Magnetic Pressure-Driven Jets from a Torus

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Abstract

We examine steady, wind-type magnetohydrodynamical (MHD) jets accelerated in a funnel formed along the rotational axis of a geometrically thick torus around a central object, incorporating the effects of mass and toroidal magnetic field injections from the torus. When the mass and magnetic field are injected from only the base of the jets (without injection from the funnel wall), the magnetic pressure of the toroidal magnetic fields accelerates the gas, whereas the magnetic tension decelerates it as long as the cross-sectional area of the jets increases. In the present case, the locations of the trans-magnetosonic points of jet flow are determined by the thermal properties of the flow and are independent of the magnetic properties. Trans-magnetosonic points shift to infinity in the cold limit, similar to the case of a spherical approximation, although this is not the general case. The terminal speed, \( v_\infty \), of MHD jets in the funnel becomes the order of \( \sqrt{\Phi^2/(4\pi M)} \) \(^{1/3} \), where \( M \) is the mass flux and \( \Phi \) is the transfer rate of the toroidal magnetic flux. When the mass and magnetic field injections from the funnel wall are taken into account, trans-magnetosonic points are located at larger distances than the scale length of mass injection. We
found that the injection of magnetic fields from the funnel wall enhances the acceleration of jets. In this case, $v_{\infty}$ is the order of $[\Phi_{\infty}^2/(4\pi M_{\infty})]^{1/3}$, where the mass flux and the magnetic flux are those at infinity.

Key words: Accretion tori; Active galactic nuclei; Astrophysical jets; Magnetohydrodynamics; Winds.

1. Introduction

In order to explain the origin of astrophysical jets, various theoretical models have been proposed [see Begelman et al. (1984) for reviews]. Among them, Lynden-Bell (1978) pointed out the possibility that jets are collimated and accelerated in a funnel formed along the rotational axis of tori around a central object.

The radiative acceleration of particles in the funnel was examined by Sikora and Wilson (1981), while it was demonstrated that the jets are accelerated hydrodynamically through transonic points like a stellar wind (Fukue 1982, 1983; Calvani and Nobili 1983). Along the line of the latter wind-type model in the funnel, several refinements have been made (Ferrari et al. 1984, 1985; Nobili et al. 1985; Lu 1986; Chakrabarti 1986; Fukue and Yamamoto 1986; Fukue 1987; Lu and Pineau 1990).

So far, the effect of the magnetic field on the acceleration of jets in the funnel has not been investigated, except by Fukue (1983), although there have been several studies on magnetohydrodynamical (MHD) jets from geometrically thin disks (Blandford and Payne 1982; Pudritz and Norman 1983, 1986; Uchida and Shibata 1985; Shibata and Uchida 1985, 1986; Lovelace et al. 1987, 1989; Maruyama and Fujimoto 1987; Sakurai 1987; Fukue 1990). In addition, for a flow confined within the funnel region, the gas (and magnetic fields) is injected continuously along the flow, not only at the center but also from the wall of the funnel. However, the influence of such injections was little studied (cf. Calvani and Nobili 1983).

As schematically illustrated in figure 1, it is expected that the gas in the funnel is accelerated by the magnetic pressure of toroidal magnetic fields as well as the thermal pressure of the gas.

In the present paper, we thus examine these effects of the magnetic fields and injections from the funnel wall on the acceleration of MHD jets in the funnel using a simple model.

In the next section, our assumptions and basic equations are presented. The effect of toroidal magnetic fields on jets from the torus is discussed in section 3. In section 4 the effect of gas and magnetic field injections from the funnel wall is examined. The final section is devoted to discussion. The basic equations for MHD flows with source terms are given in appendix 1 for the convenience of readers. In appendix 2, the terminal speed of the \textsuperscript{T}IHD funnel jets is briefly discussed.

2. Basic Equations

Let us consider a jet confined in a funnel formed along the rotational $(z)$ axis of a gaseous torus around a central object of mass $m$. The self-gravities of the torus gas and jet gas are neglected. We assume that the jet flow is steady, axisymmetric and one-dimensional along the $z$-axis.
The conservation of mass and magnetic flux are expressed as

\[ A \dot{\rho} v = \int A \dot{\rho} dz = \dot{M}(z), \]

\[ \sqrt{A} B_\varphi v = \int \sqrt{A} B_\varphi dz = \Phi(z), \]

where \( \rho \) is the density of the jet gas, \( v \) the jet velocity, and \( B_\varphi \) the toroidal magnetic field (see appendix 1). The cross-sectional area \( A(z) \) of the funnel, the mass injection rate \( \dot{\rho}(z) \), and the magnetic-flux injection rate \( \dot{B}_\varphi(z) \) are specified later in the present simple model. The mass flux rate \( \dot{M}(z) \) and the magnetic flux rate \( \Phi(z) \) are then given as a function of \( z \).

The equation of motion is

\[ v \frac{dv}{dz} + \frac{Gm}{(r^2 + z^2)^{3/2}} \left( r \frac{dr}{dz} + z \right) = -\frac{dP}{\rho dz} - \frac{B_\varphi dB_\varphi}{4\pi \rho dz} - \frac{B_\varphi^2}{4\pi \rho r dz} - \frac{\dot{\rho} v}{\rho}, \]

where \( P \) is the gas pressure and \( r \) the radius of the cross-section of the funnel (jet flow); \( r = (A/\pi)^{1/2} \). It should be noted that the gravitational force [the second term on the left-hand side of equation (3)] and the magnetic tension (the third term on the right-hand side) are both evaluated at the outer boundary of jets which is coincident with the funnel wall (cf. Fukue 1989; Fukue and Okada 1990; see also Fukue 1990). Hence, the term \( dr/dz \) arises. The final term on the right-hand side appears since we implicitly assumed that the injected gas has no net momentum (Holzer and Axford 1970; see also appendix 1).

Finally, a polytropic equation of gas is adopted:

\[ P/\rho^\gamma = K \text{ (constant)}. \]
It should be noted that the energy along the flow is not conserved since the mass and magnetic field are injected along the flow. Hence, the so-called Bernoulli equation is not used in the present analysis (see appendix 1).

Equations (1) to (4) are combined into a single ordinary differential equation of \( v \):

\[
\frac{dv}{dz} = \frac{v \left[ c_s^2 \frac{A'}{A} - \left( c_s^2 + v^2 \right) \frac{\rho'}{\rho v} - v_A^2 \frac{B_\varphi'}{B_\varphi v} - \frac{Gm}{(r^2 + z^2)^{3/2}} \right]}{v^2 - c_s^2 - v_A^2},
\]

(5)

where the prime denotes differentiation with respect to \( z \). The sound speed \( c_s \) and the Alfvén speed \( v_A \) are, respectively, defined by

\[
c_s^2 = \frac{\gamma P}{\rho} = K \gamma \left( \frac{\dot{M}}{Av} \right)^{\gamma^{-1}},
\]

(6)

\[
v_A^2 = \frac{B_\varphi^2}{4\pi \rho} = \frac{\Phi^2}{4\pi vM},
\]

(7)

where we use equations (1) and (2). In the following sections we consider the isothermal case where \( \gamma = 1 \).

Before specifying the cross section of the funnel and the injection rates, we should mention several properties of the present flow described by the wind equation (5) of the general form. The first term, expressing the geometrical expansion, in the numerator on the right-hand side of equation (5) includes \( c_s \) but not \( v_A \). The sound speed arises from the continuity equation (1). Similarly, from magnetic flux conservation (2) a similar term including the Alfvén speed appears, which is canceled out by the magnetic tension term. This happens because (for the present purpose) we assume that the magnetic flux of toroidal magnetic fields is proportional to \( \sqrt{A} \) like the magnetohydrodynamical wind from the disk under the spherical approximation (Maruyama and Fujimoto 1987). As a result, the Alfvén speed does not appear in the geometrical term and, thus, the critical points of the present flow shift to infinity in the cold limit. In general, however, this is not true (Fukue 1990).

The second and third terms in the numerator represent the source terms associated with the mass and magnetic flux injections. The apparent asymmetry between these two terms originates from the last term on the right-hand side of equation (3); that is, the injected mass reduces the momentum of the flow but the magnetic flux does not. Since we assume that the mass injected from the funnel wall brings no net momentum into the flow, the mass-injection term always suppresses the acceleration of the jet.

We now specify the cross section of the funnel and the mass and magnetic field injection rates. First, regarding the cross-sectional area \( A(z) \) of the funnel, we assume that the specific angular momentum \( L \) of a torus forming the funnel is spatially constant. Hence, in the present nonrelativistic treatment, the surface of the torus is given as an equipotential,

\[
\psi = -\frac{Gm}{(r^2 + z^2)^{1/2}} + \frac{L^2}{2r^2} = \psi_0 \text{ (constant)},
\]

(8)
where $\psi_0$ is the value of the effective potential at the surface of the torus and is assumed to be zero (Limber 1964). We thus obtain the surface of the torus or the cross-sectional area of the funnel as

$$A(z)/\pi = \frac{1 + (1 + 16G^2m^2L^2/L^4)^{1/2}}{8G^2m^2/L^4}.$$  \hspace{1cm} (9)

Regarding injection rates, we assume simple exponential forms:

$$\dot{A}\rho = (\dot{M}_\infty/\sigma_D)\exp[-(z - z_0)/\sigma_D],$$  \hspace{1cm} (10)

$$\sqrt{A}B_\varphi = (\Phi_\infty/\sigma_B)\exp[-(z - z_0)/\sigma_B],$$  \hspace{1cm} (11)

or

$$\dot{M}(z) = \dot{M}_\infty\{1 - \exp[-(z - z_0)/\sigma_D]\},$$  \hspace{1cm} (12)

$$\Phi(z) = \Phi_\infty\{1 - \exp[-(z - z_0)/\sigma_B]\}.$$  \hspace{1cm} (13)

Here the stagnation point $z_0$, the scale length of the mass injection region $\sigma_D$, the scale length of the magnetic flux injection region $\sigma_B$, the mass flux at infinity $\dot{M}_\infty$, and the magnetic flux at infinity $\Phi_\infty$ are assumed to be constants.

In what follows we use $L^2/(Gm)$ and $Gm/L$ as units of length and velocity, respectively. The wind equation (5) is then written in a dimensionless form:

$$\frac{dv}{dz} = \frac{v[c_s^2A'/A - (c_s^2 + v^2)Y_D - v_A^2Y_B - Y_z]}{v^2 - c_s^2 - v_A^2},$$  \hspace{1cm} (14)

where

$$A/(2\pi) = [(16z^2 + 41/2 + 1)/16,$$  \hspace{1cm} (15)

$$A'/(2\pi) = z/(16z^2 + 41/2),$$  \hspace{1cm} (16)

$$Y_D = \frac{\exp[-(z - z_0)/\sigma_D]}{\sigma_D\{1 - \exp[-(z - z_0)/\sigma_D]\}},$$  \hspace{1cm} (17)

$$Y_B = \frac{\exp[-(z - z_0)/\sigma_B]}{\sigma_B\{1 - \exp[-(z - z_0)/\sigma_B]\}},$$  \hspace{1cm} (18)

and

$$Y_z = \frac{A'/(2\pi) + z}{2[2A/(2\pi) + z^2]^{3/2}}.$$  \hspace{1cm} (19)

The sound speed is constant in the present case of an isothermal flow, while the Alfvén speed is

$$v_A^2 = \frac{f_A\{1 - \exp[-(z - z_0)/\sigma_B]\}^2}{v\{1 - \exp[-(z - z_0)/\sigma_D]\}}.$$  \hspace{1cm} (20)

Here $f_A$ is a nondimensional parameter defined as

$$f_A = ([L/(Gm)]^3)^{1/2}.$$  \hspace{1cm} (21)

The parameters of the present model are ultimately $z_0$, $\sigma_D$, $\sigma_B$, $c_s$, and $f_A$. We set $z_0 = 0$; that is, we do not consider the inflow inside the stagnation point (cf. Calvani and Nobili 1983).
3. MHD Funnel Jets without Source Terms

In this section we first consider the case where the mass and magnetic flux are injected only at the base of the jet flow (i.e., no mass injection along the flow; \( Y_D = Y_B = 0 \)). We also discuss the effects of the magnetic pressure and magnetic tension of the toroidal magnetic fields on the jet flow in the funnel. In this case, the zero of the numerator of equation (14) determines the location of critical points, while the zero of the denominator gives the jet speed there.

As already stated in the previous section, the location of the critical points depends only on the sound speed and is independent of the magnetic field in the present case, although this is not true in general (Fukue 1990). In figure 2, the position of the critical points, \( z_c \), is plotted as a function of the sound speed \( c_s \). In the cold limit of smaller \( c_s \), the critical points shift to infinity as \( z_c \sim 1/(2c_s^2) \). On the other hand, if the value of the sound speed is larger than unity in nondimensional units, the critical points disappear, since the gravitational force, evaluated at the funnel boundary, is not infinite at \( z = 0 \), similar to the case of disk winds (Fukue 1989). In such a hot case, the gas is supersonically flowing out, although the present one-dimensional approximation is, of course, violated near \( z_c \sim 0 \).

The flow velocity \( v_c \) at the critical points, on the other hand, depends on both \( c_s \) and \( f_A \). In figure 3, the flow velocity \( v_c \left[ = (c_s^2 + v_{Ac}^2)^{1/2} \right] \) and the Alfvén speed \( v_{Ac} \) at the critical points as well as the sound speed \( c_s \) (fixed as 0.5) are shown as a function of \( f_A \). For a sufficiently large \( f_A \), \( v_c \) approaches to \( f_A^{1/3} \) since \( v_{Ac}^2 = f_A / v_c \). Since the flow is assumed to be isothermal here, there is no terminal speed. In order to give a rough estimate of the speed at the exit of the funnel (located at about \( 10^4 - 10^5 \) in nondimensional units), the flow speed at \( z = 10^4 \) is also plotted in figure 3. A rough estimate of the terminal speed in the cold limit is discussed in appendix 2.
Fig. 3. Several characteristic speeds at the critical points as a function of the parameter $f_A$ describing the magnetic field strength. The solid curve, the dashed one, and the dotted horizontal line represent the flow speed $v_c$, the Alfvén speed $v_A$, and the sound speed $c_s$ (which is a constant parameter and fixed as $c_s = 0.5$), respectively. The flow speed at $z = 10^4$ is also shown by the solid curve for a comparison.

Fig. 4. Solutions for several values of $f_A$ in the case of no injections. The solid curves denote the flow velocity $v$, whereas the dashed ones indicate the sound speed of the fast mode $c = (c_s^2 + v_A^2)^{1/2}$. The sound speed is fixed as $c_s = 0.5$. In this case, the critical points are located at $z_c = 1.732$.

Solutions are shown for several values of $f_A$ in figure 4 as a function of $z$ from the center. In figure 4, the solid and the dashed curves denote the jet velocity $v$ and the propagation speed of the fast mode of MHD waves $c = (c_s^2 + v_A^2)^{1/2}$, respectively. The sound speed is fixed as $c_s = 0.5$. 

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Fig. 5. Positions \( z_c \) of the critical points as functions of the injection scale lengths \( \sigma_D \) and \( \sigma_B \) in the case including source terms. The parameters are fixed as \( c_s = 0.5 \) and \( f_A = 0.1 \).

4. MHD Funnel Jets with Source Terms

We now consider the effect of both mass and magnetic flux injections. When the sound speed is sufficiently larger than the Alfvén speed (i.e., \( f_A \ll 1 \)), the effect of magnetic fields can be ignored and the flow is reduced to thermally driven jets with mass injection (Calvani and Nobili 1983). On the other hand, in the cold limit of \( c_s^2 \ll v_{Ac}^2 \) (i.e., \( f_A \gg 1 \)), the critical points shift to infinity in the present model and no trans-magnetosonic (trans-fast) jets exist (cf. Fukue 1990). We thus concentrate our attention on warm jets, where \( c_s^2 \gtrsim v_{Ac}^2 \) (i.e., \( f_A \lesssim 1 \)).

First, the locations of the critical points and speeds there are also obtained from the condition that both the numerator and the denominator of equation (14) vanish simultaneously. In figure 5, for some fixed values of \( c_s (=0.5) \) and \( f_A (=0.1) \), the position \( z_c \) of the critical points is plotted as functions of \( \sigma_D \) and \( \sigma_B \).

For \( \sigma_D, \sigma_B \ll 1 \), position \( z_c \) is the same as that in the case discussed in the previous section in which the mass and magnetic flux injections take place at the center. For larger values of \( \sigma_D \) or \( \sigma_B \), the position strongly depends on the scale length \( \sigma_D \) of the mass injection, but is independent of \( \sigma_B \). This is because in the warm jets considered here the Alfvén speed is so small that the term related to magnetic injection in the numerator of equation (14) is insignificant. On the other hand, the term related to mass injection becomes the order of unity for larger \( \sigma_D \).

As a result, the critical points are always located a little bit outside the scale length \( \sigma_D \). Therefore, if mass injection occurs semi-homogeneously along the funnel wall, wind-type steady acceleration through the critical points cannot work and the gas is ejected from the exit of the funnel as a nonsteady jet. In other word, mass injection must take place deep inside the funnel in order for the supersonic steady flow to blow out from the funnel.

Although the locations of the critical points depend on \( \sigma_D \), the jet speed \( v_c \) there
Fig. 6. Typical solutions of the flow velocity $v$ for several values of $\sigma_D$ and $\sigma_B$ in the case with injections. The combinations of the values of $(\sigma_D, \sigma_B)$ are (0.01, 0.01) in (a), (0.01, 10) in (b), (10, 0.01) in (c), and (10, 10) in (d). The other parameters are fixed as $c_s = 0.5$ and $f_A = 0.1$. The short vertical bars represent the positions of critical points.

does not so strongly depend on $\sigma_D$ or $\sigma_B$. This can be understood from the energetic view point. That is, the thermal and the magnetic energies are converted into the kinetic energy of the bulk motion inside the critical point, which is located outside the main injection regions.

In figure 6, the jet speeds $v$ of typical solutions are shown as a function of $z$ for several combinations of the values of parameters $\sigma_D$ and $\sigma_B$. The other parameters are fixed as $c_s = 0.5$ and $f_A = 0.1$. The short vertical bars represent the positions of critical points. Since the values of $c_s$ and $f_A$ are fixed in figure 6, the behavior of the flow far beyond the critical points is similar, although this does not hold for flow inside the critical points. A rough estimation of the terminal speed in the cold limit is discussed in appendix 2.

In closing this section we should mention the acceleration via injection of magnetic flux, which does not appear explicitly. In the present analysis, we normalized the magnetic flux injection by $\Phi_\infty$, the transfer rate of the toroidal magnetic flux of the flow at infinity. In other words, we fix the total magnetic flux injected over the funnel wall. Hence, the behavior of the flow beyond the critical points becomes similar, irrespective of the injection scale lengths, $\sigma_D$ and $\sigma_B$, as stated. If, however, we fix the strength of the magnetic field at the flow base, the jet flow is accelerated by the magnetic field injected and the terminal speed increases with increasing $\sigma_B$. Namely, for the same condition at the central region of the funnel, the injection of magnetic fields from the funnel wall enhances the acceleration of MHD jets.
5. Discussion

In this paper we have examined the MHD acceleration of astrophysical jets in a funnel formed by an astrophysical torus around a central object, including both mass and magnetic flux injections.

When the mass and magnetic fields are injected only at the base of the jets, the position of the critical points of the flow is determined by the sound speed, and is independent of the magnitude of the magnetic flux, similar to MHD winds driven by magnetic pressure from the disk under the spherical approximation [Maruyama and Fujimoto (1987); see also related study by Lovelace et al. (1987, 1989) and Fukue (1990)]. This is because we assumed that the toroidal magnetic flux is proportional to $\sqrt{A}$, and evaluated the magnetic tension at the flow boundary. This nature, however, does not always hold, but rather naively depends on the streamline adopted [see Fukue (1990) for the formalism in the general curvilinear coordinates].

When there are both mass and magnetic flux injections along the jet flow, the injection of magnetic fields from the funnel wall enhances jet acceleration. The critical point is always located slightly outside the scale length of the mass injection, and its position does not depend much on the scale length of the magnetic flux injection. Hence, if the mass is injected semihomogeneously from the entire region along the funnel, the jet flow cannot be in a steady state, so that the gas is ejected from the exit of the funnel as a nonsteady jet.

Let us now briefly discuss the relation between our model and the previous MHD jet models, apart from the new effect of mass and magnetic flux injections. Since our model jets include only the toroidal magnetic field, a jet is accelerated only by the magnetic pressure force $\nabla (B_\varphi^2/8\pi)$ in a cold limit (Note that the magnetic field lines are perpendicular to the flow). Similar models have been developed by Maruyama and Fujimoto (1987), Lovelace et al. (1989), and Fukue (1990). The nonsteady MHD jet model (Shibata and Uchida 1985, 1986) have similar characteristics; i.e., acceleration is due to the magnetic pressure, although the centrifugal force also works to accelerate the jet in that model.

Mathematically speaking, our jet model may be classified as a special version of a steady MHD jet/wind formulated as a centrifugally driven jet/wind. Namely, when the Alfvén radius is very close to the center of the tori, the magnetic field lines are tightly wound up from very near the center, so that the jet in this case can be approximated to be a jet with pure toroidal fields. In a typical centrifugal MHD jet (Blandford and Payne 1982; Pudritz and Norman 1983, 1986; Sakurai 1987), however, it is often argued that the gas is accelerated by the centrifugal force along the magnetic field line in a corotating frame of the central object, because the gas flows along the magnetic field lines in such a frame. Hence, and initial acceleration parallel to $\mathbf{B}$ must be present to produce jets in the centrifugally driven jet if the gravitational acceleration (along $\mathbf{B}$) is larger than the centrifugal acceleration (along $\mathbf{B}$) at the surface of the central object. If the initial acceleration is due to thermal pressure, as in the solar wind, many properties of the jet sensitively depend on the thermodynamic state of gas at the central object (Pudritz and Norman 1986). One of the merits of our model is that it does not require an initial acceleration by thermal pressure and, thus, the properties of jets can be independent of the thermodynamic state of gas at
Fig. 7. Schematic picture of the cross-section of MHD jets in the funnel. A thick solid circle represents the funnel wall, whereas thin solid curves indicate the line of force of the magnetic fields. Crosses denote the reconnection points. The arrows represent the direction of the effective gravity near the funnel. See the text for details.

the center of the tori. This merit is common to the nonsteady MHD jet model by Shibata and Uchida (1986). There is, however, one apparent demerit in our model: a jet with pure toroidal magnetic fields is very unstable for both sausage (interchange) and kink MHD instabilities. In order to stabilize these instabilities, there should be a strong poloidal field near the $z$-axis.

In this paper we have assumed that the gas as well as the toroidal magnetic field are supplied from the surrounding torus. We now discuss the mechanism of injection. At the first stage, in the torus, from the seed magnetic field, the toroidal component of the magnetic field is generated by the differential rotation of the torus gas. The toroidal field becomes stronger and stronger and then escapes from the torus through, e.g., the Parker instability (Parker 1966; Horiuchi et al. 1988; Matsumoto et al. 1988). When the generation rate and the escape rate of the toroidal magnetic field are balanced, an MHD torus where the toroidal magnetic field plays an important role is formed (Okada et al. 1989).

Moreover, in such a quasi-steady state, the escape of the toroidal magnetic field takes place into the funnel region, since the direction of the effective gravity is outwards (into the torus) in the vicinity of the inner region of the torus where the centrifugal force dominates the gravitational force (figure 7). As schematically illustrated in figure 7, the magnetic loops (arches) will rise into the funnel region by the buoyancy force against the effective gravity. Since the magnetic reconnection (denoted by crosses in figure 7) occurs between adjacent loops (arches) of magnetic fields, global toroidal magnetic fields are formed again. In these processes, since most of the mass inside the loops slides down the loop onto the torus surface (cf. Parker 1979; Shibata et al. 1990), the injection rate is much larger for toroidal magnetic fields than for the gas.
Eventually, these escaping magnetic fields (or injected magnetic fields) will accelerate jets in the funnel described in the present paper. Even if the ejection of magnetic flux from the torus into the funnel is very transient (or explosive; e.g., Shibata et al. 1990), it is possible that the jet has the quasi-steady nature because the explosive magnetic flux ejection is expected to occur quasi-periodically (Shibata et al. 1990).

Finally, the physical quantities such as the density and the strength of magnetic fields are evaluated. From the continuity equation (1), at the exit of the funnel, the density of jet gas becomes \( \rho \sim \dot{M} / (A_\infty v_\infty) \). Since the height \( z \) is sufficiently larger than \( L^2/(Gm) \) at the exit of the funnel [which is located at about \( 10^4 - 10^5 L^2/(Gm) \)], the cross-sectional area \( A(z) \) given by equation (9) is approximately \( A_\infty \sim \pi L^2 z/(2Gm) \). Hence, the density at \( z \) becomes \( \rho \sim GmM_\infty / (\pi L^2 v_\infty z) \).

For a marginally bound torus, the specific angular momentum \( L \) is about \( 2r_g c = 4Gm/c \), where \( r_g = 2Gm/c^2 \) is the Schwarzschild radius of the central black hole, \( c \) being the light speed. Thus, the density at the exit of the funnel is roughly estimated as

\[
\rho \sim 3 \times 10^{-20} \left( \frac{m}{10^8 M_\odot} \right)^{-1} \left( \frac{\dot{M}_\infty}{0.01 M_\odot \text{ yr}^{-1}} \right)^{-2} \left( \frac{L}{2r_g c} \right)^{-1} \left( \frac{v_\infty}{c} \right)^{-1} \left( \frac{z}{1 \text{ ly}} \right)^{-1} \text{ g cm}^{-3}.
\]

(22)

Similarly, from the conservation of magnetic flux (2) and equation (21), the strength of magnetic fields \( B_\varphi \) at the exit of the funnel is roughly

\[
B_\varphi \sim 3f_A^{1/2} \left( \frac{m}{10^8 M_\odot} \right)^{-1/2} \left( \frac{\dot{M}_\infty}{0.01 M_\odot \text{ yr}^{-1}} \right)^{1/2} \left( \frac{L}{2r_g c} \right)^{-5/2} \left( \frac{v_\infty}{c} \right)^{-1} \left( \frac{z}{1 \text{ ly}} \right)^{-1/2} \text{ Gauss.}
\]

(23)

For a low-energy jet, where the specific angular momentum of the torus is sufficiently larger than the marginally bound value and the terminal speed is smaller than the light speed, equations (22) and (23) are also applicable. For example, when the typical rotation speed of a torus with a size of 1ly is \( 10^7 \text{ cm s}^{-1} \), \( L \) is about \( 10^{25} \text{ cm}^2 \text{ s}^{-1} \), while \( 2r_g c \) for a \( 10^6 M_\odot \) black hole is about \( 2 \times 10^{22} \text{ cm}^2 \text{ s}^{-1} \). Hence, if the jet speed \( v_\infty \) is about \( 10^7 \text{ cm s}^{-1} \), the typical strength of the magnetic fields is on the order of 10 milli-Gauss. This is consistent with the observational value of the magnetic field in Our Galactic Center (e.g., Aitken 1986).

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Appendix 1. Basic Equations and the Bernoulli Equation with Source Terms

In the text we consider the MHD flow with source terms. For the convenience of readers, we present here the basic equations describing such a flow in more general form [see Holzer and Axford (1970) under the spherical approximation].
The continuity equation along the flow with the source term is
\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = \dot{\rho}, \] (A1)
where \( \dot{\rho} \) is the mass injection rate and the other notations have their usual meanings. For a steady, axisymmetric flow along the z-axis, this equation (A1) is reduced to equation (1) in the text, after integrating it over the cross-section of the flow and further along the flow.

The induction equation, on the other hand, is written as
\[ \frac{\partial \mathbf{B}}{\partial t} = \text{rot}(\mathbf{v} \times \mathbf{B}) + \dot{\mathbf{B}}, \] (A2)
where \( \dot{\mathbf{B}} \) is the magnetic-flux injection rate. Integrating the azimuthal component of equation (A2) in the direction perpendicular to the flow, we obtain equation (2) in the text.

The equation of motion with source terms becomes
\[ \rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \text{grad}) \mathbf{v} \right] = \rho \mathbf{K} - \text{grad} P + \frac{1}{c} \mathbf{J} \times \mathbf{B} + \dot{\mathbf{q}} - \dot{\rho} \mathbf{v}, \] (A3)
where \( \mathbf{K} \) is the external force, the current \( \mathbf{J} \) is \( \mathbf{J} = (c/4\pi) \text{rot} \mathbf{B} \) in the MHD approximation, and \( \dot{\mathbf{q}} \) is the momentum-flux injection rate associating with mass injection. The final term \( \dot{\rho} \mathbf{v} \) on the right-hand side arises from the continuity equation (A1). In the text we assume that the momentum is not brought into the funnel flow from the funnel wall associated with mass injection. The external force exerted from the central object as well as the magnetic tension of toroidal magnetic fields is evaluated at the funnel wall. Under these assumptions, the component of the equation of motion (A3) along the direction of the flow is reduced to equation (3) [see also Fukue (1989), (1990), Fukue and Okada (1990)].

Finally, conservation of the total energy, \( E = \rho (U + v^2/2 + \phi) + B^2/8\pi \), is expressed as
\[ \frac{\partial}{\partial t} \left[ \rho \left( U + \frac{v^2}{2} + \phi \right) + \frac{B^2}{8\pi} \right] + \text{div} \left[ \rho \mathbf{v} \left( H + \frac{v^2}{2} + \phi \right) + \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} + \mathbf{F} \right] = \dot{E}, \] (A4)
where \( U = P/[(\gamma - 1)\rho] \) is the internal energy, \( H = U + P/\rho \) the enthalpy, and \( \phi \) the potential of the central object. The electric field in the term of the Poynting flux \( (c/4\pi) \mathbf{E} \times \mathbf{B} \) is written as \( \mathbf{E} = -\mathbf{v} \times \mathbf{B}/c \) in the ideal MHD approximation. The flux term \( \mathbf{F} \) is not considered in the text. The right-hand side of equation (A4) is the energy injection associated with the mass and magnetic flux injections.

If the flow is steady and axisymmetric and, thereby, one-dimensional along the z-axis, after integrating equation (A4) over the cross-section of the flow and along the flow, we formally obtain the "so-called" Bernoulli equation for the present case:
\[ A\rho v \left( \frac{v^2}{2} + \frac{\gamma - 1}{\gamma - 1} \frac{P}{\rho} + \phi + \frac{B^2}{4\pi \rho} \right) = \int A\dot{E}dz, \] (A5)
Appendix 2. Terminal Speed of MHD Funnel Jets

For the isothermal flow discussed in the text, the terminal speed of jets does not exist. Hence, we show the flow speed at \( z = 10^4 \) as a measure of the terminal speed. However, if the flow is adiabatic or sufficiently cold, there exists a terminal speed far beyond the critical point (i.e., beyond the injection region). In this appendix, we shall briefly estimate the terminal speed for such a case.

When the flow is sufficiently cold, the Bernoulli equation (A5) becomes

\[
\frac{v^2}{2} + v_A^2 + \phi = E, \tag{A6}
\]

where \( v_A \) is the Alfvén speed defined in equation (7) as

\[
v_A^2 = B_c^2 / (4\pi \rho) = \Phi^2 / (4\pi \dot{M} v), \tag{A7}
\]

and \( E \) is the Bernoulli “constant.”

If there are both mass and magnetic flux injections, the mass flux \( \dot{M} \) and the magnetic flux transfer rate \( \Phi \) as well as the Bernoulli “constant” \( E \) vary along the flow. In a region far beyond the injection region, however, (or in the entire region for the case without injections), they are constant.

In such a case, the terminal speed \( v_\infty \), if it exists, is determined by

\[
\frac{v_\infty^2}{2} + F/v_\infty = E, \tag{A8}
\]

where \( F = \Phi^2 / (4\pi \dot{M}) \) is constant. Hence, the condition that there should be a real positive root in this equation (A8) is

\[
\Phi^2 / (4\pi \dot{M}) < (2E/3)^{3/2}. \tag{A9}
\]

In particular, if \( F \) is very close to \((2E/3)^{3/2}\), we roughly have

\[
v_\infty = (2E/3)^{1/2} = [\Phi^2 / (4\pi \dot{M})]^{1/3} = f_A^{1/3}/(Gm/L). \tag{A10}
\]

This is consistent with the results obtained in section 3, where \( v_\infty \) approaches \( f_A^{1/3} \) for a sufficiently large \( f_A \) in the case without source terms.

References


