Nonmagnetic polarization of the Doppler cores of Fraunhofer resonance lines

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The peaks of linear polarization observed in the Doppler cores of resonant Fraunhofer lines near the solar limb are investigated. The polarization is due to resonant scattering. The main assumptions are as follows: a two-level atom, complete frequency redistribution, and a semi-infinite isothermal atmosphere. It is shown that a well-known result of the scalar theory of line formation, the so-called \( \sqrt{1-\lambda} \) law, where \( \lambda \) is the probability of survival of a photon in scattering, carries over to the vector case. A vector generalization of Rybicki’s quadratic integrals of the radiative transfer equation is also given. The polarization of radiation at the limb is expressed in terms of the components of the vector source function for \( \tau = 0 \), i.e., at the boundary of the atmosphere. In the limiting case of conservative scattering (\( \lambda = 1 \)), the degree of polarization at the limb is 9.5% for the Doppler core of a line.

1. INTRODUCTION

Profiles of the polarization of a number of Fraunhofer resonance lines have recently been obtained from observations of the solar limb.\(^1\text{–}^5\) The typical pattern is given in Fig. 1 (from Ref. 6). It shows the degree of polarization (in \%) as a function of wavelength for the Ca I \( \lambda \) 4227 line in the spectrum of a section of the quiet solar atmosphere located 10” from the limb in the polar region. At the center of the line, within its Doppler core, there is a narrow peak of polarization, which will be the subject of our analysis. In addition, in the wings there are two more extended polarization maxima that are symmetric relative to the center of the line. We shall not consider them.

The height of the central polarization peak increases toward the limb. Because of the finite spectral resolution and scintillation of the image, however, the true height of the central peak and its limiting value at the limb cannot be obtained from observations. In the present paper we discuss the theoretical questions associated with this.

We are dealing here with plane polarization. It has a nonmagnetic nature and originates in resonant scattering. This polarization is related to that of radiation escaping from an electron-scattering atmosphere. It is produced essentially by the anisotropy of the radiation field near the boundary of the atmosphere. A specific feature for the frequencies of lines is that two effects must be taken into account simultaneously: a change in the state of polarization and frequency redistribution in scattering.

Theoretical investigations of polarization in lines with allowance for multiple scattering have, until very recently, been based exclusively on numerical solutions of the corresponding vector equation of radiative transfer (ERT).\(^7\text{–}^{16}\) A recent paper by Faurébert-Scholl and Frisch\(^17\) has initiated the use of analytical methods in this field. In particular, the asymptotic properties of fields of polarized radiation are analyzed in the approximation of complete frequency redistribution (CFR) (this approximation is applicable only to the cores of lines; see below).

Results that supplement those of Ref. 17 are obtained in the present paper. We find simple exact relationships that must be satisfied by the solutions of the vector ERT for the frequencies of lines with CFR. These are apparently the first exact (rather than asymptotic) results in this field of the theory of radiative transfer.

2. STATEMENT OF THE PROBLEM. EQUATION OF TRANSFER

We consider the standard model problem in the theory of line formation: to calculate the radiation field at the frequencies of a line formed in a homogeneous, plane, semi-infinite isothermal atmosphere consisting of two-level atoms. Both radiative and collisional transitions between levels are taken into account. Absorption in the continuum is unimportant. There is no magnetic field. The only difference from the usual statement of the problem (see, e.g., Ref. 18, Chap. 11, and Ref. 19, Chap. 6) is that the polarization originating in resonant scattering in the line is completely taken into account.

We shall assume that line broadening is due only to the Doppler effect. This limits the applicability of our results to the vicinity of the Doppler core. Another important approximation is the assumption that CFR occurs in scattering. A comparison of numerical solutions of the ERT obtained in the CFR approximation and without it\(^8\text{–}^{12}\text{,}^{13}\) showed that for line cores this approximation provides good accuracy, not only in intensity (which is well known) but also in polarization. (Strange as it may seem, the question of why this is so has not been raised with respect to polarization. See the author’s Ref. 20 for an explanation.)

The stated problem reduces to the solution of the vector ERT

\[
\mu \frac{\partial I(\tau, \mu, z)}{\partial \tau} = \phi(z) I(\tau, \mu, z) - \frac{\lambda}{2} \int dz' \int \tilde{R}(\mu, z; \mu', z') \\
\times I(\tau, \mu', z') d\mu' - \phi(z) (1-\lambda) B.
\]

(1)

Here \( I = (I_1, Q) \) is the two-component Stokes vector (\( T \) denotes matrix transposition), \( \tilde{R} \) is the phase–frequency matrix, which describes the redistribution of the radiation in direction, frequency, and polarization in resonant scattering, \( B = B(1, 0) \) is the Planck function (we assume that it does not depend on depth, since the atmosphere is isothermal: \( B = \text{const} \)), and \( \phi(z) \) is the...
normalized absorption profile \( (f\varphi(x')dx' = 1) \). For purely Doppler broadening we have \( \varphi(x) = \exp(-x^2/\sigma^2) \). The parameter \( \lambda \) is the probability of survival of a photon in scattering (\( \lambda = 1 \) for conservative scattering, i.e., in the absence of collisional deexcitation). Further, \( \tau \) is the optical depth averaged over the line (so that \( \phi(0) = \) the optical depth at the center of the line), \( \mu = \cos \theta \), where \( \theta \) is the angle between the ray and the outward normal to the layers, and finally, \( x \) is the frequency reckoned from the center of the line in units of Doppler width. Here and everywhere below, integration over \( x' \) runs from \(-\infty \) to \(+\infty \). The desired solution of Eq. (1) must satisfy the conditions

\[
\mathbf{I}(0, \mu, z) = 0, \quad \mu < 0; \quad \mathbf{I}(\tau, \mu, z) \to \mathbf{B}, \quad \tau \to \infty,
\]

the first of which expresses the absence of radiation incident from outside, and the second expresses the approach of the radiation field to equilibrium in deep layers.

In accordance with the assumption of CFR, we take

\[
\mathbf{R}_d(\mu, \mu'; z, z') = \varphi(\tau) \mathbf{J}(\mu, \mu') \mathbf{R}(\mu, \mu'),
\]

where \( \mathbf{R}(\mu, \mu') \) is the phase matrix of resonant scattering (Ref. 21, Secs. 17 and 19). It is conveniently represented in the form

\[
\mathbf{R}(\mu, \mu') = \mathbf{F} + \mathbf{F}_d(\mu, \mu'),
\]

where

\[
\mathbf{F} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{F}_d(\mu, \mu') = \frac{3}{8} \begin{pmatrix} \frac{1}{2} (1 - 3\mu^2)(1 - 3\mu'^2) & (1 - 3\mu^2)(1 - 3\mu'^2) \\ (1 - \mu^2)(1 - 3\mu'^2) & 3(1 - \mu^2)(1 - \mu'^2) \end{pmatrix}
\]

and \( W \) is the depolarization parameter. The matrix \( \mathbf{F} \) describes isotropic scattering without polarization and \( \mathbf{F} + \mathbf{F}_d = \mathbf{F} \) is the phase matrix for Rayleigh scattering. The depolarization parameter \( W, 0 \leq W \leq 1 \), is determined by the quantum numbers \( j \) for the lower \((j)\) and upper \((j')\) states (Ref. 21, Sec. 19; Ref. 23). Depolarization is greatest for the transition \( j_j = 0 \to j'_{\mu} = 1 \) (dipole scattering, \( W = 1 \)). An example of a line with \( W = 1 \) is the Ca I \( \lambda 4227 \) A resonant line — an observational standard, in a certain sense, for the polarization effects in question.

### 3. BASIC INTEGRAL EQUATION

The phase matrix can be factored in the form \( 24 \) (other factorings are also possible)

\[
\mathbf{R}(\mu, \mu') = \mathbf{A}(\mu) \mathbf{A}'(\mu'),
\]

where

\[
\mathbf{A}(\mu) = \begin{pmatrix} \frac{W}{8} & 0 \\ 0 & \frac{1 - 3\mu^2}{3(1 - \mu^2)} \end{pmatrix}.
\]

In accordance with (3) and (6), all the variables in matrix \( \mathbf{R} \) are separable, which enables us to greatly simplify the ERT.

Introducing (3) and (6) into the ERT (1) and using \( \mathbf{A}(\mu) = \mathbf{B} \), we obtain

\[
\frac{\partial \mathbf{I}(\tau, \mu, z)}{\partial \tau} = \varphi(\tau) \mathbf{I}(\tau, \mu, z) \mathbf{A}(\mu) \mathbf{S}(\tau),
\]

where

\[
\mathbf{S}(\tau) = \begin{pmatrix} \mathbf{A}(\mu') \mathbf{I}(\tau, \mu', z') + (1 - \lambda) \mathbf{B} \end{pmatrix}.
\]

Here \( \langle \rangle \) is the operator for averaging over direction and frequency (with a weight \( \phi \)), defined as follows:

\[
\langle \mathbf{I}(\mu', z') \rangle = \frac{1}{2} \int \phi(x')dx' \int \mathbf{I}(\mu', x')d\mu'.
\]

The vector \( \mathbf{S} \equiv (S_x S_y)^T \) is the vector source function. In many papers (Refs. 9, 13, 17, 25, and 26) \( \mathbf{P} = (3/8) \mathbf{S} \) is used instead of our \( \mathbf{S} \) (for \( W = 1 \)). A factoring of the phase matrix slightly different from (6)-(7) is actually performed (explicitly) in those papers. Other factorings are also widely used in problems of monochromatic scattering.

In the standard way, by substituting the formal solution (8) into (9), we arrive at the basic integral equation of the problem,

\[
\mathbf{S}(\tau) = \lambda \int \mathbf{K}(\tau - \tau') \mathbf{S}(\tau')d\tau' + (1 - \lambda) \mathbf{B},
\]

where the kernel matrix \( \mathbf{K} \) is defined to be

\[
\mathbf{K}(\tau) = \frac{1}{2} \int \varphi(x)dx \int \mathbf{I}(\mu, x')d\mu \mathbf{F}(\mu, \mu')d\mu'.
\]

and the characteristic matrix \( \mathbf{F} \) is

\[
\mathbf{F}(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & 7W/10 \end{pmatrix}.
\]

We note that

\[
\int \mathbf{K}(\tau)d\tau = \int \mathbf{F}(\mu)d\mu = \begin{pmatrix} 1 & 0 \\ 0 & 7W/10 \end{pmatrix}.
\]

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Equation (11) is the basic integral equation for the problem under consideration. This is the vector generalization of the usual integral equation for the source function of the scalar (without polarization) theory of line formation with CFR. We emphasize that (11) is a system of two Wiener–Hopf integral equations. In contrast to a single equation, there is no general method of solving such systems in explicit form. As we show below, however, one important property of the solution of the scalar analog of Eq. (11) carries over directly to the vector case.

4. ESCAPING RADIATION. POLARIZATION

AT THE LIMB

From the vector source function $S(\tau)$ we easily calculate the Stokes vector $I(0, \mu, x)$ of the escaping radiation, which yields both the intensity $I(0, \mu, x)$ of the escaping radiation and its degree of polarization

$$\delta(\mu, x) = -\frac{Q_0(0, \mu, x)}{I(0, \mu, x)}.$$  \hspace{1cm} (15)

In accordance with (8), we have

$$I(0, \mu, x) = \hat{\Lambda}(\mu) \int_0^\infty S(\tau') \exp(-\phi(\tau'/\mu)) \phi(\tau') d\tau' \mu,$$

$$\mu > 0.$$  \hspace{1cm} (16)

The radiation escaping from the limb is of particular interest, since polarization is greatest there. Since

$$\exp(-\phi(\tau'/\mu)) \phi(\tau') \mu \rightarrow \delta(0) \text{ as } \mu \rightarrow +0$$

where $\delta(x)$ is the Dirac delta function, from (15) we find

$$I(0, \mu, x) \rightarrow \hat{\Lambda}(0) S(0), \quad \mu \rightarrow 0.$$  \hspace{1cm} (17)

For the factoring (6)-(7) that we are using, this equation in components has the form ($\mu \rightarrow 0$)

$$I(0, \mu, x) = S_x(0) + \left(\frac{W}{8}\right) S_y(0),$$

$$Q(0, \mu, x) \rightarrow -3\left(\frac{W}{8}\right) S_y(0).$$  \hspace{1cm} (17')

From the latter equations it follows, in particular, that for radiation escaping from the limb itself ($\mu = 0$), the degree of polarization should be the same within the entire Doppler core (it does not depend formally on $x$). For small $\mu$ the degree of polarization at the center of the line is actually close to this limiting value. We emphasize that the polarization at the limb is expressed directly in terms of the components of the vector source function $S = (S_x, S_y)^T$ at the boundary of the atmosphere. In accordance with (15) and (17'), in our factoring of the phase function we have

$$\delta(\mu, x) \rightarrow -3S_x(0) / [(8/W) S_y(0) + S_y(0)], \quad \mu \rightarrow 0.$$  \hspace{1cm} (18)

The accurate calculation of $S(0)$ is thus of considerable interest.

5. THE $(1 - \lambda)^{\frac{3}{2}}$ LAW FOR THE VECTOR CASE

The main result of the present paper consists in the statement that the following relationship holds:

$$|S(0)| = \sqrt{1 - \lambda} B,$$

where $|S|$ is the Euclidean norm of vector $S$, i.e.,

$$|S(\tau)|^2 = S^T(\tau) S(\tau) = S^x(\tau) + S^y(\tau).$$  \hspace{1cm} (20)

We note that

$$|S(\infty)| = B.$$  \hspace{1cm} (21)

Equation (19) is a generalization of a well-known result in scalar theory, often called the $(1 - \lambda)^{\frac{3}{2}}$ law (Ref. 18, Sec. 11:2; Ref. 19, Secs. 6.1 and 6.2; Refs. 30-32). In contrast to the scalar case, unfortunately, Eq. (19) does not enable us to find the vector source function for $\tau = 0$. Nevertheless, it is very useful. Experience in numerical calculations has shown that Eq. (19) can serve as a sensitive test to check the accuracy of a numerical solution of the vector ERT. One must bear in mind that $\tau = 0$ is a "bad" point, since the derivative $dS(\tau)/d\tau$ diverges as $\tau \rightarrow 0$.

The proof of (19) is given in Sec. 7.

6. THE MILNE PROBLEM. LIMITING POLARIZATION

In studying the dependence of the solution of the basic integral equation (11) on the parameter $\lambda$, instead of comparing solutions for a fixed $B$ and different $\lambda$, as is usually done (see, e.g., Ref. 25), it is far more productive to proceed differently. We use (19). Setting $B = (1 - \lambda)^{\frac{3}{2}}$, we ascertain that the solution of the equation

$$S(\tau) = \lambda \hat{\tilde{K}}(\tau - \tau') S(\tau') d\tau' + (1 - \lambda)^{\frac{3}{2}} I_0,$$

where $I_0 = (1, 0)^T$, is normalized so that

$$|S(0)| = 1,$$

with $|S(\infty)| = (1 - \lambda)^{\frac{3}{2}}$ by virtue of (21).

A comparison of solutions of Eq. (22) for different $\lambda$ immediately reveals that existence of a surface boundary layer with a structure that does not depend on $\lambda$ for $1 - \lambda << 1$. In the limit as $\lambda \rightarrow 1$, (22) changes into the homogeneous equation

$$\tilde{S}_x(\tau) = \hat{\tilde{K}}(\tau - \tau') S_x(\tau') d\tau'.$$

Its solution, normalized by the condition

$$|S_x(0)| = 1,$$

is the limit to which the solution of (22) tends as $\lambda \rightarrow 1$.

The determination of the vector function $S_x(\tau)$ and the corresponding, radiation field, i.e., the Stokes vector $I_x(\tau)$ such that

$$\mu \frac{\partial I_x(\tau)}{\partial \tau} = \phi(\tau) I_x(\tau), \mu, x = \phi(\tau) \hat{\Lambda}(\mu) S_x(\tau),$$

$$I_x(0, \mu, x) = 0, \quad \mu < 0,$$

comprises the Milne problem for the type of scattering under consideration.
The vector function \( S_k(\tau) \) generates the Stokes vector of the escaping radiation,

\[
I_\perp(0, \mu, \tau) = \hat{A}(\mu) \int_0^\infty \frac{S_k(\tau) \exp(-\phi(x)/\tau)}{\mu \phi(x) d\tau} d\tau, \mu > 0.
\]

(28)

It determines the limiting form that the peaks of polarization in line cores assume as \( \lambda \to 1 \) (for an isothermal atmosphere).

The investigation of \( I_\perp(0, \mu, x) \) for monochromatic scattering (i.e., for a rectangular profile: \( \phi(x) = 1 \) for \( |x| \leq 1/2 \) and \( \phi(x) = 0 \) for \( |x| > 1/2 \)) has been the subject of an extensive literature, starting with the classic work of Chandrasekhar and Sobolev in the 1940s. The problem of finding \( I_\perp(0, \mu, x) \) for other profiles of \( \phi(x) \), of which the Doppler profile is the most interesting, of course, has not been raised before. We have published detailed numerical data pertaining to it elsewhere. One number deserves mention, however. For monochromatic scattering, the degree of polarization at the limb for the Milne problem is 11.7% (the so-called Chandrasekhar–Sobolev polarization limit), as is well known. For a Doppler profile, the analogous limiting polarization at the limb is 9.5%, as calculations have shown. This is the theoretical upper limit for the height of polarization peaks observed in the cores of resonant lines (within the framework of the model under consideration, of course).

7. RYBICKI’S EQUATION

Another important rigorous result of the scalar theory that carries over directly to the vector case is the so-called Rybicki’s quadratic integral.

We designate

\[
I_\perp = I(\tau, \mu, \tau) ; \quad L = L^\perp(\tau, -\mu, \tau).
\]

(29)

We have the equation

\[
|S|^2 = \lambda I(\perp I_\perp) + (1-\lambda) B^2
\]

(30)

or, in more detail,

\[
S^\perp(\tau) + S^\perp(\tau) = (1-\lambda) B^2 + \frac{\lambda}{2} \int \phi(x') dx' \int \left[ I(\tau, \mu', \tau') \times I(\tau, -\mu', \tau') \right] dx'.
\]

(30')

The proof of (30) is analogous to the derivation of the scalar Rybicki’s equation (obtained from (30') with \( S_Q = Q = 0 \). In fact, the equation (29) and the fact that \( A(\mu) = A(\mu) \), yields

\[
\frac{\partial L}{\partial \tau} = \phi(x) L - \phi(x) A(\mu) S,
\]

\[
-\frac{\partial L}{\partial \tau} = \phi(x) L - \phi(x) S^\perp A(\mu).
\]

(1')

Cross-multiplying these equations, i.e., multiplying the left-hand side of the first by the right-hand side of the second (from the left) and the right-hand side of the first by the left-hand side of the second (also from the left), we obtain

\[
\frac{\partial}{\partial \tau} (I L) = S^\perp A(\mu) \frac{\partial L}{\partial \tau} + \frac{\partial S}{\partial \tau} A(\mu) S.
\]

(31)

To this equation we apply the operator of averaging over frequency and direction, defined by (10). Since \( S^\perp \) and \( S \) depend only on \( \tau \), these factors can be taken outside the averaging operator \( \langle \rangle \). We then have

\[
\langle \hat{A}(\mu) \frac{\partial L}{\partial \tau} \rangle = \frac{d}{d\tau} \hat{A}(\mu) L \langle I \rangle.
\]

From (9) it follows that

\[
\frac{d}{d\tau} \langle \hat{A}(\mu) I \rangle = \frac{1}{\lambda} \frac{dS}{d\tau}.
\]

Similarly, we establish that

\[
\langle \frac{\partial S}{\partial \tau} \hat{A}(\mu) \rangle = \frac{1}{\lambda} \frac{dS}{d\tau}.
\]

The action of the \( \langle \rangle \) operator on (28) therefore has the result

\[
\lambda \frac{d}{d\tau} \langle I L \rangle = S^\perp \frac{dS}{d\tau} + \frac{dS}{d\tau} S
\]

from which we have

\[
S^\perp S = \lambda \langle I L \rangle + C,
\]

(32)

where \( C \) is the integration constant. It is found from the condition that the radiation field in ion infinitely deep layers must be in equilibrium, so that \( I \to B \) and \( S \to B \) as \( r \to \infty \). Going to the limit \( r \to \infty \) in (32), we find \( C = (1-\lambda) B^2 \), which completes the derivative of (30).

Equation (19), which expresses the \( (1-\lambda)^{1/2} \) law, is a special case of (30) for \( r = 0 \). Here the first term on the right-hand side of (30) vanishes by virtue of the first condition of (2).

8. SUMMARY

Let us summarize our main results. The purpose of the present work has been to discuss the vector analog of the basic model problem of the theory of line formation (a semi-infinite isothermal atmosphere, a two-level atom, CFI; see Ref. 18, Chap. 11). The basic integral equation for the source function (11) has the same form as in the scalar case. The only difference is the replacement of scalar quantities by vectors and matrices.

The best-known analytical result of the scalar theory, which has gone into textbooks (see, in particular, Ref. 18, Sec. 11.2), is expressed by the equation \( S(0)S(\infty) = (1-\lambda)^{1/2} \), where \( S(\tau) \) is the source function. As is well known, this equation provides a sensitive test for checking numerical solutions of the ERT. In the present paper we showed that this result is also valid in the vector case, if we understand \( S(\tau) \) to be the Euclidean norm of the vector source function. The vector analog of the so-called Rybicki’s quadratic integral serves us as a means of proving this beautiful result (although there are other ways of proving it).
The problem of finding the theoretical degree of polarization of radiation at the center of a line for the limb was raised. It was shown that for a Doppler profile and an isothermal atmosphere, the upper limit of polarization at the limb is 9.5%. It corresponds to conservative scattering ($\lambda = 1$). This is less than the classical Chandrasekhar–Sobolev polarization limit (11.7%), which pertains to monochromatic Rayleigh scattering, because of the influence of frequency redistribution.

Detailed numerical data pertaining to the problem considered here will be published separately.

I am grateful to Dr. Helene Frisch, who revived my interest in problems of the theory of line formation with her visit to Leningrad in June-July 1988. All the results of the present paper (except for the figure of 9.5%) were obtained from 7 to 11 July 1988. I also thank V. M. Serbin for permission to cite the limiting polarization at the limb (9.5%) before the publication of all the other numerical data.


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