A NEW SOLAR CYCLE MODEL INCLUDING MERIDIONAL CIRCULATION

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ABSTRACT

We present a kinematic model for the solar cycle which includes not only the transport of magnetic flux by supergranular diffusion and a poleward bulk flow at the Sun's surface, but also the effects of turbulent diffusion and an equatorward "return flow" beneath the surface. As in the earlier models of Babcock and Leighton, the rotational shearing of a subsurface poloidal field generates toroidal flux that erupts at the surface in the form of bipolar magnetic regions (BMRs). However, such eruptions do not result in any net loss of toroidal flux from the Sun (as assumed by Babcock and Leighton); instead, the large-scale toroidal field is destroyed both by "unwinding" as the local poloidal field reverses its polarity, and by diffusion as the toroidal flux is transported equatorward by the subsurface flow and merged with its opposite hemisphere counterpart. The inclusion of meridional circulation allows stable oscillations of the magnetic field, accompanied by the equatorward progression of flux eruptions, to be achieved even in the absence of a radial gradient in the angular velocity. We describe an illustrative case in which a subsurface flow speed of order 1 m s\(^{-1}\) and subsurface diffusion rate of order 10 km\(^2\) s\(^{-1}\) yield 22 yr oscillations in qualitative agreement with observations.

Subject headings: hydromagnetics — Sun: activity — Sun: interior — Sun: magnetic fields — Sun: rotation

1. INTRODUCTION

In their seminal papers of the 1960s, Babcock (1961) and Leighton (1964, 1969) were able to construct a coherent picture of the solar cycle using newly available observations of the Sun's magnetic fields and surface motions. Despite the wealth of additional data acquired during the intervening 22 yr cycle and improved capabilities for numerical modeling, their work continues to underlie much of our current understanding of the solar cycle. Although Babcock's model was largely empirical, Leighton gave it a physical and quantitative foundation by introducing the concept of supergranular diffusion (1964) and performing detailed numerical calculations (1969).

The main features of Leighton's \(\alpha\omega\) dynamo may be summarized as follows:

1. A subsurface poloidal field is wound up by differential rotation, producing a toroidal field which erupts at the Sun's surface in the form of bipolar magnetic regions (BMRs). The eruption of BMRs destroys the subsurface toroidal flux.

2. The BMRs are assumed to emerge with their "leading" (in the direction of the Sun's rotation) poles located equatorward of their "trailing" poles, so that their dipole moments have the opposite sense to that of the original poloidal field. Nonstationary supergranular convection gives rise to an effective diffusion which transports net quantities of trailing-polarity flux poleward, thereby reversing the Sun's polar fields.

3. Stable oscillations of the magnetic field are found to require a radial gradient in the angular velocity; moreover, the rotation rate must increase inward in order to produce the observed equatorward progression of flux eruptions. If a purely latitudinal shear is assumed, the oscillations damp out unless the BMRs are assigned unrealistically large axial dipole moments.

Despite its ability to reproduce the general properties of the solar magnetic cycle, recent observational and theoretical results suggest a number of potential difficulties with Leighton's model:

1. As noted by Parker (1984), it is unlikely that any significant amount of net toroidal flux can be expelled from the Sun, since this would require BMRs to erupt in chains extending over 360° of longitude. The main effect of erupting and transporting flux at the solar surface is to rearrange the topology of the subsurface field, with the toroidal flux tubes continually reconnecting with each other and with the surface loops in such a way as to reduce the magnetic stresses (see Wang & Sheeley 1991). Thus, contrary to the assumption of both Babcock (1961) and Leighton (1969), the eruption of BMRs does not of itself reduce the toroidal field, which must be destroyed by other mechanisms.

2. There is now considerable observational evidence for the presence of a poleward flow of order 10 m s\(^{-1}\) on the Sun's surface (Duvall 1979; Howard & LaBonte 1981; Ulrich et al. 1988). Such a flow is required not only to explain the poleward surges of magnetic flux which are especially prominent just after sunspot maximum, but also to account for the strength and concentration of the polar fields and the areal sizes of polar coronal holes around sunspot minimum (see Wang, Nash, & Sheeley 1989a, b, and references therein). By conservation of mass, a return flow toward the equator should exist below the Sun. Leighton's model does not include such a meridional circulation.

3. Recent helioseismological measurements suggest that the radial gradient of angular velocity remains small in the convection zone (see, e.g., Duvall, Harvey, & Pomerantz 1986; Harvey 1988; Goode et al. 1991). Large radial shears may be present near ~0.7 \(R_\odot\) (where \(R_\odot\) is the Sun's radius), but their sense is such that the rotation rate decreases with depth at low latitudes, contrary to Leighton's prediction.

In this paper, we modify the model of Leighton (1969) in a way that addresses these inconsistencies. In particular, we no longer assume that BMR eruptions deplete the toroidal field, and we include the effects of both a poleward surface flow and...
an equatorward subsurface flow. For simplicity, we also suppose that the surface rotation profile extends to the depth where the toroidal field is generated, so that there is no radial shear in the present model. After describing our equations and assumptions in § 2, we present numerical simulations of the solar cycle in § 3. We then summarize our results in § 4.

2. The Model

2.1. Basic Equations

We employ spherical coordinates $(r, \theta, \phi)$, with radius $r$ measured from the Sun’s center, colatitude $\theta$ defined with respect to the rotation axis, and longitude $\phi$ increasing westward (in the direction of the solar rotation); the time coordinate will be denoted by $t$. The longitudinally averaged magnetic field, $\bar{B}(r, \theta, t)$, is assumed to obey an induction equation of the form

$$\frac{\partial \bar{B}}{\partial t} = \nabla \times \left[ v \times \bar{B} - \kappa \nabla \times \bar{B} \right] + \nabla \times \left[ \omega(r) \sin \theta \bar{e}_\phi + v_\theta(r, \theta, t) \bar{e}_\phi \times \bar{B} \right] - \nabla \times \left[ \kappa(r) \nabla \times \bar{B} \right] + S(\theta, \phi, t). \tag{1}$$

Here the angular rotation rate $\omega$ is taken to be a function of $\theta$ only, while the meridional flow speed $v_\theta$ depends on both $r$ and $\theta$; we also assume that the contribution of the radial component of the flow velocity can be neglected. The effective diffusion rate due to turbulent motions is denoted by $\kappa$ and is allowed to vary with radius. Finally, $S$ is a (divergence free) “source term” which will be used to describe the eruption of BMRs at the solar surface and whose form will be specified below.

We make the following assumptions about the radial variation of $\bar{B}, S, v_\theta, \kappa$:

1. The magnetic field vanishes inside the radius $r = R_0$, where $R_0$ is the solar radius.
2. The toroidal component of the field, $B_\phi$, is confined to the subsurface layer $R_0 < r < R_0 + h$, within which it is a function of $\theta$ and $t$ only. The thickness $h$ of the layer is small compared with $R_0$.
3. The meridional component of the field, $B_\theta$, vanishes at the solar surface $r = R_0$, where observations indicate that the large-scale field is predominantly radial (Howard & LaBonte 1981). In addition, assumption (2) implies that $|B| \ll |B_\phi|$ outside the layer $R_0 < r < R_0 + h$, since the toroidal field is generated from $B_\phi$.
4. The source term $S$ is purely radial at the solar surface. Because BMR eruptions do not reduce the net toroidal flux, we also set $S_\phi = 0$ for all $r$.
5. Within the subsurface layer $R_0 < r < R_0 + h$, the meridional flow velocity $v_\theta$ is a function of $\theta$ alone, antisymmetric about the equator, and directed equatorward in both hemispheres. At the surface $r = R_0$, $v_\theta$ is antisymmetric about the equator and directed poleward in both hemispheres.
6. The turbulent diffusion rate $\kappa$ has a constant value $\kappa_0$ within the subsurface layer $R_0 < r < R_0 + h$, but vanishes for $r \leq R_0$ (i.e., the interior region has perfect conductivity). At the surface $r = R_0$, we identify $\kappa_0$ with the effective diffusion rate due to supergranular motions, $\kappa_s$.

We now proceed to write separate, radially averaged equations for the three components of $\bar{B}$. For this purpose, it is convenient to replace $B_\phi(r, \theta, t)$ by the quantity $B_m(\theta, t)$, defined by

$$B_m(\theta, t) = \frac{\int_{R_0}^{R_0 + h} r B_\phi(r, \theta, t) \, dr}{\int_{R_0}^{R_0 + h} r \, dr}. \tag{2}$$

Since $B_m$ is assumed to be small in the region $R_0 + h \leq r \leq R_0$, $B_m$ essentially represents an average of $B_\phi$ taken over the subsurface layer $R_0 < r < R_0 + h$. Integrating the flux-conservation condition $\nabla \cdot B = 0$ from $r = R_0$ to $r = R_0 + h$, recalling that $B_m(r, \theta, t) = 0$ from assumption (1), and neglecting higher order terms in $h/R_0$, we obtain the following relation between $B_m(\theta, t)$ and the surface field $B_s(\theta, t)$:

$$B_m(\theta, t) = -\frac{R_0}{h R_0} \int_{0}^{\theta} B_s(\theta', t) \sin \theta' \, d\theta', \tag{3}$$

where the lower limit of the integral has been chosen so that $B_m(0, t) = 0$.

In order to obtain an equation for $B_m$, which vanishes outside the layer $R_0 < r < R_0 + h$, we multiply the $\phi$-component of equation (1) by $r$ and integrate from $r = R_0$ to $r = R_0 + h$, with the result

$$\frac{\partial B_m}{\partial t} = \frac{\partial}{\partial \theta} \left[ \frac{\omega}{R_0} \sin \theta \frac{\partial B_m}{\partial \theta} \right] + \frac{\kappa_0}{R_0} \frac{\partial}{\partial \theta} \left[ \frac{1}{\sin \theta} \frac{\partial B_m}{\partial \theta} \right] + S(\theta, t). \tag{4}$$

Here $v_\phi(\theta)$ and $\kappa_0$ represent the meridional flow velocity and diffusion rate within the subsurface layer, and we have neglected higher-order terms in $h/R_0$.

Finally, the evolution of the radial field at the solar surface, $B_s(\theta, t)$, is given by the $r$-component of equation (1): setting $B_m(R_0, \theta, t)$ and its radial derivative to zero in accordance with assumption (3), we obtain

$$\frac{\partial B_s}{\partial t} = -\frac{1}{R_0} \frac{\partial}{\partial \theta} \left[ \sin \theta v_\phi(\theta) B_m \right] + \frac{\kappa_0}{R_0} \frac{\partial}{\partial \theta} \left[ \frac{\sin \theta}{\sin \theta} \frac{\partial B_s}{\partial \theta} \right] + S(\theta, t). \tag{5}$$

Here $v_\phi(\theta)$ and $\kappa_0$ denote the meridional flow velocity and supergranular diffusion rate at the solar surface.

Following arguments given by Leighton (1969), we assume that the source term has the form

$$S(\theta, \phi, t) = \frac{\epsilon h R_0}{2 \pi R_0^2} \sin \theta \frac{\partial}{\partial \theta} \left( \frac{B_m a \sin \gamma}{\tau} \right). \tag{6}$$

Here $a$ is the linear pole separation of a BMR formed from the subsurface toroidal flux, $\gamma$ is an angle of tilt relative to the east-west line, $\tau$ is the time scale for the BMR to erupt, and $\epsilon$ is a numerical constant which is at most of order unity. This expression is obtained by representing a BMR with meridipole moment $eB_m h R_0 \Delta \phi(\sin \gamma)$ by a ring doublet with moment $(eB_m h R_0 \sin \theta \Delta \phi(\sin \gamma))$, and noting that neighboring ring doublets contribute flux of opposite signs at a given colatitude $\theta$ (see Leighton 1969). We have inserted the factor $\epsilon$ to allow for the possibility that only a small fraction of the underlying toroidal flux erupts in the form of a single BMR emerging on the time scale $\tau$ (in § 3.1 we find that $\epsilon \sim 0.05$). It should also be noted that, unlike Leighton (1969), we do not assume or require a “critical” value of $|B_m|$ below which $S = 0$.

Equations (3)–(6) are the basic equations of our kinematic dynamo. The essential differences between our formulation and that of Leighton (1969) are contained in equations (4) and (5). In equation (5) describing the transport of surface flux $B_m$ we have added meridional flow to the supergranular diffusion.
term introduced by Leighton. In equation (4) for the toroidal field \( B_\phi \), we have included not only a subsurface flow but also a turbulent diffusion term, both of which effects were omitted by Leighton. On the other hand, Leighton included in his \( B_\phi \) equation a term describing the loss of toroidal flux due to BMR eruptions. We have omitted this term because flux eruptions do not of themselves reduce the net toroidal field (Parker 1984; see also the discussion in Wang & Sheeley 1991).

### 2.2. Parameters of the Model

In order to solve for the evolution of the magnetic field, it is necessary to specify \( \omega(\theta), v_0(\theta), v_0, \kappa, \kappa_\phi, \gamma, R_s, \) and \( h \). Our assumptions about these quantities will now be discussed.

For simplicity, we neglect radial gradients in the rotation rate and adopt the empirical rotation profile of Newton & Nunn (1951): \( \omega(\theta) = 13.39 - 2.77 \cos^2 \theta \) deg day\(^{-1} \). The shearing rate appearing in equation (4) is then given by

\[
\frac{d\omega}{d\theta} = 5.54 \sin \theta \cos \theta \text{ deg day}^{-1},
\]

and has its maximum value at \( \theta = 45^\circ \).

Using the results described in Wang et al. (1989a), we adopt a supergranular diffusion rate of \( \kappa = 600 \text{ km}^2 \text{s}^{-1} \) and a poleward surface flow given by

\[
v_0(\theta) = \pm 10 \sin \theta |\cos \theta|^{0.01} \text{ m s}^{-1},
\]

for which the peak speed occurs 5:7 from the equator. These values of the surface transport parameters were inferred by modeling a number of observed properties of the Sun’s large-scale field, including the dispersal of active region flux, the structure and evolution of the polar fields, and the rotation of the photospheric coronal fields.

Following Leighton (1969), who in turn cited the sunspot group measurements of Brunner (1930), we assume that the mean BMR tilt angle varies with colatitude as

\[
\sin \gamma = 0.5 \cos \theta.
\]

This empirical relation is supported by a recent statistical study of some 2700 BMRs that erupted during sunspot cycle 21 (Wang & Sheeley 1989). As discussed in Wang & Sheeley (1991), both the magnitude and latitudinal dependence of the average tilt angle can be explained by the action of Coriolis forces on rising, laterally expanding toroidal flux loops. The self-consistent incorporation of such dynamical effects is beyond the scope of the present kinematic model.

We shall also suppose that the mean pole separation of a BMR is proportional to the strength of the toroidal field, and that its eruption time scale is inversely proportional to \( |B_\phi| \): thus

\[
a(\theta, t) = a_0 \frac{|B_\phi|}{B_0},
\]

\[
\tau(\theta, t) = \tau_0 \frac{B_0}{|B_\phi|},
\]

where \( a_0, \tau_0, \) and \( B_0 \) are constants independent of \( \theta \) and \( t \). The relation (10) is suggested by the observed tendency for the pole separation of a newly erupted BMR to scale almost linearly as its flux (Wang & Sheeley 1989). The latter study also showed that the mean pole separation peaks around sunspot maximum, which is qualitatively consistent with equation (10) and the evolution of \( B_\phi \) as calculated in the next section. The basis for equation (11) is the idea that magnetic buoyancy, with a characteristic time scale determined by the Alfvén speed, is responsible for lifting flux to the surface (Parker 1955). It should again be emphasized that, unlike Babcock (1961) and Leighton (1969), we do not need to postulate a critical value of \( |B_\phi| \) below which no eruptions can occur.

Substituting the relations (9)-(11) into equation (6), we may rewrite the source term as

\[
S(R_s, \theta, t) = \frac{1}{\sigma} \left( \frac{h R_s}{R_{s0}^2} \right) \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} B_\phi^2 \cos \theta,
\]

where

\[
\sigma = \frac{4\pi R_{s0} B_0^2 v_0}{\epsilon_0}.
\]

The parameter \( \sigma \), which has dimensions of \( G^2 \text{ yr} \), will be used to specify the rate at which flux erupts at the solar surface.

The toroidal and meridional components of the field are assumed to be concentrated into a layer of thickness \( h \), whose inner boundary is located at \( r = R_s \). We note that equations (3)-(5), with source term given by equation (12), are left unchanged by the set of transformations \( h \rightarrow h/\xi, B_\phi \rightarrow \xi B_\phi, B_\phi \rightarrow B_\phi, B^\prime \rightarrow B^\prime, \) and \( \sigma \rightarrow \xi^2 \sigma \). Thus, reducing \( h \) by a factor of \( \xi \) intensifies the subsurface fields \( B_\phi \) and \( B^\prime \) by a factor of \( \xi \) but leaves the surface field \( B \), unchanged, provided the eruption “rate coefficient,” proportional to \( h/\sigma \), is decreased by a factor of \( \xi^2 \). Likewise, the equations are invariant under the transformations \( R_s \rightarrow R_s/\xi, B_\phi \rightarrow B_\phi/\xi, B^\prime \rightarrow B^\prime/\xi, \) and \( \sigma \rightarrow \xi^2 \sigma \). For definiteness, we shall suppose that \( h = 0.1 R_s \) and \( R_s = 0.7 R_\odot \). Our choice of \( R_s \) is motivated by the helioseismological result (cited in §1) that the surface-like rotation profile (7) persists down to the base of the convection zone, located at \( r \sim 0.7 R_\odot \).

We shall assume that the subsurface meridional flow is directed equatorward in both hemispheres and that it has the general form

\[
v_\phi(\theta) = \pm v_0 \sin^q \theta |\cos \theta|^q,
\]

where \( v_0, p, \) and \( q \) are constants.

The quantities \( \sigma, \kappa, v_0, p, \) and \( q \), which determine the rates of flux eruption at the surface and of diffusion and flow in the subsurface layer, will henceforth be regarded as the free parameters of the model. In the next section, we present some illustrative solutions and discuss the general dependence of the model on these parameters.

### 3. Model Calculations

In the present investigation, we consider only configurations that are symmetric or antisymmetric about the equator, so that the computations may be restricted to the hemisphere \( 0^\circ \leq \theta \leq 90^\circ \). The boundary conditions on the magnetic field are then that \( \partial B_\phi/\partial \theta, B_\phi, \) and \( B^\prime \) vanish at the pole \( (\theta = 0^\circ) \), and that \( B_\phi, B^\prime, \) and \( \partial B_\phi/\partial \theta \) vanish at the equator \( (\theta = 90^\circ) \). For the initial configuration we take \( B(\theta, 0) = 12 G \cos^8 \theta, B_\phi(\theta, 0) = 0, \) and (from eq. [3]) \( B^\prime(\theta, 0) = -19 G (1 - \cos^9 \theta)/\sin \theta \). Our choice of \( B_\phi(\theta, 0) \) is similar to the observed surface flux distribution near sunspot minimum (see Svalgaard, Duvall, & Scherrer 1978; Wang et al. 1989b).
For computational purposes, it is convenient to define
\[ B_m(\theta, t) = B_m(\theta, t) \cos \theta \]
and to rewrite equations (3)-(5) in the form
\[
\frac{\partial B_m}{\partial t} = - \frac{v(\theta) \partial B_m}{R} + \frac{\kappa}{\sigma} \sin \theta \left( \frac{1}{\sin \theta} \frac{\partial B_m}{\partial \theta} \right) - \frac{1}{\sigma} B_m^2 \cos \theta;
\]
\[
\frac{\partial B_s}{\partial t} = \frac{d\omega}{d\theta} B_s - \frac{1}{\sigma} \frac{\partial}{\partial \theta} \left[ \frac{v(\theta) B_s}{R} \right] \cos \theta + \frac{\kappa_s}{\sigma} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( B_s \sin \theta \right);
\]
where expression (12) has been substituted for the source term \( S \). Since equations (15) and (16) do not involve the surface field \( B_s \), they can be numerically integrated to yield \( B_m \) and \( B_s \) independently of \( B_m \), and the latter can then be obtained from equation (17).

3.1. The “Reference Case”

As an illustrative example, we have taken \( \alpha = 3.5 \times 10^7 \text{ G}^2 \text{ yr}^{-1}, k_\alpha = 11 \text{ km}^2 \text{ s}^{-1}, \nu_0 = 1.3 \text{ m s}^{-1}, \) and \( \rho = q = 0.2 \). (The global time scales corresponding to the adopted subsurface transport rates are, for diffusion, \( \tau_{th} = R_\alpha/k_\alpha = 655 \text{ yr} \), and for flow, \( \tau_{th} = R_\alpha/\nu_0 = 12 \text{ yr} \).) This choice of parameters will be designated the “reference case.” Figure 1 shows \( B_m, B_s, \) and \( B_m \) as a function of latitude and time over a 66 yr interval beginning at \( t = 90 \text{ yr} \). (The behavior of the solutions in fact remained largely unchanged after \( t \approx 25 \text{ yr} \).) For comparison, Figure 2 displays the observed evolution of the longitudinally averaged surface field, as derived from magnetograph measurements taken during 1976-1990 at the Wilcox Solar Observatory. The data sequence has been transposed and plotted 3 times in succession as described in the figure legend.

The oscillations of the calculated surface flux distribution \( B_s \) (Fig. 1a) qualitatively reproduce the observed solar cycle behavior (Fig. 2), with the centroid of flux eruptions migrating from midlatitudes toward the equator and the polar field reversing its polarity approximately every 11 yr. The simulated “sunspot belt” fields reach their maximum (longitudinally averaged) strength of 3-4 G at latitudes of 15°-20° just prior to the time of polar field reversal. (The observed fields are slightly weaker, and their centroid is located at somewhat lower latitudes.) The erupted flux appears to have predominantly the leading polarity, even though the source term given by equation (12) vanishes when integrated over the hemisphere. This apparent imbalance arises because the higher latitude, trailing-polarity flux is partially canceled by the “old cycle” polar field as it spreads poleward. The peak values of 10-11 G that \( B_s(\theta, t) \) attains near the pole at sunspot minimum considerably exceed the strength of the low-latitude fields at sunspot maximum. The dominance of the polar field, which is evident in both the simulation and the observations, is due to the concentrating effect of the poleward surface flow.

Comparison of Figure 2 with Figure 1a shows that the observed polar field undergoes a more sudden reversal than the simulated one; moreover, the poleward surge of trailing-polarity flux that accompanies this reversal is only weakly present in the simulation. Previous studies suggest that the large poleward surges observed around sunspot maximum are accompanied by an increase in the meridional flow speed (see Howard & LaBonte 1981; Wang et al. 1989b, and references therein). Thus these discrepancies may be the result of our assumption that \( v(\theta) \) and \( v(\theta) \) are independent of time.
As indicated by Figure 1b, the centroid of the toroidal field distribution, like that of the erupted surface flux, migrates from midlatitudes toward the equator during each 11 yr cycle. The maximum toroidal field strengths of 1600–1700 G are attained midlatitudes toward the equator during each 11 yr cycle. The model. Thus BMRs with a given Hale orientation erupt in critical field strength for eruptions to occur in the present model. Thus BMRs with a given Hale orientation erupt in "extended" cycles that attain their maximum near the time of polar field reversal 11 yr later. Harvey, Harvey, & Martin (1975) and Martin & Harvey (1979) have provided observational evidence for an extended activity cycle involving small bipoles or "ephemeral regions," which make their first appearance at high latitudes several years before sunspot minimum.

The meridional component \( B_m \), like the surface field \( B_s \), changes sign first at midlatitudes and soon thereafter at the poles, as the erupted trailing-polarity flux spreads poleward by flow and diffusion; the reversal of \( B_m \) is completed at the equator ~2 yr after polar field reversal, as the eruptions of trailing-polarity flux progress to low latitudes (see Fig. 1c). The polarity reversal of \( B_s \) begins at high latitudes ~2 yr after polar field reversal and is completed near the equator ~2 yr before the next polar field reversal. A total of ~18 yr elapses between the initial appearance of toroidal field of a given sign at high latitudes and its final disappearance near the equator. This same field gives rise to BMRs with a corresponding east-west ("Hale") polarity orientation throughout its 18 yr existence, since there is no critical field strength for eruptions to occur in the present model. Thus BMRs with a given Hale orientation erupt in "extended" cycles that attain their maximum near the time of polar field reversal 11 yr later. Harvey, Harvey, & Martin (1975) and Martin & Harvey (1979) have provided observational evidence for an extended activity cycle involving small bipoles or "ephemeral regions," which make their first appearance at high latitudes several years before sunspot minimum.

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We now study how the evolution of the magnetic field depends on the choice of the parameters \( \sigma, \kappa_s, v_0, \rho, \) and \( q \).

The time scale for erupting the subsurface toroidal flux in the form of BMRs is controlled by the value of \( \sigma \). In general, we find that decreasing \( \sigma \) (while keeping the other parameters of the reference model fixed) shortens the cycle period while at the same time reducing the peak strength attained by the polar field. On the other hand, increasing \( \sigma \) prolongs the time between polar field reversals, weakens the erupted fields, and causes the oscillations to damp. These effects are illustrated in Figure 3, which shows the evolution of the surface field distribution \( B_s \) when \( \sigma = 5 \times 10^6 \) G\(^2\) yr (Fig. 3a) and \( \sigma = 1 \times 10^6 \) G\(^2\) yr (Fig. 3b).

The solutions are strongly dependent on the value of the turbulent diffusion rate \( \kappa_s \). For example, reducing \( \kappa_s \) from 11 to 7 km\(^2\) s\(^{-1}\) (with the other parameters of the reference model held fixed) shortened the interval between polar field reversals to ~6 yr and greatly increased the strength of the erupted flux and of the polar field. On the other hand, increasing \( \kappa_s \) to 15 km\(^2\) s\(^{-1}\) caused the magnetic field and its oscillations to gradually decay. The evolution of \( B_s \) for these two cases is shown in Figures 4a and 4b, respectively.

The solutions are also sensitive to the amplitude \( v_0 \) of the equatorward subsurface flow. As shown in Figure 5a, increasing \( v_0 \) by as little as 25% (from 1.3 to 1.6 m s\(^{-1}\)) with the other parameters fixed at their "reference" values) causes the oscillations to damp out after a number of cycles. Decreasing \( v_0 \) prolongs the interval between polar field reversals, when \( v_0 \) is reduced by 50% (to 0.6 m s\(^{-1}\)), the oscillations undergo a slow, progressive decay (Fig. 5b). These calculations suggest that
there is an optimum flow speed corresponding to a given value of the diffusion rate: if the equatorward flow is too fast or too slow, the toroidal field is unable to build up to a sufficient strength to sustain the oscillations.

As illustrated by Figure 6, stable solutions can be "recovered" by simultaneously changing the subsurface flow and diffusion rates. Here we have reduced $v_0$ from 1.3 to 0.5 m s\(^{-1}\) and $\kappa_h$ from 11 to 3 km\(^2\) s\(^{-1}\), leaving the remaining parameters fixed at their reference values. As in the simulation of Figure 1, the polar field reverses every 11 yr and attains a maximum strength of about 10 G. However, because of the smaller diffusion rate, both the subsurface and erupted fields are much stronger than before. Also, because of the slower transport of toroidal flux toward the equator, the eruption rate peaks well after the polar field reverses, contrary to observations (see Fig. 2).

The parameters $p$ and $q$ determine the latitudinal variation of the subsurface flow speed; our choice $p = q = 0.2$ implies a relatively uniform flow which attains its maximum speed at $\theta = 45^\circ$. By experimenting with different values of $p$ and $q$, we found that the solutions are quite sensitive to the assumed flow profile. Thus, if the equatorward velocity declines too rapidly between midlatitudes and the equator, the toroidal flux accumulates and the erupted fields are too strong. On the other hand, if the flow speed increases too rapidly toward the equator, too little flux is erupted and the solution damps out. A relatively flat profile was found to yield the best agreement with the observed surface flux evolution.

It is interesting to consider the effect of setting both $v_0$ and $\kappa_h$ to zero (leaving $\sigma$ fixed at its reference value). The resulting evolution of $B_s(\theta, t), B_a(\theta, t)$, and $B_m(\theta, t)$ during the interval $t = 0-88$ yr is shown in Figure 7. Each component still undergoes a series of polarity reversals, but the oscillations rapidly damp out. The initial oscillations are driven by the flux eruptions, which reverse the sign of $B_m$ and cause the local toroidal field to unwind. However, in the absence of a subsurface flow that steadily convects the toroidal flux to low latitudes, the centroid of the eruptions does not migrate equatorward rapidly enough to maintain the oscillations; in particular, the lag between the polarity reversals at high and low latitudes becomes ever longer. The introduction of subsurface diffusion would cause the oscillations to damp out even sooner. This simulation suggests that the "static" dynamo of Babcock (1961), in which the equatorward progression of sunspot activ-
ity is attributed solely to the latitudinal dependence of the shearing rate (given by eq. [7]), does not in fact give rise to self-sustaining oscillations.

In order to emphasize the role of surface transport in maintaining the oscillations, we display in Figure 8 the result of omitting supergranular diffusion and the poleward surface flow from the reference calculation of Figure 1. Since the erupted flux is no longer transported across latitudes, the polar field is unable to reverse its polarity, and eventually each latitude zone becomes permanently dominated by one polarity or the other.

In § 1 of the Appendix we discuss the limiting case in which all (surface and subsurface) latitudinal transport processes are omitted, leaving only the effect of rotational shearing and flux eruptions. We obtain closed-form solutions in which the three components of the field oscillate with a latitude dependent frequency; the oscillations are driven by the flux eruptions, which act to reverse the underlying meridional field and thus cause the toroidal component to unwind. The oscillation period depends on the shearing and eruption time scales and on the initial field distribution; it has its minimum value at midlatitudes and becomes infinitely long at the equator and the pole. In § 2 of the Appendix we perturb these solutions by introducing meridional flow and requiring the characteristic flow time scales to be long compared with an oscillation period. We find that an equatorward (subsurface or surface) flow acts to shorten the oscillation period at low latitudes but to lengthen it at high latitudes, consistent with the early evolution of the field seen in Figure 8, where the neutral lines slant steeply toward the equator at low latitudes but become nearly horizontal at high latitudes. On the other hand, a poleward flow has the opposite effect, shortening the oscillation period at high latitudes but increasing it at low latitudes, as suggested by Figure 7. We also show analytically that an equatorward subsurface flow and a poleward surface flow both tend to strengthen the toroidal field near the equator.

4. DISCUSSION

In the kinematic dynamo model presented in this paper, a number of processes combine to maintain the oscillations of the Sun’s magnetic field:

1. The latitudinal gradient in the rotation rate, which is steepest at midlatitudes, generates toroidal flux from the subsurface meridional field.

2. The toroidal flux erupts in the form of BMRs, whose axial tilts are assumed to obey the empirical relation \( \sin \gamma = 0.5 \cos \theta \) (the “\( \alpha \)-effect” in both the present model and that of Leighton 1969). The neutral line of the erupting flux is centered about the latitude where \(|B_\theta(\theta, t)|^3 \cos \theta\) has its maximum. As discussed in Wang & Sheeley (1991), the axial inclinations of the BMRs have their physical origin in the action of Coriolis forces on rising, laterally expanding flux loops.

3. The erupted flux acts to reverse the underlying meridional field, which in turn causes the toroidal field to unwind at that latitude (see Leighton 1964). As shown in earlier studies (see Wang et al. 1989b, and references therein), the reversal of the polar field is due to the combined action of the surface transport processes: the diffusive annihilation of leading-polarity flux around the equator leaves a surplus of trailing-polarity flux, which is carried to the poles and concentrated there by the surface flow.

4. A subsurface flow that steadily convects toroidal flux from midlatitudes toward the equator is required to maintain globally periodic oscillations. In the absence of such a flow, the centroid of flux eruptions drifts equatorward too slowly to reverse the low-latitude meridional field in a timely manner.

5. The toroidal field is intensified as it converges toward the equator. A small amount of turbulent diffusion is required to limit the growth of \( B_\theta \) by merging it with its opposite hemisphere counterpart.

The present model differs from those of Babcock (1961) and Leighton (1969) in a number of important respects. First, it incorporates meridional circulation, which is found to play an essential role in sustaining the magnetic field oscillations. Second, it does not require toroidal flux to be expelled from the Sun. Third, it does not assume a critical toroidal field strength below which no eruptions can occur; as a result, it is consistent with the concept of “extended,” overlapping activity cycles initiated by weak, high-latitude eruptions. Finally, unlike the model of Leighton (1969), it yields stable oscillations even in the absence of a radial gradient in the angular velocity.

On the basis of the picture presented here, it would appear that latitudinal transport processes occurring both at and below the Sun’s surface are essential for maintaining the solar cycle. We obtain qualitative agreement with observations by assuming a subsurface flow speed \( v_0 \sim 1 \text{ m s}^{-1} \) and a turbulent diffusion rate \( \kappa_\tau \sim 10 \text{ km}^2 \text{ s}^{-1} \), while assigning to the surface transport rates the values inferred from earlier modeling of the Sun’s large-scale field (\( v_\tau \sim 10 \text{ m s}^{-1}, \kappa_\tau \sim 600 \text{ km}^2 \text{ s}^{-1} \); see Wang et al. 1989a).

While we have considered here only the effect of a latitudinal gradient in the rotation rate, the model can in principle be extended to include radial gradients which may become significant at the base of the convection zone. A more realistic treatment would also allow for fluctuations in the flux eruption...
rate, longitudinal variations, asymmetries between the northern and southern hemispheres, and time-dependent flows. These effects will be explored in future studies.

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APPENDIX

A1. EVOLUTION OF THE FIELD IN THE ABSENCE OF LATITUDINAL TRANSPORT

It is instructive to consider how the magnetic field would evolve in the limiting case in which all latitudinal transport processes are omitted, leaving only the effect of differential rotation and flux eruptions. For this purpose, we again assume that the rotational shear is given by the Newton-Nunn formula

$$\frac{d\sigma}{d\theta} = \frac{1}{\tau_w} \sin \theta \cos \theta ,$$

(A1)

where $\tau_w = 0.028 \text{ yr}$. In addition, we generalize the source term (12) by allowing the eruption rate to scale as an arbitrary odd power $n$ of $B_\phi(\theta, t)$: thus

$$S(R_\odot, \theta, t) = \frac{1}{\sigma} \left( \frac{hR_\odot}{R_\odot^2} \right) \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (B_\phi^2 \cos \theta) ,$$

(A2)

where the constant $\sigma$ has dimensions of $G^{-1} \text{ yr}$. (This generalization will allow us to compare the nonlinear case $n = 3$ adopted in the text with the linear case $n = 1$.) The evolution of the field is then determined by the equations

$$\frac{\partial B_x}{\partial t} = - \frac{1}{\sigma} B_\phi \cos \theta ;$$

(A3)

$$\frac{\partial B_\phi}{\partial t} = \frac{1}{\tau_w} B_\phi \sin \theta \cos \theta ;$$

(A4)

$$B_x = - \left( \frac{hR_\odot}{R_\odot^2} \right) \frac{1}{\sin \theta} \frac{\partial B_x}{\partial \theta} ,$$

(A5)

where $B_\phi(\theta, t) \equiv B_{\phi}(\theta, t) \sin \theta$.

Multiplying both sides of equation (A3) by $B_\phi$ and then eliminating $B_\phi$ from the right-hand side using equation (A4), we obtain

$$\frac{\partial}{\partial t} \left[ B_\phi^2 + \frac{2}{(n+1)} \left( \frac{\tau_w}{\sigma} \right) B_\phi^{n+1} \right] = 0 .$$

(A6)

For simplicity, we assume that initially $B_\phi(\theta, 0) = 0$. Then equation (A6) may be integrated to yield

$$\left[ \frac{B_\phi(\theta, t)}{B_\phi(\theta, 0)} \right]^2 + \left[ \frac{B_\phi(\theta, t)}{B_\phi^{\text{max}}(\theta)} \right]^{n+1} = 1 ,$$

(A7)

where

$$B_\phi^{\text{max}}(\theta) = \left[ \left( \frac{n+1}{2} \right) \left( \frac{\sigma}{\tau_w} \right) B_\phi^2(\theta, 0) \sin \theta \right]^{1/(n+1)} \sin^{n/(n+1)} \theta \cos \theta .$$

(A8)

Equation (A7) shows that the fields are periodic at any given latitude. In particular, as $B_\phi(\theta, t)$ declines in magnitude from $B_\phi(\theta, 0)$ to 0, $B_\phi(\theta, t)$ increases from 0 to its maximum value, given by equation (A8).

Using equation (A7) to eliminate $B_\phi(\theta, t)$ from equation (A4), we obtain an integral equation for $B_\phi(\theta, t)$:

$$\Omega(\theta) t = \int_0^{\psi/2} \frac{dx}{(1 - x^{n+1})^{1/2}} ,$$

(A9)

where

$$\Omega(\theta) \equiv \frac{1}{\tau_w} B_\phi(\theta, 0) \sin \theta \cos \theta = \left( \frac{2}{n+1} \left[ \left( \frac{1}{\sigma \tau_w} \right) B_\phi^{-1}(\theta, 0) \right]^{1/(n+1)} \sin^{n/(n+1)} \theta \cos \theta \right) .$$

(A10)

At each colatitude $\theta$, equation (A9) expresses $\Omega t$ as a monotonic function of $B_\phi/B_\phi^{\text{max}}$ over the interval [0, 1]. The inverse function of $\Omega t$ exists over the interval [0, $\psi/2$], where

$$\frac{\psi}{2} = \int_0^{\gamma_1} \frac{dx}{(1 - x^{n+1})^{1/2}} ;$$

(A11)
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denoting this function by \( s(\theta t) \), we may write

\[
B_\phi(\theta, t) = B_\phi^{\text{max}}(\theta)s(\theta t) \tag{A12}
\]

Like the sine function to which it reduces when \( n = 1 \), \( s(\theta t) \) may be extended to cover the entire interval \([0, \infty)\) periodically. The latitude-dependent oscillation period, \( \tau(\theta) \), can be derived by recognizing that \( B_\phi(\theta, t) = B_\phi^{\text{max}}(\theta) \) when \( t = \tau(\theta)/4 \); thus, from equation (A11),

\[
\tau(\theta) = \frac{2\psi}{\Omega(\theta)} = \frac{4}{\Omega(\theta)} \int_0^1 \frac{dx}{(1 - x^{n+1})^{1/2}} . \tag{A13}
\]

When \( n = 1 \), \( \psi \) is equal to \( \pi \). For the case \( n = 3 \), numerical evaluation of the integral yields \( \psi = 2.62 \).

The meridional field \( B_\phi(\theta, t) \) may be expressed in terms of the function \( c(\Omega t) \), which is defined by

\[
c(y) = \frac{ds}{dy} \tag{A14}
\]

and reduces to the cosine function when \( n = 1 \). From equations (A9) and (A12), it is evident that \( c(\Omega t) \) also satisfies

\[
c(y) = \pm \left[ 1 - s^{n+1}(y) \right]^{1/2}; \tag{A15}
\]

\[
\frac{dc}{dy} = -\left( \frac{n + 1}{2} \right) s(y) . \tag{A16}
\]

By substituting equation (A12) into equation (A7) and using equation (A15), we deduce that \( B_\phi(\theta, t)/B_\phi(\theta, 0) = c(\Omega t) \). The surface field \( B_\phi(\theta, t) \) can then be expressed in terms of \( c(\Omega t) \) and \( s(\Omega t) \) using equation (A5) and the relation (A16).

In summary, the solutions to equations (A3)–(A5) may be written in the form

\[
B_m(\theta, t) = B_m(\theta, 0)c(\Omega t); \tag{A17}
\]

\[
B_\phi(\theta, t) = \left[ \left( \frac{n + 1}{2} \right) \frac{\sigma}{\tau_w} \right]^{1/(n+1)} B_m^{2(n+1)}(\theta/\sigma, 0) \sin^{3/(n+1)} \theta s(\Omega t); \tag{A18}
\]

\[
B_\phi(\theta, t) = B_\phi(\theta, 0)c(\Omega t) + \left( \frac{n + 1}{2} \right) \frac{hR_0}{R_0^2} B_m(\theta, 0) \sin^2(\Omega t) \frac{d\Omega}{\partial \theta} , \tag{A19}
\]

where \( \Omega = \Omega(\theta) \) is given by equation (A10). At each colatitude \( \theta \), the amplitude of \( B_m(\theta, t) \) is fixed by the initial condition \( B_m(\theta, 0) \), whereas that of \( B_\phi(\theta, t) \) depends also on the ratio \( \sigma/\tau_w \): the longer the time scale for eruption compared with that for rotational shearing, the larger the maximum strength attained by the toroidal field during each oscillation. It can also be seen that the surface field \( B_\phi(\theta, t) \) has a component whose amplitude grows linearly with time, whereas \( B_m \) and \( B_\psi \) oscillate with constant amplitude.

As noted above, \( s(\Omega t) \rightarrow \sin (\Omega t) \) and \( c(\Omega t) \rightarrow \cos (\Omega t) \) when the source term is linear in \( B_\phi \) (\( n = 1 \)). The behavior of \( s(\Omega t) \) for the nonlinear case \( n = 3 \), shown in Figure 9. In general, as \( n \) increases, \( s(\Omega t) \) takes on a more “saw-toothed” appearance, while \( c(\Omega t) \) increasingly resembles a square-wave. This behavior reflects the fact that as the eruption rate and \( B_\phi \)—whose secular component is proportional to \( s'(\Omega t) \) of equation (A19)—become more sharply peaked in time, \( B_\phi \) reverses its polarity more suddenly and \( B_\phi \) consequently undergoes a more abrupt transition from amplification to “unwinding.”

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![Figure 9](https://example.com/figure.png)

**Fig. 9.**—Behavior of the periodic functions \( s(\Omega t) \) (thick solid line), \( c(\Omega t) \) (thin solid line), and \( s^3(\Omega t) \) (dotted line) for the case of a source term proportional to \( B_\phi^2 \). Argument \( \Omega t \) is expressed in units of \( \psi = 2.62 \), where \( 2\psi/\Omega \) represents a full oscillation period.
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The oscillation period, $\tau(\theta) = 2\Psi/\Omega(\theta)$, becomes infinitely long at $\theta = 0$ and $\theta = \pi/2$, where $\Omega = 0$. The minimum period $\tau_{\text{min}}$ occurs at the colatitude $\theta_c$, where $d\Omega/d\theta = 0$. From equation (A19), we note that $\theta = \theta_c$ also coincides with the neutral line of the secularly growing contribution to $B_s(\theta, t)$. For the linear case $n = 1$, equation (A10) reduces to

$$\Omega(\theta) = \frac{1}{(\sigma \tau_0)^{1/2}} \sin^{1/2} \theta \cos \theta,$$

so that $\theta_c = 35.3^\circ$ and $\tau_{\text{min}} = 10.1(\sigma \tau_0)^{1/2}$. When $n > 1$, $\Omega(\theta)$ and $\theta_c$ depend on $B_s(\theta, 0)$, which is related to $B_s(\theta, 0)$ through equation (A5). Let us consider the case $n = 3$ and suppose in addition that $B_s(\theta, 0) = B_s^0 \cos^k \theta$. Then equation (A10) becomes

$$\Omega(\theta) = \frac{1}{2 \sigma \tau_0^{3/4}} \left[ \frac{1}{k+1} \left( \frac{R_s^2}{h R_k} \right) B_s^0 (1 - \cos^{k+1} \theta) \right]^{1/2} \sin^{3/4} \theta \cos \theta.$$  

For $k = 1, 8, \infty$, we find that $\theta_c = 52.9^\circ, 44.5^\circ, \text{and} 40.9^\circ$, respectively.

Finally, we illustrate the behavior of the solutions for the particular choice of parameters $n = 3, k = 8, B_s^0 = 12 \text{ G}, h/R = 0.1, R_k = 0.7 R_\odot, \text{and} \sigma = 3.5 \times 10^7 \text{ G}^2 \text{ yr}$. This case corresponds to that of the “reference model” of § 3.1 with all latitudinal transport processes omitted. Figure 10 displays the evolution of $B_s(\theta, t), B_m(\theta, t), \text{and} B_m(\theta, t)$ during the interval $t = 0-88 \text{ yr}$. As anticipated above, $B_s$ oscillates $90^\circ$ out of phase with $B_m$; the polarity reversals occur gradually for $B_s$ and abruptly for $B_m$, reflecting the behavior of the functions $s(\Omega t)$ and $c(\Omega t)$, respectively. Except near $t = 0$ (when the initial field still dominates over the secularly growing component in eq. [A19]), the surface field $B_s$ vanishes near $\theta_c = 44.5^\circ$, where the period has its minimum value $\tau_{\text{min}} = 14.2 \text{ yr}$. Because of the latitudinal variation of the period, each component of the field “winds up” into an increasing number of latitudinal polarity bands, with a new pair of bands forming after the lapse of every $\Delta t = \tau_{\text{min}}/2$.

A2. INCLUSION OF MERIDIONAL FLOW: LINEAR PERTURBATION ANALYSIS

Adopting a perturbation approach, we now consider how the solutions obtained above are modified when a slow meridional circulation is introduced. We specialize to the case of a linear source term (setting $n = 1$ in eq. [A2]) and express the subsurface and surface flow velocities in the general form

$$v_s(\theta) = \frac{R_k}{\tau_{fs}} f_s(\theta),$$

$$v_s(\theta) = -\frac{R_s}{\tau_{fs}} f_s(\theta).$$
Here \( \tau_{fh} \) and \( \tau_{fs} \) denote the global time scales for the subsurface and surface flows, respectively; the functions \( f_\delta(\theta) \) and \( f_\phi(\theta) \) are assumed to be positive for \( 0 < \theta < \pi/2 \) and to vanish at \( \theta = 0 \) and \( \theta = \pi/2 \); the minus sign in equation (A23) indicates that the surface flow is directed poleward. In place of equations (A3) and (A4) we now have

\[
\frac{\partial B_\phi}{\partial t} = - \frac{1}{\tau_{fs}} B_\phi \cos \theta + \frac{1}{\tau_{fh}} f_\phi(\theta) \frac{\partial B_\phi}{\partial \theta},
\]

(A24)

\[
\frac{\partial B_\phi}{\partial t} = \frac{1}{\tau_w} \sin \theta \cos \theta - \frac{1}{\tau_{fh} \tau_{fs}} \frac{\partial}{\partial \theta} \left[ f_\phi(\theta) B_\phi \right].
\]

(A25)

Equations (A24) and (A25) may be combined in the form

\[
\frac{\partial^2 B_\phi}{\partial t^2} + \Omega^2(\theta)B_\phi = \left( \frac{f_\phi(\theta)}{\tau_{fs}} - \frac{f_\phi(\theta)}{\tau_{fh}} \right) \frac{\partial^2 B_\phi}{\partial t \partial \theta} - \cos \theta \frac{d}{d \theta} \left[ \frac{f_\phi(\theta)}{\cos \theta} \frac{\partial B_\phi}{\partial \theta} \right] + \cos \theta \frac{\partial}{\partial \theta} \left[ f_\phi(\theta) f_\phi(\theta) \frac{\partial B_\phi}{\partial \theta} \cos \theta \frac{\partial}{\partial \theta} \right],
\]

(A26)

where \( \Omega(\theta) \) is given by equation (A20). We now suppose that the right-hand side of this equation is small compared with either term on the left-hand side, and write

\[
B_\phi(\theta, t) = B_\phi^{(0)}(\theta, t) + B_\phi^{(1)}(\theta, t) = B_\phi^{(0)}(\theta, 0) \cos (\Omega t) + B_\phi^{(1)}(\theta, t).
\]

(A27)

Here \( B_\phi^{(0)} \) is the solution to equation (A26) when the flow terms are set to zero, while \( B_\phi^{(1)} \) represents a first-order correction due to the presence of flow. (The other components of the field may be expanded in a similar way.) Substitution of equation (A27) into (A26) yields

\[
\frac{\partial^2 B_\phi^{(1)}}{\partial t^2} + \Omega^2(\theta)B_\phi^{(1)} = \left[ \frac{f_\phi(\theta)}{\tau_{fs}} - \frac{f_\phi(\theta)}{\tau_{fh}} \right] \frac{\partial^2 B_\phi^{(0)}}{\partial t \partial \theta} - \cos \theta \frac{d}{d \theta} \left[ \frac{f_\phi(\theta)}{\cos \theta} \frac{\partial B_\phi^{(0)}}{\partial \theta} \right] + \cos \theta \frac{\partial}{\partial \theta} \left[ f_\phi(\theta) f_\phi(\theta) \frac{\partial B_\phi^{(0)}}{\partial \theta} \cos \theta \frac{\partial}{\partial \theta} \right],
\]

(A28)

where only the lowest order contributions to the right-hand side have been retained.

The solution to equation (A28) may be expressed as

\[
B_\phi^{(1)}(\theta, t) = B_\phi^{(0)}(\theta, 0) \left[ F(\theta) t^2 \sin (\Omega t) - G(\theta) t \cos (\Omega t) + \delta(\theta) \sin (\Omega t) \right],
\]

(A29)

where

\[
F(\theta) = \frac{1}{4} \left[ \frac{f_\phi(\theta)}{\tau_{fs}} - \frac{f_\phi(\theta)}{\tau_{fh}} \right] \frac{d \Omega}{d \theta};
\]

(A30)

\[
G(\theta) = \frac{1}{4} \left[ \frac{f_\phi(\theta)}{\tau_{fh}} - \frac{f_\phi(\theta)}{\tau_{fs}} \right] \frac{d \Omega}{d \theta} + \frac{1}{2} \frac{f_\phi(\theta)}{\tau_{fs}} B_\phi(\theta, 0) \frac{d B_\phi(\theta, 0)}{d \theta} + \frac{1}{2} \frac{f_\phi(\theta)}{\tau_{fs}} \cos \theta \frac{d}{d \theta} \left[ f_\phi(\theta) \right] \frac{d B_\phi(\theta, 0)}{d \theta};
\]

(A31)

\[
\delta(\theta) = \frac{1}{\Omega(\theta)} \left[ G(\theta) + \frac{f_\phi(\theta)}{\tau_{fs}} B_\phi(\theta, 0) \frac{d B_\phi(\theta, 0)}{d \theta} \right].
\]

(A32)

In determining the coefficients of the solution, we have required that both \( B_\phi^{(1)}(\theta, t) \) and \( B_\phi(\theta, t) \) (given to first order through eq. [A24]) vanish at \( t = 0 \).

The zeroth- and first-order contributions to \( B_\phi(\theta, t) \) may be combined into an expression of the form

\[
B_\phi(\theta, t) \approx B_\phi(\theta, 0) \exp \left[ - G(\theta) t \cos (\Omega t) - (F(\theta) t - \delta(\theta)) \right].
\]

(A33)

This equation, and the perturbation method used to derive it, is valid provided \( \Omega(\theta) \tau_{fh} \) and \( \Omega(\theta) \tau_{fs} \) are much greater than \( 1, \Omega(\theta) t \), and \( [\Omega(\theta) t]^2 \): thus the flow must be sufficiently slow that a large number of oscillations occur during the global flow time scales \( \tau_{fh} \) and \( \tau_{fs} \).

Equation (A24) may be used to deduce \( B_\phi \) from \( B_\phi = B_\phi^{(0)} + B_\phi^{(1)} \): we obtain

\[
B_\phi(\theta, t) \approx \left( \frac{\sigma}{\tau_w} \right)^{1/2} \sin^{1/2} \theta B_\phi(\theta, 0) \exp \left[ - G(\theta) t - \frac{1}{2} \left( \frac{f_\phi(\theta)}{\tau_{fh}} + \frac{f_\phi(\theta)}{\tau_{fs}} \right) \right] \frac{d \Omega}{d \theta} t \sin \left[ \Omega(\theta) t - (F(\theta) t - \delta(\theta)) \right].
\]

(A34)

The surface field \( B(\theta, t) \) can be evaluated in a similar manner from equation (A5).

From the form of these solutions, we conclude that "switching on" meridional flow alters the oscillation frequency: thus

\[
\Omega(\theta) \rightarrow \Omega(\theta) - F(\theta) t = \Omega(\theta) - \frac{1}{4} \frac{f_\phi(\theta)}{\tau_{fh}} \frac{f_\phi(\theta)}{\tau_{fs}} \frac{d \Omega}{d \theta} t.
\]

(A35)

As an example, let us suppose that \( f_\phi(\theta) = 0 \), so that only the equatorward subsurface flow is present. Then the frequency shift has the opposite sign to that of \( d \Omega/d \theta \). Since \( \Omega(\theta) \), given by equation (A20), attains its maximum value at \( \theta = \theta_c = 35/3 \), the oscillation frequency decreases with time for \( \theta < \theta_c \) (where \( d \Omega/d \theta > 0 \)), remains unchanged at \( \theta = \theta_c \) (where \( d \Omega/d \theta = 0 \)), and increases with time for \( \theta > \theta_c \) (where \( d \Omega/d \theta < 0 \)). The result is that \( \Omega(\theta) \rightarrow \Omega(\theta) \) at low latitudes but \( \Omega(\theta) \rightarrow 0 \) at high latitudes. On the other hand, if
we now include a poleward surface flow with $|v_s| \geq v_h$ for all $\theta$, we obtain the opposite result: the oscillation frequency increases poleward of $\theta = \theta_0$ but decreases equatorward of it.

The introduction of meridional flow also changes the amplitude of the oscillations at each latitude. According to equation (A34), $B_\theta(\theta, t)$ grows or decays exponentially depending on whether the quantity

$$
\gamma(\theta) \equiv -G(\theta) - \frac{1}{2} \left[ \frac{f_h(\theta)}{\tau_{fh}} + \frac{f_s(\theta)}{\tau_{fs}} \right] \frac{1}{\Omega(\theta)} \frac{d\Omega}{d\theta}
$$

is positive or negative, respectively. Near the equator, where $(1/\Omega)d\Omega/d\theta$ is large and negative but $(1/B_\theta)dB_\theta/d\theta$ is small, an equatorward subsurface flow and a poleward surface flow both yield positive contributions to $\gamma$, and thus act to increase the amplitude of $B_\theta$. The growth rate is further enhanced if the subsurface flow speed declines more steeply than $\cos \theta$ toward the equator, since this makes the term proportional to $d(f_s/\cos \theta)/d\theta$ positive. Near the pole, where $(1/\Omega)d\Omega/d\theta$, $(1/B_\theta)dB_\theta/d\theta$, and $d(f_s/\cos \theta)/d\theta$ are all positive, an equatorward subsurface flow tends to reduce $|B_\theta|$, while the effect of a poleward surface flow depends on the form of the initial field. If we assume that $B_\theta(\theta, 0) = B_{\theta 0} \cos^k \theta$, then the positive surface-flow term proportional to $(1/B_\theta)dB_\theta/d\theta \sim 2/\theta$ dominates over the negative contribution proportional to $(1/\Omega)d\Omega/d\theta \sim 1/(2\theta)$, and the poleward flow acts to increase the amplitude of $B_\theta$ near $\theta = 0$.

By similar reasoning, it can be shown from equation (A33) that an equatorward subsurface flow increases $|B_\theta|$ near the equator but reduces it near the pole, whereas a poleward surface flow has the opposite effect. While it might at first seem surprising that a poleward surface flow should both weaken $B_\theta$ and strengthen $B_\theta$ near the equator, we recall that such a flow also acts to increase the oscillation period at low latitudes: the longer period allows $B_\theta$ more time to become amplified by differential rotation before $B_\theta$ changes sign. This effect more than offsets the reduction in the amplitude of $B_\theta$, which in turn is due to the greater amount of flux erupted at low latitudes.

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