissipative equations to be solved look like

$$s \Delta - i \omega (v + \eta) \frac{d^2}{ds^2} \frac{d y}{ds} = R,$$

(50)

where $y$ stands for $\xi_r$ and $P'$ respectively and $R$ denotes the constant right-hand member: $R = g_B C_A / \rho B^2$ for $y = \xi_r$, $R = 2 f_B B_\varphi B_z C_A / \mu r \rho B^2$ for $y = P'$, respectively. The differential equation is of second-order in terms of $dY/ds$, by virtue of the conservation law (32). Dissipation is important when

$$\left| s \Delta \frac{d y}{ds} \right| \quad \text{and} \quad \left| \omega (v + \eta) \frac{d^2 Y}{ds^2} \right|$$

are comparable. The thickness of the dissipation layer can then be measured by the quantity $\delta_A$ defined as

$$\delta_A = \left[ \frac{\omega}{\Delta} (v + \eta) \right]^{1/3}.$$

(51)

The thickness of the dissipation layer is therefore proportional to $((v + \eta) / |\omega_A^2|)^{1/3}$, a result already obtained by Kappraaf and Tataronis (1977) and Hollweg and Yang (1988) and numerically verified by Poedts, Goossens, and Kerner (1990). Now we introduce
scaled variables

\[ x = \varepsilon \frac{s}{\delta_A}, \quad y = \frac{\delta_A A}{R} \frac{dY}{ds}, \]  

(52)

to rewrite Equation (50) as

\[ \frac{d^2y}{dx^2} + ixy = i\varepsilon \]  

(53)

with \( \varepsilon = \frac{\text{sign}(\omega/\Delta)}{\Delta} \). Equation (53) was previously obtained by Hollweg (1987b). The first term on the left-hand side represents the effect of dissipation; in the ideal case Equation (53) reduces to \( xy = \varepsilon \). Equation (53) indicates that \( \text{Re}(y) \) is anti-symmetric whereas \( \text{Im}(y) \) is symmetric with respect to \( x \). We are concerned with \( \int_{-\infty}^{+\infty} y \, dx \) since this integral is related to the jumps in the solutions. The integral of \( y \) from \( x = -\infty \) to \( x = +\infty \) leads to a purely imaginary number.

The quantity to be determined is the jump in \( Y \), i.e., the difference of \( Y \) between slightly positive and negative values of \( s \):

\[ [Y] = Y(r_A + \rho) - Y(r_A - \rho) = \int_{r_A - \rho}^{r_A + \rho} \frac{dY}{dr} \, dr, \]

where \( \rho \) satisfies \( \delta \ll \rho \). In terms of \( x \) and \( y \), \([Y]\) can be rewritten as

\[ [Y] = i \frac{R}{\Delta} \int_{-\rho/\delta}^{\rho/\delta} \text{Im} \, y \, dx. \]

Since \( \delta \ll \rho \), we can replace the integration limits \( -\rho/\delta \) and \( \rho/\delta \) by \( -\infty \) and \( +\infty \), respectively, so that

\[ [Y] = i \frac{R}{\Delta} \int_{-\infty}^{+\infty} \text{Im} \, y \, dx. \]

(54)

In the Appendix we show that

\[ \int_{-\infty}^{+\infty} \text{Im} \, y \, dx = -\pi\varepsilon. \]

(55)

Going back to the definitions for \( Y \) and \( R \), we obtain the jump relations for \( \xi_r \) and \( P' \) across the singular point:

\[ [\xi_r] = -i\pi \frac{\text{sign} \omega}{|\Delta|} \frac{g_B}{\rho B^2} \frac{C_A}{}, \]

(56)
\[
[P'] = -i\pi \frac{\text{sign } \omega}{|A|} \frac{2B_\phi B_z f_B}{\rho B^2 \mu r} C_A. \tag{57}
\]

The jump relations are not unexpected, because \(\ln s = \ln (-s) \pm i\pi\) for the complex \(\ln\) function, but the \(\pm\) sign is not arbitrarily but uniquely determined by the inclusion of dissipative effects. We can use Equations (12b) in combination with jump relations (56) and (57) to obtain the jump in \(\xi_\parallel\) as

\[
[\xi_\parallel] = \pi \frac{\text{sign } \omega}{|A|} \frac{2kB_\phi f_B}{\mu r \rho^2 B_0} \frac{c^2}{c^2 + V_A^2} C_A. \tag{58}
\]

The jumps in \(P'\) and \(\xi_\parallel\) are both proportional to \(B_\phi\). In an equilibrium with straight magnetic field lines (\(B_\phi = 0\), both \(P'\) and \(\xi_\parallel\) are continuous across the resonant point. Also when \(B_\phi = 0\), \(\xi_\parallel\) does not jump for waves with azimuthal wave number \(m = 0\), so that waves with \(m = 0\) are not resonantly absorbed in the Alfvén continuum in that case. However, when \(B_\phi \neq 0\), \(\xi_\parallel\) does jump for waves with azimuthal wave number \(m = 0\), and the jump in \(\xi_\parallel\) is proportional to both \(B_\phi\) and the longitudinal wave number \(k\).

4.2. Connection Formulae at the Slow Resonance Point

The jump conditions at the cusp resonance point can be determined in a similar way. Close to the slow singularity \(r = r_S\), the equations can be rewritten when dissipation is included, as

\[
\left[ sA - i\omega \left( v + \frac{\omega_c^2}{\omega_A^2} \eta \right) \frac{d^2}{ds^2} \right] \frac{d\xi_r}{ds} = \frac{\mu \omega_c^4}{B^2 \omega_A^2} \left( P' - 2B_\phi^2 \xi_r/\mu r \right), \tag{59}
\]

\[
\left[ sA - i\omega \left( v + \frac{\omega_c^2}{\omega_A^2} \eta \right) \frac{d^2}{ds^2} \right] \frac{dP'}{ds} = 2 \frac{\omega_c^2 B_\phi^2}{rB^2 \omega_A^2} \left( P' - 2B_\phi^2 \xi_r/\mu r \right). \tag{60}
\]

The thickness of the dissipative layer is now

\[
\delta_c = \left[ \frac{\omega}{|A|} \left( v + \frac{\omega_c^2}{\omega_A^2} \eta \right) \right]^{1/3}, \tag{61}
\]

and the jumps in \(\xi_r\) and \(P'\) are now found to be

\[
[\xi_r] = -i\pi \frac{\text{sign } \omega}{|A|} \frac{\mu \omega_c^4}{B^2 \omega_A^2} C_S, \tag{62}
\]

\[
[P'] = -i\pi \frac{\text{sign } \omega}{|A|} \frac{2 \omega_c^4 B_\phi^2}{rB^2 \omega_A^2} C_S. \tag{63}
\]

We can use Equation (12a) in combination with jump relations (62) and (63) to obtain
the jump in $\xi_\perp$ as

$$\left[ \xi_\perp \right] = -\pi \frac{\text{sign } \omega}{|\Delta|} \frac{2kB_\varphi \omega_c^4}{r\omega_\Lambda^2} \frac{C_S}{(\omega_c^2 - \omega_\Lambda^2)\rho B}.$$  \hspace{1cm} (64)

In an equilibrium with a straight magnetic field ($B_\varphi = 0$) $P'$ and also $\xi_\perp$ (see Equation (64)) are conserved quantities. The jump in $\xi_r$ reduces to

$$\left[ \xi_r \right] = -i\pi \frac{\text{sign } \omega}{\rho |\Delta|} \left( \frac{c^2}{e^2 + V_\Lambda^2} \right)^2 k^2 P'.$$

It is independent of the azimuthal wave number $m$ and for given $P'$ proportional to $k^2$. In particular axisymmetric ($m = 0$) waves are resonantly absorbed at the slow resonance point in an equilibrium with $B_\varphi = 0$. This is in contrast with the result for the Alfvén resonance point.

5. Conclusions

In this paper we have established a basic procedure for studying the resonance problems that appear in MHD. The procedure uses connection formulae across the resonance point and avoids solving the full fourth-order differential equation near the resonance point. Under the plausible assumptions that the effects of dissipation are small and confined to a thin dissipation layer, the equations to be solved are the ideal MHD equations (9)–(10). If an Alfvén resonance point exists in the system, the solution near the resonance point behaves as in Equations (33)–(34). The jumps in the solutions across the thin dissipation layers are given in Equations (56)–(57). If a slow resonant point exists, the solutions near the resonance point are then given by Equations (45)–(46). The jumps in the solutions across the resonance point are specified in Equations (62) and (63). With proper boundary conditions, the problem can then be solved as an ordinary boundary value problem. The values of resistivity and viscosity only affect the thickness of the dissipation layer and the behaviour of the solutions in the dissipation layers; they do not affect the differences (jumps) in the solutions across the dissipation layer.

An application of this procedure will be presented in a subsequent paper where we investigate the interaction between sunspots and waves in the surrounding medium (Sakurai, Goossens, and Hollweg, 1991, Paper II). Equations (33)–(35) and jump relations (56)–(58) can be used for interpretation of numerical results on Alfvén wave heating in dissipative MHD of coronal loops by Poedts, Kerner, and Goossens (1989) and Poedts, Goossens, and Kerner (1990) (see, e.g., Goossens, 1991).

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Appendix. Derivation of Equation (55)

We consider Equation (53), namely

\[
\frac{d^2y}{dx^2} + ixy = i\varepsilon .
\]  

(A.1)

This inhomogeneous differential equation can be solved by the standard Green’s function method. If the Green’s function \(G(x, x_0)\) satisfying

\[
\frac{d^2G}{dx^2} + iG = \delta(x - x_0)
\]  

(A.2)

is found, then the solution is written as

\[
y = \int_{-\infty}^{\infty} i\varepsilon G(x, x_0) \, dx_0 .
\]  

(A.3)

The Green’s function \(G\) can be constructed from the solutions to the homogeneous equation

\[
\frac{d^2y}{dx^2} + ixy = 0 ,
\]  

(A.4)

which is satisfied by

\[
y = x^{1/2} Z_{1/2} \left( \frac{2}{3} i^{1/2} x^{3/2} \right) ,
\]  

(A.5)

where \(Z\) stands for any cylindrical function (Watson, 1922, p. 97). The boundary condition requires \(G(x, x_0) \to 0\) as \(|x| \to \infty\). Therefore, we obtain

\[
G(x > x_0) = G_+ x^{1/2} H^{(1)}_{1/3} \left( \frac{2}{3} i^{1/2} x^{3/2} \right) \quad (\to 0 \text{ as } x \to \infty) ,
\]  

(A.6)

\[
G(x < x_0) = G_- x^{1/2} H^{(2)}_{1/3} \left( \frac{2}{3} i^{1/2} x^{3/2} \right) \quad (\to 0 \text{ as } x \to -\infty) .
\]  

(A.7)

Further, \(G\) should be continuous and \(dG/dx\) should have a unit jump at \(x = x_0\). These conditions are satisfied if

\[
G_+ = -\frac{\pi i}{6} x_0^{1/2} H^{(2)}_{1/3} \left( \frac{2}{3} i^{1/2} x_0^{3/2} \right) ,
\]  

(A.8)
\[ G_+ = -\frac{\pi i}{6} x_0^{1/2} H_{1/3}^{(1)} \left(\frac{2}{3} i^{1/2} x_0^{3/2}\right). \]  

(A.9)

Therefore the solution to Equation (A1) is

\[ \frac{y}{\varepsilon} = -\int_{-\infty}^{\infty} \frac{\pi}{6} x_0^{1/2} x^{1/2} H_{1/3}^{(1)} \left(\frac{2}{3} i^{1/2} x^{3/2}\right) H_{1/3}^{(2)} \left(\frac{2}{3} i^{1/2} x_0^{3/2}\right) \, dx_0 + \]

\[ + \int_{x}^{\infty} \frac{\pi}{6} x_0^{1/2} x^{1/2} H_{1/3}^{(1)} \left(\frac{2}{3} i^{1/2} x^{3/2}\right) H_{1/3}^{(2)} \left(\frac{2}{3} i^{1/2} x_0^{3/2}\right) \, dx_0. \]  

(A.10)

Next we will evaluate the integral \( \int_{-\infty}^{\infty} y \, dx \). The double integral (with respect to \( x \) and \( x_0 \)) can be converted to the integral in cylindrical coordinates \( (r, \theta) \) defined by

\[ r \cos \theta = \frac{2}{3} x_0^{3/2}, \quad r \sin \theta = \frac{2}{3} x^{3/2}. \]  

(A.11)

We obtain

\[ \int_{-\infty}^{\infty} \frac{y}{\varepsilon} \, dx = \frac{\pi}{3} \int_{\pi/4}^{5\pi/4} d\theta \int_{0}^{\infty} H_{1/3}^{(1)} \left(i^{1/2} r \sin \theta\right) H_{1/3}^{(2)} \left(i^{1/2} r \cos \theta\right) r \, dr. \]  

(A.12)

The integration with respect to \( r \) can be carried out (Watson, 1922, p. 134) and gives

\[ \int_{-\infty}^{\infty} \frac{y}{\varepsilon} = \int_{\pi/4}^{5\pi/4} d\theta \left[ \frac{r}{i \left(\sin^2 \theta - \cos^2 \theta\right)} \times \right. \]

\[ \left. \times \left\{ i^{1/2} \cos \theta H_{1/3}^{(1)} \left(i^{1/2} r \sin \theta\right) H_{1/3}^{(1)} \left(i^{1/2} r \cos \theta\right) - \right. \right. \]

\[ \left. \left. - i^{1/2} \sin \theta H_{1/3}^{(1)} \left(i^{1/2} r \sin \theta\right) H_{2/3}^{(1)} \left(i^{1/2} r \cos \theta\right) \right\} \right]_{0}^{\infty}. \]  

(A.13)

The contribution from \( r \to 0 \) is evaluated by using a power series expansion of the Hankel functions near the origin. We find

\[ \text{contribution from } r \to 0 = \]

\[ \frac{2i}{3} \int_{\pi/4}^{5\pi/4} d\theta \left[ \frac{\cos \theta}{\sin^2 \theta - \cos^2 \theta} \times \right. \]

\[ \left. \times \left[ \left(i + \frac{1}{\sqrt{3}}\right) \left(\frac{\cos \theta}{\sin \theta}\right)^{2/3} + \left(i - \frac{1}{\sqrt{3}}\right) \left(\frac{\sin \theta}{\cos \theta}\right)^{2/3} \right] \right]. \]  

(A.14)

The real part diverges, but the imaginary part is equal to \( -4\pi i/3 \).
The contribution from \( r \to \infty \) is evaluated by an asymptotic expansion of the Hankel functions. We obtain

\[
\text{(contribution from } r \to \infty ) = \frac{2}{3} \int_{\pi/4}^{5\pi/4} \frac{d\theta}{\sin \theta - \cos \theta (\sin \theta \cos \theta)^{1/2}} \exp \left[ i^{3/2} r (\sin \theta - \cos \theta) \right].
\]

(A.15)

Finite contributions result only if \( \sin \theta \approx \cos \theta \), namely \( \theta \approx \pi/4 \) and \( \theta \approx 5\pi/4 \). Power series expansion near these values of \( \theta \) leads to the integrated value of \( \pi i/3 \). Therefore,

\[
\int_{-\infty}^{\infty} \text{Im}(y) \, dx = -\pi \varepsilon.
\]

(A.16)

References