THE DYNAMICS OF SOLAR SAILS WITH A NON-POINT
SOURCE OF RADIATION PRESSURE

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Abstract. The form of the solar radiation pressure on a heliocentric orbiting solar sail is obtained for a finite
angular sized and limb darkened solar disk by the use of the radiation pressure tensor. It is found that the
usual inverse square variation of the solar radiation pressure is modified by the finite angular size, and to a
lesser extent by the solar limb darkening. The actual magnitude of the modification is in itself small, except
at close heliocentric distances. However, its existence has implications for the dynamical stability of solar
sails both in parked and circular orbital configurations and for the accuracy of trajectory calculations,
particularly for sails in the inner solar system.

Keywords. Solar sails, radiation pressure, stability.

1. Introduction

The concept of using solar radiation pressure as a means of spacecraft propulsion
appears to have its origins in the writings of the Soviet pioneers Tsio1kovskii (1921)
and Tsander (1924), who described spacecraft with large mirrors being driven by the
pressure of sunlight. These ideas were later revived and the term ‘solar sailing’ coined
by Garwin (1958) outlining possible sail designs and their expected performances. The
first actual trajectory calculations were carried out by Tsu (1959) and later London
(1960) who obtained a logarithmic spiral solution to the equations of motion,
characterised by the sail velocity vector maintaining a constant angle with respect to
its instantaneous radius vector, the sail orientation being fixed with respect to the
incoming solar radiation. This type of solution however ignored the two-point
boundary conditions of the problem in that an initial and final impulse were required
for departure and rendezvous.

During the 1970's extensive time optimal trajectory calculations were performed by
Sauer (1976) as part of an ongoing research program for the proposed comet Halley
rendezvous mission using a solar sailed spacecraft. These high precision calculations,
using the Pontryagin maximum principle of the calculus of variations, relaxed many
of the assumptions made in previous studies and in particular took account of the
eccentricity and inclination of the initial and final orbits. Other authors, in particular
Van der Ha and Modi (1979) have obtained high order analytic series solutions to the
three-dimensional equations of motion of a solar sail with a fixed, arbitrary sail
orientation. With the aid of these solutions it was shown that the optimal method of
increasing the orbital inclination of a sail is through a strategy of switching the sail
orientation at each node of the orbit so as to maximise the out of plane component of


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the solar radiation pressure force, a so-called "cranking orbit". Such manoeuvres are of course executed more quickly at close heliocentric distances (i.e. \( <0.3 \) AU).

In all of these trajectory calculations it has been assumed in the equations of motion that the solar radiation pressure force has an inverse square variation allowing, in some cases, a closed analytic solution such as the logarithmic spiral. In this paper it will be shown that for a plane sail this assumption is in fact not valid when account is taken of the finite angular size and limb darkening of the solar disk. Starting from the fundamental definition of radiation pressure through the radiation pressure tensor, to take account of the varying direction of incidence of solar radiation from different parts of the solar disk, it is shown in Section 2 that the solar radiation pressure is modified from an inverse square law by a function of the sail heliocentric distance and the solar radius. A more precise calculation is also carried out using a limb darkened solar disk which gives a closed, but complex form of the limb darkened solar radiation pressure. It should be noted that these effects are small and comparable to other sources of perturbation such as an error in the sail orientation.

It is found that the resulting expression for the solar radiation pressure gives a small modification to the inverse square variation at all but the closest heliocentric distances, but as will be seen in Section 3 using the simpler non-limb darkened radiation pressure, this modification results in the destabilisation of the previously assumed neutral stability of 'levitated' sails (sails in equilibrium with solar radiation pressure balancing solar gravity). Further, the sail parameters required for levitation will no longer be independent of the sail heliocentric distance, as is the case for an inverse square variation of the solar radiation pressure.

Levitated sails have been proposed by Drexler (1979) and more recently Forward (1986), for solar sail missions to observe the Sun either from an arbitrary stationary point above the solar equator or the solar poles, a task impossible for any other type of spacecraft. Further, it has been proposed that for a sail of a suitably temperature resistant material it would be possible to follow the 25 day solar equatorial rotation from a heliocentric distance less than that required for a Keplerian orbit under the action of solar gravity only (i.e. \( <0.167 \) AU). This is possible as the sail orbital period may be decoupled from its heliocentric distance, since the effect of solar radiation pressure is essentially to modify the magnitude of the gravitational constant so that the solar rotation may be followed from any desired close heliocentric distance. In Section 4 it will be shown that sails in circular heliocentric orbits, such as are described above, are linearly stable except for a well defined, narrow range of sail parameters and heliocentric distances so that for most sails the heliocentric distance can be stabilised if the sail is given even a small orbital angular momentum.

Finally, in Section 5, the effect of the deviation of the solar radiation pressure from an inverse square form will be examined for some simple semi-optimal sail trajectories by comparing the trajectories obtained with an inverse square variation of the solar radiation pressure and a variation modified by the uniformly bright finite angular sized solar disk.
2. Solar Radiation Pressure with an Extended Source

In this first section the form of the solar radiation pressure on a plane, homogeneous, perfectly reflecting sail will be obtained using the concept of specific intensity. We will define the specific intensity $I_s(r, \mathbf{n}; t)$ of a radiation field as the energy in the frequency range $(\nu, \nu + d\nu)$, flowing through unit solid angle $d\Omega$, about direction $\mathbf{n}$, across unit area $dA$, in unit time $dt$, at a position $r$, with $I_s(r, \mathbf{n}; t)$ being conserved along rays (Rybicki and Lightman (1979)). The radiation pressure tensor, $P(r)$, is defined as the second angular moment of the specific intensity of the radiation field integrated over the entire frequency range, viz.

$$P(r; t) = \frac{1}{c} \int_0^{\infty} \int_{4\pi} I_s(r, \mathbf{n}; t) \mathbf{n} \cdot \mathbf{n} d\Omega dv.$$  \hspace{1cm} (1)

This tensor equation will now be used to obtain the radiation pressure from a uniformly bright and limb darkened finite sized solar disk.

a. UNIFOMLY BRIGHT SOLAR DISK

In the case of the non-limb darkened solar radiation field where the specific intensity is time independent and isotropic across the solar disk, we may write for the radiation pressure on a radially oriented, perfectly reflecting sail at a heliocentric distance $r$

$$P(r) = \frac{2}{c} \int_0^{\infty} \int_0^{2\pi} \int_0^{\theta_0} I_\nu \cos^2 \theta d\Omega d\nu; \hspace{0.5cm} d\Omega = \sin \theta d\theta d\phi,$$  \hspace{1cm} (2)

with $c$ the speed of light and the geometry specified in Figure 1. If we make use of the conservation of specific intensity along rays so that $I_\nu$ is independent of $r$, then due to the azimuthal symmetry of the geometry Equation (2) reduces to the integral

$$P(r) = \frac{4\pi}{c} I_0 \int_{\eta_0}^{1} \eta^2 d\eta; \hspace{0.5cm} \eta = \cos \theta, \hspace{0.5cm} \eta_0 = \cos \theta_0,$$  \hspace{1cm} (3)

![Diagram of solar disk and sail geometry](image.png)

**Fig. 1.** Sun–sail geometry with a finite angular sized solar disk.
where $I_0$ is the frequency integrated specific intensity. Performing this integration and substituting for $\eta_0$ we obtain

$$P(r) = \frac{4\pi}{3c} I_0 \left\{ 1 - \left\{ 1 - \left\{ \frac{R_0}{r} \right\}^2 \right\}^{3/2} \right\}.$$  \hspace{1cm} (4)

Equation (4) may be expanded and for $r \gg R_0$ we may write, to first order

$$P(r) = \frac{2\pi}{c} I_0 \left\{ \frac{R_0}{r} \right\}^2 + O((R_0/r)^4).$$  \hspace{1cm} (5)

However, at large values of $r$ this expansion must match asymptotically with the expression for the radiation pressure from a point source, viz.

$$P^*(r) = \frac{2}{c} \left\{ \frac{L_0}{4\pi r^2} \right\},$$  \hspace{1cm} (6)

where $L_0$ is the solar luminosity. Hence, by comparing Equations (5) and (6) we identify $I_0$ as

$$I_0 = \frac{L_0}{4\pi^2 R_0^2}.$$  \hspace{1cm} (7)

Substituting for $I_0$ in Equation (4) we obtain the expression for the solar radiation pressure on a radially oriented, perfectly reflecting sail at a heliocentric distance $r$ from a uniformly bright, finite sized solar disk

$$P(r) = \frac{L_0}{3\pi c R_0^2} \left\{ 1 - \left\{ 1 - \left\{ \frac{R_0}{r} \right\}^2 \right\}^{3/2} \right\}.$$  \hspace{1cm} (8)

A more useful way of representing the solar radiation pressure is to express it in terms of the point source, inverse square variation $P^*(r)$ defined by Equation (6), viz.

$$P(r) = P^*(r)F(r); \hspace{1cm} F(r) = \frac{2}{3} \left\{ \frac{r}{R_0} \right\}^2 \left\{ 1 - \left\{ \frac{R_0}{r} \right\}^2 \right\}^{3/2}.$$  \hspace{1cm} (9)

We see that the function $F(r)$ attains its minimum value at $r = R_0$, where $F(r) = 2/3$, giving the greatest deviation of the solar radiation pressure from an inverse square variation. Further, as $r \to \infty$, $F(r) \to 1$ as the solar disk becomes more point-like, as shown in Figure 2.

b. LIMP DARKENED SOLAR DISK

A more accurate model of the solar radiation pressure may now be obtained by the inclusion of solar limb darkening in the functional form of the specific intensity. Using the grey solar atmosphere model (Mihalas and Mihalas (1984)), the specific intensity of the radiation field may now be written as

$$I = \frac{I_0}{4} (2 + 3 \cos \psi),$$  \hspace{1cm} (10)
where \( I_0 \) is defined in Equation (7). Thus the solar limb will be darker than the centre of the solar disk by a factor of 0.4 using the grey atmosphere approximation. The angle \( \psi \) may be related to the integration variable \( \theta \) through the relation
\[
\cos \psi = \left\{ 1 - \left( \frac{r}{R_0} \right)^2 \sin^2 \theta \right\}^{1/2},
\]
so that the required integral now becomes
\[
P(r) = \frac{2}{c} \int_0^{2\pi} \int_0^{\theta_0} \frac{1}{4} I_0 \left\{ 2 + 3 \left( 1 - \left( \frac{r}{R_0} \right)^2 \sin^2 \theta \right)^{1/2} \right\} \cos^2 \theta \, d\Omega.
\]
After some lengthy integration we may obtain the expression for the solar radiation pressure from a finite angular sized, limb darkened solar disk as
\[
P(r) = \frac{2\pi}{3} I_0 \left\{ 1 - \left( 1 - \left( \frac{R_0}{r} \right)^2 \right)^{3/2} \right\} + \frac{3\pi}{2} I_0 \left[ \frac{R_0}{r} \right]^3 \left\{ 1 + \left( \frac{r}{R_0} \right)^2 \right\}
\]
\[
+ \frac{1}{8} \left\{ 1 - \left( \frac{r}{R_0} \right)^2 \right\}^2 \ln \left\{ \frac{(r/R_0)^2 - 1}{(r/R_0)^2 + 2(r/R_0) + 1} \right\}.
\]
This can again be written in terms of the point source radiation pressure, \( P^*(r) \), viz.
\[
P(r) = P^*(r) G(r),
\]
where the function \( G(r) \) is defined as

\[
G(r) = \frac{1}{3} \left[ \frac{r}{R_0} \right]^2 \left( 1 - \left[ 1 - \left( \frac{R_0}{r} \right)^{2/3} \right] \right) + \frac{3}{4} \left[ \frac{R_0}{r} \right] \left[ 1 \left( \frac{r}{R_0} \right)^2 \right] \left( 1 + \left( \frac{r}{R_0} \right)^2 \right) + \frac{1}{8} \left( 1 - \left( \frac{r}{R_0} \right)^2 \right)^2 \ln \left( \frac{(r/R_0)^2 - 1}{(r/R_0)^2 + 2(r/R_0) + 1} \right).
\]

(15)

The form of Equation (15) is shown in Figure 2. It can be seen that the functions \( F(r) \) and \( G(r) \) have the same behaviour but differ somewhat in the precise numerical values, their fractional difference being \( \approx 10^{-2} \) for all but the closest heliocentric distances. For this reason we will use the uniformly bright solar disk approximation with its much simpler functional form.

3. The Dynamical Stability of a Levitated Sail

Using the results of the previous section the linear stability of a levitated sail will be investigated by perturbing the equations of motion and examining the form of the resulting sail motion. Consider now a radially oriented sail of total mass \( m \), cross-sectional area \( A \), at a heliocentric distance \( r \), under the influence of solar gravity and an inverse square radiation pressure given by \( P^*(r) \). The sail has zero angular velocity so that we may consider a one-dimensional equation of motion, given simply by

\[
\frac{d^2 r}{dt^2} = -\frac{\mu}{r^2} + \frac{L_0}{2\Pi c \sigma r^2}; \quad \sigma = \frac{m}{A} = \frac{G(M_0 + m)}{c}, \quad \mu = \frac{G M_0}{c},
\]

(16)

where \( \sigma \) is the sail loading, \( G \) is the gravitational constant and \( M_0 \) the solar mass. From Equation (16) it can be seen that there is a unique value of \( \sigma \) which gives \( (d^2 r/dt^2) = 0 \) independent of \( r \). We will denote this critical sail loading by \( \sigma^* \) which is given by

\[
\sigma^* = \frac{L_0}{2\Pi G M_0 c}; \quad (\sigma^* = 1.529 \times 10^{-3} \text{ kg m}^{-2}).
\]

(17)

Thus, if we have a sail initially at rest at some heliocentric distance \( r_0 \), with \( \sigma = \sigma^* \) and apply a small perturbation of \( \xi \), such that \( r_0 \rightarrow r_0 + \xi \), the resulting sail motion is of the form

\[
\xi(t) = \xi_{01} + \xi_{02} t,
\]

(18)

where \( \xi_{01} \) and \( \xi_{02} \) are constants of the motion. From Equation (18) it can be seen that for an inverse square variation of solar radiation pressure a levitated sail has neutral stability.

Consider now the one-dimensional equation of motion with the modified solar radiation pressure of a uniformly bright, finite angular sized solar disk given by Equation (9)

\[
\frac{d^2 r}{dt^2} = -\frac{\mu}{r^2} + \frac{L_0}{2\Pi c \sigma r^2} F(r).
\]

(19)
Clearly there is no longer a unique value of \( \sigma \) giving equilibrium at all distances \( r \), but the required value of \( \sigma \) will now be a function of the sail heliocentric distance given by

\[
\sigma_c(r) = \sigma^* F(r). \tag{20}
\]

Suppose now that the sail is in equilibrium at a heliocentric distance \( r_0 \), with \( \sigma(r_0) = \sigma_c(r_0) \) and we apply a small perturbation of \( \xi \), such that \( r_0 \rightarrow r_0 + \xi \). Then, expanding Equation (19) in powers of \( \xi \), the equation of the subsequent sail motion is given by

\[
\frac{d^2 \xi}{dt^2} = -\frac{\mu}{r_0^2} \left( 1 - 2 \frac{\xi}{r_0} \right) + \frac{L_0}{2\Pi C_0 \sigma_0^2} \left( 1 - 2 \frac{\xi}{r_0} \right) \left( F(r_0) + \left. \frac{dF}{dr} \right|_{r=r_0} \right). \tag{21}
\]

Substituting for \( \sigma_c(r_0) \) and retaining linear terms only we obtain

\[
\frac{d^2 \xi}{dt^2} - D(r_0) \xi = 0, \tag{22}
\]

where the function \( D(r_0) \), defining the eigenfrequencies and form of the solution to Equation (22) is given by

\[
D(r_0) = \frac{\mu}{r_0^2} \frac{1}{F(r)} \left. \frac{dF}{dr} \right|_{r=r_0}. \tag{23}
\]

Evaluating the derivative of \( F(r) \) and making the substitution, \( x = (R_0/r_0)^2 \), Equation (23) reduces to

\[
D(r_0) = \frac{2\mu}{r_0^3} M(x); \quad M(x) = 1 - \frac{3}{2} x(1-x)^{1/2}, \tag{24}
\]

where the sign of the function \( M(x) \) determines whether Equation (22) has real or purely imaginary eigenvalues.

We now examine the asymptotic behaviour of the function \( M(x) \) in the limits of \( x \rightarrow 1 \), \( (r_0 \rightarrow R_0) \) and \( x \rightarrow 0 \), \( (r_0 \rightarrow \infty) \). Firstly, as \( x \rightarrow 1 \) we see that

\[
M(x) \rightarrow 1, \quad x \rightarrow 1, \tag{25}
\]

and as \( x \rightarrow 0 \) we may expand the fractional powers to obtain

\[
M(x) = 1 - \frac{1}{4} x + O(x^2) \quad \rightarrow 0, \quad x \rightarrow 0. \tag{26}
\]

Hence, since it can be shown that the function \( M(x) \) has only one real root at \( x = 0 \), we may write \( M(x) > 0; \forall x \in (0, 1] \), so that Equation (22) has two real eigenvalues of opposite sign and an exponential solution of the form

\[
\xi(t) = \xi_{01} e^{\omega t} + \xi_{02} e^{-\omega t}, \tag{27}
\]

where \( \xi_{01}, \xi_{02} \) are constants and the eigenfrequencies, \( \pm \omega \), are defined by

\[
\omega^2 = 2\omega_0^2 M(r/R_0); \quad \omega_0^2 = \frac{\mu}{r_0^3}, \tag{28}
\]

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with \( \omega_0 \) the usual Keplerian angular frequency of a circular orbit at a heliocentric distance \( r \).

It is seen, then, that by considering the Sun as a uniformly bright extended source of radiation pressure that a sail of a given loading has only one possible equilibrium point and that this point is \textit{unstable}, the instability being independent of the sail parameters. The time scale of this instability will, however, be dependent on the sail heliocentric distance and is given by

\[
T = T_0 (2M(r/R_0))^{-1/2},
\]

where \( T_0 \) is the period of a Keplerian orbit at a heliocentric distance \( r \) under the action of solar gravity only. This time scale is itself large (e.g. at 1/3 AU \( T \approx 20 \) years) but the existence of the instability adds to the need for active sail station keeping and means that the dynamics of a levitated solar sail is not as elementary a problem as is at first thought.

The instability may readily be understood physically as shown schematically in Figures 3a and 3b. In Figure 3a, showing the variation of the solar gravitational force, \( F_G \), and the solar radiation pressure force, \( F_R \), with an inverse square variation of solar radiation pressure, it can be seen that the solar gravitational force may be balanced at all points with a suitable choice of sail loading by the solar radiation pressure force thus giving a neutral equilibrium as discussed above. However, in Figure 3b with the deviation of the solar radiation pressure force from an inverse square form, given by Equation (9), it can be seen that, for a given sail loading, there is only one heliocentric distance, \( r_0 \), where \( F_G \) and \( F_R \) intersect thus giving equilibrium.

![Fig. 3(a). Schematic form of the solar gravitational and inverse square radiation pressure forces.](image1)

![Fig. 3(b). Schematic form of the solar gravitational and modified radiation pressure forces with equilibrium at \( r = r_0 \).](image2)
For $r > r_0$ we see that $F_R > F_G$ so that the sail is accelerated outward by the solar radiation pressure force and for $r < r_0$ we see that $F_G > F_R$ so that the sail falls sunward, accelerated by the solar gravitational force.

4. The Dynamical Stability of a Sail in a Heliostationary Orbit

It was seen in the last section that the modification to the inverse square form of the solar radiation pressure yielded unstable equilibria for the one-dimensional problem of a levitated solar sail. In this section we will examine the stability of sails in circular heliocentric orbits, such as the 25-day orbit following the solar equatorial rotation.

Consider now a sail with heliocentric plane polar coordinates $(r, \theta)$ and a sail loading $\sigma$. For a radial sail orientation the equations of motion are of the form

$$\frac{d^2r}{dt^2} - r \frac{d}{dt} \left\{ \frac{d\theta}{dt} \right\}^2 = -\frac{\mu}{r^2} + \beta \frac{\mu}{r^2} F(r),$$

(30a)

$$\frac{1}{r} \frac{d}{dt} \left\{ r^2 \frac{d\theta}{dt} \right\} = 0,$$

(30b)

where $\beta$ is the ratio of the solar inverse square radiation pressure force, to the solar gravitational force on the sail. Thus, the parameter $\beta$ is not in fact the ratio of the actual forces acting on the sail but, as is clear from Equation (17), is equivalent to the ratio $\sigma^* / \sigma$. It is this meaning that will be used in the subsequent analysis. From Equation (30b) it is seen that we may immediately obtain an angular momentum integral, viz.

$$r^2 \left\{ \frac{d\theta}{dt} \right\} = H,$$

(31)

with $H$ being the sail’s angular momentum per unit mass. Using Equation (31) to eliminate $(d\theta/dt)$ from Equation (30a) we obtain a radial equation of the form

$$\frac{d^2r}{dt^2} - \frac{H^2}{r^3} = -\frac{\mu}{r^2} (1 - \beta F(r)).$$

(32)

Firstly, consider the case of a purely inverse square variation of solar radiation pressure (i.e. $F(r) = 1$) so that we may define a reduced gravitational constant $\mu^* = \mu(1 - \beta)$. Suppose now that the sail is in an initially circular orbit with $(d^2r/dt^2) = 0$ at $r = r_0$ and so $H^2 = \mu^* r_0$. If we again apply a small perturbation of $\xi$, such that $r_0 \rightarrow r_0 + \xi$ and linearise Equation (32) with respect to $\xi$ we find

$$\frac{d^2\xi}{dt^2} + \frac{\mu^*}{r_0^3} \xi = 0,$$

(33)

which has a solution

$$\xi(t) = \xi_{01} e^{i\Omega_0 t} + \xi_{02} e^{-i\Omega_0 t}, \quad \Omega_0^2 = \frac{\mu^*}{r_0^3},$$

(34)
where \( \xi_{01}, \xi_{02} \) are constants and \( \Omega_0 \) is the angular frequency of a circular orbit at a heliocentric distance \( r \) with a reduced gravitational constant \( \mu^* \). Therefore, as expected, it can be seen that sail orbits with an inverse square variation of solar radiation pressure are linearly stable.

We now repeat this analysis with the modified, uniformly bright form of the solar radiation pressure as defined by Equation (9) with the sail angular momentum per unit mass now being given by \( H^2 = \mu r_0(1 - \beta F(r_0)) \). A small perturbation of \( \xi \) is applied to Equation (32) and, expanding in powers of \( \xi \), we obtain

\[
\frac{d^2 \xi}{dt^2} - \frac{H^2}{r_0^3} \left\{ 1 - 3 \frac{\xi}{r_0} \right\} = -\frac{\mu}{r_0^2} \left\{ 1 - 2 \frac{\xi}{r_0} \right\} \left\{ 1 - \beta F(r_0) - \beta \left. \frac{dF}{dr} \right|_{r=r_0} \right\} \xi. \tag{35}
\]

Substituting for \( H^2 \) and retaining linear terms only, Equation (35) reduces to

\[
\frac{d^2 \xi}{dt^2} - \frac{\mu}{r_0^3} \left\{ \beta F(r_0) + r_0 \beta \left. \frac{dF}{dr} \right|_{r=r_0} - 1 \right\} \xi = 0, \tag{36}
\]

which may be written as

\[
\frac{d^2 \xi}{dt^2} - P(r_0) \xi = 0. \tag{37}
\]

Evaluating the derivative of \( F(r) \) and making the substitution, \( x = (R_0/r_0)^2 \), the function \( P(r_0) \) may be written as

\[
P(r_0) = \frac{\mu}{r_0^3} Q(x); \quad Q(x) = 2\beta \left\{ \frac{1}{x} (1 - (1 - x)^{3/2}) - (1 - x)^{1/2} \right\} - 1. \tag{38}
\]

For stability we require that \( Q(x) < 0 \); \( \forall x \in (0, 1] \). This condition may be written in terms of an inequality on \( \beta \), viz.

\[
\beta(x) < \left\{ \frac{2}{x} (1 - (1 - x)^{3/2}) - 2(1 - x)^{1/2} \right\}^{-1}. \tag{39}
\]

Equation (39) divides the parameter space into two distinct regions of stability and instability, as shown in Figure 4, with a further hyperbolic region defined by \( \sigma < \sigma_c(r) \). The solutions to the perturbed equations of motion in the regions of stability and instability are of the form

\[
\xi(t) = \xi_{01} e^{\Omega t} + \xi_{02} e^{-\Omega t}; \quad \beta > \beta_c, \tag{40a}
\]
\[
\xi(t) = \xi_{01} e^{i\Omega t} + \xi_{02} e^{-i\Omega t}; \quad \beta < \beta_c, \tag{40b}
\]

where \( \beta_c \), the limiting value of \( \beta \), is defined by an equality in Equation (39) and the eigenfrequencies, \( \pm \Omega \), are defined by

\[
\Omega^2 = \omega_0^2 Q(r/R_0). \tag{41}
\]

It can be seen that for all but the closest heliocentric distances the condition for instability is confined to a very narrow region of the sail parameter space close to \( \beta = 1 \) (i.e. to a narrow range of long orbital periods close to levitation). Thus, in general a very specific set of sail parameters is required for an unstable orbit.
Substituting for $H$ in Equation (31) we may write

$$\frac{d\theta}{dt} = \omega_0(1 - \beta F(r))^{1/2}. \quad (42)$$

For $r \gg R_0$ we have that $F(r) \approx 1$ and so for $\beta = \beta_c \approx 1, (d\theta/dt) \ll 1$ for a circular orbit at a heliocentric distance $r$. If the sail then orbits with a higher angular frequency we see from Equation (42) that a smaller value of $\beta$, within the stable region of the parameter space will be required to maintain a heliocentric distance $r$. The value of the sail orbital angular frequency to obtain stability (i.e. $\beta < \beta_c$), can be obtained by substituting for $\beta_c$ in Equation (42), viz.

$$\left\langle \frac{d\theta}{dt} \right\rangle_c = \omega_0(1 - \beta_c(r)F(r))^{1/2}. \quad (43)$$

However, we may require a sail orbital angular frequency of less than that required for stability and so will need a value of $\beta$ within the unstable region of the sail parameter space. For the case of a sail in a heliostationary orbit following the 25-day solar equatorial rotation the value of $\beta$ required to obtain this orbit is shown as a function of the sail heliocentric distance in Figure 5. It can be seen that $\beta \rightarrow 0$ as $r \rightarrow 35.76R_0$, corresponding to a Keplerian orbit of period 25 days at 0.167 AU under the influence of solar gravity only. For a heliostationary orbit at closer distances the required value of $\beta$ increases to reach a maximum value of 1.5 at the solar surface,
corresponding to $\sigma = \sigma_c(R_0)$. We note however that the required value of $\beta$ crosses into the unstable region of the sail parameter space at a heliocentric distance of $6.7R_0$ (0.03 AU) so that any 25-day sail orbit at a closer heliocentric distance will be necessarily unstable. Similarly for a sail in an earth synchronous 1-year orbit the region of instability is bounded by a heliocentric distance of $19.1R_0$ (0.09 AU). Therefore, we conclude that if a sail orbits with even a small angular frequency it will require a value of $\beta < \beta_c$ to maintain the required heliocentric distance and so will be dynamically stable except when some particular orbital period is required, such as the heliostationary or earth synchronous orbits.

It is seen then that a dynamically unstable levitated sail may be stabilised, depending on the required orbital period and heliocentric distance, if it has even a small orbital angular momentum and that any perturbations will result merely in periodic oscillations about its nominal circular orbit with a frequency given by Equation (41). For stationary, levitated sails in the ecliptic plane this would be acceptable or even necessary for many purposes, such as solar observing with the 25-day heliostationary orbit. However, for sails levitated above the solar poles where the objective was to have continuous observations of the poles it would not be desirable to have the sail orbiting and so the sail would have to be actively controlled to remain in a dynamically unstable equilibrium.
5. The Effect of $F(r)$ on Semi-optimal Sail Trajectories

In this final section a simple, heliocentric, semi-optimal sail transfer orbit will be examined comparing the trajectories obtained with an inverse square variation of solar radiation pressure and a variation modified by the function $F(r)$, to briefly illustrate the effect of $F(r)$ on solar sail trajectories. For simplicity and ease of illustration we will use a tangential thrusting approximation so that the solar radiation pressure force vector is directed along the sail velocity vector at all points along the trajectory. This ensures that the rate of energy gain of the sail is maximised throughout the trajectory, but ignores the two-point boundary conditions so that an initial and final impulse will be required to complete the transfer.

Using a high performance sail parameter of $\sigma = 10^{-3} \text{ kg m}^{-2}$ a 1 AU–0.1 AU trajectory was generated with an inverse square solar radiation pressure force, Figures 6a, b, and the modified, finite sized and uniformly bright solar disk radiation pressure force as given by Equation (9). The resulting velocity difference is shown in Figure 6c where it can be seen that an error of as large as 80 m s$^{-1}$ occurs. Hence, although the deviation from an inverse square form is small, it can lead to noticeable effects.

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Fig. 6(a). Polar plot of semi-optimal 1 AU–0.1 AU trajectory using a tangential thrusting approximation.
6. Conclusions

We have seen that, due to the finite angular size and limb darkening of the solar disk, the radiation pressure force on a solar sail spacecraft does not have an inverse square variation, as assumed in previous studies. In general the effect of this deviation is to introduce errors in the sail position and velocity into any calculations of sail trajectories. Thus, by identifying that this deviation of the form of the solar radiation pressure exists we may obtain more accurate nominal sail trajectories. This is particularly true for sail trajectories in the inner solar system and for any future sail missions with advanced sail materials making extremely close passes (<0.1 AU) by the Sun, such as is seen in time optimal trajectories to the outer planets. Although the actual magnitude of the deviation is small, except at close heliocentric distances, it has been shown that the 1-dimensional problem of a levitated sail is not as simple a matter as had at first been thought, with a sail of a given loading now having only one possible equilibrium position, which is unstable. Further, by having the sail orbiting it has been shown that, for a sufficiently great orbital angular momentum, the sail
Fig. 6(c). Sail velocity error due to the deviation of the solar radiation pressure from an inverse square form.

heliocentric distance would become linearly stable against any perturbations. However, with specific requirements on the sail orbital period there exists a region within which the sail orbit necessarily becomes unstable.

Another problem to be considered in future work are the errors encountered for a sail in a heliocentric cranking orbit which, for the comet Halley mission, was to have been of 1.15 years’ duration at a heliocentric distance of 0.25 AU. Due to the increased deviation of the solar radiation pressure at close heliocentric distances it is anticipated that the effect of the finite angular size and limb darkening of the solar disk will be of considerable importance in this problem. The determination of the form of the solar radiation pressure by considering the Sun as an extended, limb darkened source is perhaps just one aspect of the astrophysical modelling of the Sun which can be carried out to obtain a precise form of the solar radiation pressure force on a solar sail spacecraft. Whereas much work has been carried out on the modelling of sail reflection coefficients and other physical properties, the modelling of the actual source of the radiation pressure has been largely ignored. Other astrophysical effects to be
modelled are the effect of oscillations of the solar energy output, which directly relates to the momentum flux across a sail, and other non-radiative forces, such as the interaction of an electrically charged sail with the solar wind plasma. These questions will be addressed in future work.

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