THE BEHAVIOR OF BEAMS OF RELATIVISTIC NONTHERMAL ELECTRONS UNDER THE INFLUENCE OF COLLISIONS AND SYNCHROTRON LOSSES

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ABSTRACT

For many astrophysical situations, such as in solar flares or cosmic gamma-ray bursts, continuum gamma rays with energies up to hundreds of MeV have been observed and can be interpreted to be due to bremsstrahlung radiation by relativistic electrons. The region of acceleration for these particles is not necessarily the same as the region in which the radiation is produced, and the effects of the transport of the electrons must be included in the general problem. Hence it is necessary to solve the kinetic equation for relativistic electrons, including all the interactions and loss mechanisms relevant at such energies. The resulting kinetic equation for nonthermal electrons, including the effects of Coulomb collisions and losses due to synchrotron emission, has been solved analytically in some simple limiting cases and numerically for the general cases including constant and varying background plasma density and magnetic field. New approximate analytic solutions are presented for collision-dominated cases, for small pitch angles and all energies, synchrotron-dominated cases, both steadystate and time dependent, for all pitch angles and energies, and for cases when both synchrotron and collisional energy losses are important, but for relativistic electrons. These analytic solutions are compared to the full numerical results in the proper limits. These results will be useful for calculation of spectra and angular distribution of the radiation (X-rays, γ-rays, and microwaves) emitted via synchrotron or bremsstrahlung processes by the electrons. We shall examine these properties and their relevance to observations in subsequent papers.

Subject headings: plasmas — radiation mechanisms — Sun: flares — Sun: X-rays — X-rays: bursts

I. INTRODUCTION

In many astrophysical situations, the observed electromagnetic radiation is produced by accelerated electrons with nonthermal or non-Maxwellian distributions (typically with power-law energy spectra and anisotropic momentum distributions). Interaction of these electrons with ambient plasma generally with varied particle, photon, and magnetic field densities produces the observed radiation through synchrotron, Compton, or bremsstrahlung processes. In general, the acceleration site, normally a region of low-density, high-plasma turbulence or electric field, can be different than the region where the bulk of the radiation is produced. The distribution of particles derived from modeling of the emission process is not necessarily that of the accelerated particles but is modified during the transport from one region to another. The two distributions are related by the particle kinetic equation. It is imperative then to understand the effects of transport in order to determine the distribution of the accelerated particles and to gain insight into the acceleration mechanism.

In most analyses of nonthermal sources, the transport effects are either ignored or treated in an approximate manner primarily because of the complexity of the problem. Many interactions with the ambient plasma such as Coulomb collisions, inverse Compton scattering, synchrotron cooling, and interaction with both small-scale electromagnetic field fluctuations (plasma waves) and large-scale electric and magnetic fields can be simultaneously be important. Some simplified cases have been analyzed. For example, in recent studies the synchrotron effects were considered by Brainerd and Lamb (1987), Coulomb collision effects by Leach and Petrosian (1981, hereafter LP) and Petrosian (1985), and synchrotron and magnetic field effects by Ho (1986) and MacKinnon and Brown (1988). In this paper, we consider the effects of all three of these processes. In subsequent papers we shall apply the results from this study to microwave, X-ray, and γ-ray production in solar flares, and X-ray too γ-ray production in gamma-ray bursts.

We use the Fokker-Planck method for solution of the kinetic equation and determination of the distribution $f(x, p, t)$ in phase space. Because of the presence of strong fields, in general the particle drift across field lines is negligible so that only the spatial coordinate $s$, the distance along the field line, and two components of momentum (parallel and perpendicular to the field lines) are needed. In most cases discussed below, we replace the two momenta with the kinetic energy $E$ and $\mu = \cos \alpha$, where $\alpha$ is the particle pitch angle. We shall also consider the steady state case which is a good approximation when the time scale of the modulation of the observed radiation is longer than the typical time scales of the transport processes. Thus we solve the kinetic equation for the distribution $f(E, \mu, s)$, where $f \, dE \, d\mu \, ds$ is the number density of particles. We utilize the Fokker-Planck treatment of the collisional effects and neglect the inverse Compton effects (which are in many ways similar to synchrotron effects), the effects of electric fields, plasma turbulence, and self-absorption of synchrotron radiation.

The fully relativistic equation including collisional synchrotron effects as well as the effects of the field inhomogeneities

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can then be written as

\[ \frac{\partial \Phi}{\partial s} - \frac{d \ln B}{2 ds} \frac{\partial}{\partial \mu} [(1 - \mu^2) \Phi] = \frac{1}{\beta^2} \frac{\partial}{\partial E} \left[ (C + S \beta^3 \gamma^2 (1 - \mu^2) ) \Phi \right] - \frac{S}{\beta \gamma} \frac{\partial}{\partial \mu} \left[ \frac{\mu(1 - \mu^2)}{\sigma^2} \right] + \xi C \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial \Phi}{\partial \mu} \right] + \frac{\Sigma}{\sigma^2}, \]  

(1)

where \( \Phi = f/\gamma, \gamma = E + 1 \) is the total energy, \( \beta c = c(1 - 1/\gamma^2)^{1/2} \) is the electron velocity, \( B \) is the magnetic field strength, and \( \Sigma \) is a source term for the injected electrons. [See, e.g., LP; Leach 1984; Petrosian 1985. Note that the convergence term is different here than in the equations in LP and Leach 1984, with the factor of \((1 - \mu^2)\) being inside the derivative \( \partial \Phi/\partial \mu \). This results in differences of less than 5% in comparison with the earlier results.]

The steps leading to equation (1) and the definition of the collisional energy loss and diffusion coefficients \( C \) and \( C' = \xi C \), and the synchrotron coefficient \( S \), are given in LP and Petrosian (1985). For a background plasma of fully ionized hydrogen, \( \xi = 1 \) and

\[ C = 4\pi r_0^2 n \ln \Lambda = 2 \times 10^{-13} \left( \frac{\ln \Lambda}{20} \right) \left( \frac{n}{10^{10} \text{ cm}^{-3}} \right) \text{ cm}^{-1}, \]  

(2)

where \( r_0 = e^2/m_e c^2 \) is the classical electron radius, \( n \) is the ambient proton or electron density, and \( \ln \Lambda \) is the Coulomb logarithm. This simple relation is not true for a partially ionized plasma or a neutral gas. In these cases, the ratio \( \xi \) depends on the energy; e.g., for a neutral background, \( \xi \) varies from \( \sim 1/12 \) to \( \sim 1/8 \) for energies from 10 keV to 10 MeV.

The synchrotron energy loss and pitch angle change terms are proportional to

\[ S = \frac{2e^2}{3} \left( \frac{B^2}{m_e c^2} \right) = 6.5 \times 10^{-16} \left( \frac{B}{100 \text{ G}} \right)^2 \text{ cm}^{-1}. \]  

(3)

The coefficients \( S, C \), and \( C' \) have the units of inverse length and are useful scales. We will find it convenient to define the ratios \( \Sigma_0 = S/C (\Sigma_0 = 3230 B^2/n \ln \Lambda = 20) \), and

\[ R = \beta^3 \gamma^2 (1 - \mu^2) \Sigma_0, \]  

(4)

as a measure of relative importance of the synchrotron and collisional energy losses (see Table 1).

For the source term \( \Sigma \), we shall assume that the electrons are injected at one point, the origin of the spatial coordinate \( s = 0 \), so that \( \Sigma \propto \delta(s) \) is zero everywhere except at \( s = 0 \). Consequently, we solve the equation with \( \Sigma = 0 \) and use the injected distribution as a boundary condition. Furthermore, we will present the results in terms of the particle flux integrated over the cross-sectional area of the loop \( A \), given by

\[ F = \beta c A = \beta c \Phi A, \]  

which has units of \( s^{-1} \text{ keV}^{-1} \text{ sr}^{-1} \). The source, or the boundary condition at \( s = 0 \), is assumed to have the form

\[ F(\mu, s = 0) = F_0(\epsilon) G(\mu) = 2\sigma_0^{-2} F_0(\epsilon) e^{-(s^2 - s_1^2)/\sigma_0^2}. \]  

(5)

Here \( \sigma_0^2 \) is the dispersion in pitch angle and \( s_1 \) is the peak direction. For lower values of \( s_1 \) and \( s_0 \), the distribution becomes beamed along the field lines. For \( s_1 = \pi/2 \) and small \( s_0 \), the distribution will be of pancake form. In some cases we shall replace \( \sigma^2 \) with \( \sin^2 \sigma = (1 - \mu^2) \), and when necessary we shall assume a power-law energy spectrum given by

\[ F_0(\epsilon) = K \epsilon^{-\delta}. \]  

(6)

In the next section we discuss some analytic solutions for equation (1). In § III we describe the results from numerical solutions of this equation for a variety of injected pitch angle distributions and field configurations. A brief summary is presented in § IV.

II. ANALYTIC SOLUTIONS

In some limiting cases, we find that analytic solutions are possible. The analytic solutions are useful in many ways. They can give good quantitative estimates within their domain of applicability, they can be used to test accuracy of the complex numerical results, and they provide a qualitative guide for more complicated cases outside the range of applicability.

a) Collision-dominated Solutions (R \( \ll 1 \))

For high densities and low values of the field strength and electron energy, the electron transport is dominated by Coulomb collisions, and we can set \( \Sigma = 0 \). As shown by LP, analytic solutions are then possible for small pitch angles. If \( \angle_0 \ll 1 \), the injected electrons are strongly beamed along the field lines and we can set \( \mu = (1 - \angle_0^2/2) \) and \( (1 - \mu^2) = \angle_0^2 \) in equation (1). If we ignore the \( O(\angle_0^2) \) terms, and define a normalized collisional column depth \( \tau_c \) and energy parameter \( \eta \) by

\[ d \tau_c = C ds, \quad d \eta = \beta^2 d \epsilon, \]  

(7)

then the flux at any depth is given by equation (18) of LP:

\[ F(\epsilon, \mu, \tau_c) = \left\{ \frac{F_0(\epsilon)}{[\beta(\epsilon \eta + \tau_c)]} \right\}^2 2e^{-\angle_0^2/\eta^2} F_0(\epsilon \eta + \tau_c) \]  

(8)

The pitch angle distribution is Gaussian at all \( \epsilon_c \) with a dispersion \( \angle_0 \) given by

\[ \angle_0^2 = \angle_0^2 + \angle(\epsilon, \tau_c), \]  

(9)

where

\[ \angle(\epsilon, \tau_c) = 2\angle \ln \left[ \frac{\epsilon(\eta + \tau_c)}{\epsilon(\eta)} \right] \times 2 + \angle(\eta) \]  

(10)

From equation (7), we have \( \eta = \epsilon^2/(\epsilon + 1) \), so that

\[ E(x) = x \left( 1 + \sqrt{1 + \frac{4}{x}} \right). \]  

(11)

Thus \( E(\eta) = E \) and \( [\beta(\epsilon)]^2 = 1 - 1/(x + 1) \) relates the velocity \( \beta \) to energy.

1. Nonrelativistic limit.—For nonrelativistic particles \( \epsilon \ll 1 \) and \( \angle \approx \angle_c \). For \( \angle_c \ll 1 \), which will be the case for these particles because they lose most of their energy by \( \tau_c \approx \eta \approx \angle_c \),

\[ E(\eta + \tau_c) = E \sqrt{1 + \tau_c/E^2}, \]  

(12)

and

\[ \angle(\epsilon, \tau_c) = \angle \ln (1 + \angle_c/E^2). \]  

As shown by LP, this solution is a good approximation up to very large values of \( \tau_c/E^2 \) and for injected pitch angle distributions with values of \( \angle_0^2 \) up to 0.40, much larger than expected considering the assumptions made.

2. Relativistic limit.—For the extreme relativistic \( (\epsilon \gg 1) \) electrons, we have \( \eta \approx \epsilon \) and \( E(\eta + \tau_c) = \epsilon + \tau_c \). The diffusion in pitch angle is small and according to equations (9) and (10), the
dispersion $a_t^2$ does not change appreciably with depth:

$$a_t^2 = a_0^2 + \frac{4\epsilon\tau_e}{E(1 + \tau_e)}.$$  (13)

This implies an increase in dispersion with depth from $a_0^2$ to $a_0^2 + 4\epsilon_E/E$ which is a small effect, except for highly beamed injection with $a_0^2 E \ll 1$.

Equation (13), however, often overestimates the dispersion in pitch angle. We obtained the approximate solution equation (8) by setting $\mu = 1$ in front of the $\delta \Phi / \delta \tau$ term on the left-hand side of equation (1). This is reasonable for the nonrelativistic case where the neglected term is of order $a^2$ and is insignificant

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Thus for relativistic electrons we need a more accurate treatment of the $\mu \delta \Phi / \delta \tau$ term. As shown in § I of the Appendix, in this limit the diffusion term can be treated as a perturbation leading to the approximate solution (A4) for extreme relativistic electrons and all angles,

$$F(E, \mu, \tau_e) = \left( \frac{\beta(E)}{\beta(\tau + \tau_e/\mu)} \right)^2 F_0 \frac{E + \tau_e}{\mu} G(\mu),$$  (14)

and to a more accurate solution for intermediate and high energies but for the small–pitch angle regime (eq. [A12]):

$$F(E, \mu, \tau_e) = \left( \frac{\beta(E)}{\beta(\tau + \tau_e/\mu)} \right)^2 \frac{2e^{-x_\mu x_\mu^2}}{x_\mu^2} F_0 \frac{E(\eta + \tau_e)}{\eta},$$  (15)

where $a_\mu^2$ is given by equations (9) and (10) and the dispersion as a function of depth is

$$\tilde{a}_\mu^2 = \frac{a_\mu^2}{[1 + \delta_\mu + \tau_e/2(\eta + \tau_e)]} < a_\mu^2.$$  (16)

Here $\delta_\mu = \frac{d}{d\tau} F_0/E(\mu) + \frac{d}{d\tau} E(\tau + \tau_e)$ is equal to the spectral index $\delta$ if $F_0$ is a power law. Note that equations (15) and (16) reduce to the nonrelativistic limits of equations (8)–(12) in the proper limit $\eta \sim \tau_e \ll 1$. Hence we may use this corrected solution for all energies.

Table 1 gives the values for $\tilde{a}_\mu^2$ and $a_\mu^2$ along with $a_0^2$, the dispersion obtained from numerical solutions of equation (1) including only the collision terms, but without the small–pitch angle approximation, for four values of $a_0^2$ at $\tau_e/\eta = 1$. For $a_0^2 = 0.04$, the diffusion effects are more important (4/E0^2 ≈ 5 and $\delta a_0^2/2 \approx 0.10$). Consequently, the dispersion increases with depth, while for $a_0^2 = 0.40$ the reverse is true (4/E0^2 ≈ 0.5 and $\delta a_0^2/2 \approx 1.0$) and the dispersion decreases. In all cases except for isotropic injection ($a_0^2 = 40$), $\tilde{a}_\mu^2$ provides an excellent approximation to the dispersion $a_\mu^2$. For $a_0^2 \geq 1$, it is obvious that $a_\mu^2$ is not a good approximation, but $\tilde{a}_\mu^2$ provides a reasonable approximation.

3. Flux integrated over pitch angle.—In certain problems, for example those with straight magnetic field lines, and processes with isotropic cross sections, knowledge of the pitch angle distribution is not necessary. We define total flux of electrons of a given energy at a given depth to be $F_\mu(E, \tau_e) \equiv \int_0^\infty F(E, \mu, \tau_e) d\mu$ and integrated the solutions from equation (15) to obtain

$$F_\mu(E, \tau_e) = \frac{\beta^2 E(\eta + \tau_e)}{\beta^2 [E(\eta + \tau_e)][1 + \delta_\mu + \tau_e/2(\eta + \tau_e)]}.$$  (17)

which, as shown by LP, is independent of $\tilde{a}_\mu^2$ for nonrelativistic electrons for all values of $a_0^2$. For relativistic electrons, $F_\mu$ depends on $\tilde{a}_\mu^2$ due to the presence of $a_0^2$ in the correction to the dispersion in equation (16).

The qualitative behavior of $F_\mu$ is similar for all energies. The flux is constant from $\tau_e = 0$ to $\tau_e \sim \eta$ and decreases afterward with increasing depth. For a power-law–injected flux, $F_0(E) = KE^{-\delta}$, we find

$$F_\mu \approx \begin{cases} KE^{-\delta}/(\eta + \tau_e) & \text{for } E \ll 1, \\ K(\eta + \tau_e)^{-\delta}/[1 + \delta_\mu + \tau_e/2(\eta + \tau_e)] & \text{for } E \gg 1. \end{cases}$$  (18)

For large values of $E$, the flux falls off as $\tau_e^{-\delta}/(\eta + \tau_e)$ for $E \ll 1$ and as $\tau_e^{-\delta}$ for $E \gg 1$.

4. Spatially integrated flux.—For a magnetic field which is both uniform and straight, a useful quantity is $F(E, \mu)$, the flux integrated over column depth, which is defined as $F(E, \mu) \equiv \int_0^\infty F(E, \mu, \tau_e) d\tau_e$. This quantity can be sufficient for the study of
spatially unresolved sources. The zeroth-order approximation, equation (14), integrated over \( \tau \) gives

\[
F(E, \mu) = \mu G(\mu) \left( \frac{b}{E} \right)^{\beta} F(0) dE.
\]

We cannot obtain an analytic expression for \( F \), by integrating equations (8) or (15) due to the complex \( \tau \) dependence of the dispersions \( \sigma_E^2 \) and \( \sigma_\beta^2 \). It is much easier to integrate the original equation over \( 0 < \tau < \infty \). The integral of the source term \( \Sigma \) is equal to

\[
\int_0^\infty \Sigma d\tau = c F(\mu) = c b^2 \Phi(\mu),
\]

and the equation becomes

\[
\beta^2 \gamma^3 \frac{\partial \Phi}{\partial t} + \xi \frac{\partial}{\partial t} \left( 1 - \mu^2 \right) \frac{\partial \Phi}{\partial \mu} = \mu b^4 \gamma^2 \Phi = 0,
\]

where \( \Phi(\mu, t) \equiv \int_0^\infty \Phi(\mu, \tau, E) d\tau \).

For relativistic energies, the pitch angle diffusion term is small, and as above we can treat it as a perturbation. We expand \( \Phi \) in terms of \( 1/E \) and include the first-order correction due to diffusion. The zeroth-order solution for the flux is (since \( F_x = c \Phi \), for \( E \gg 1 \))

\[
F_x(E, \mu) = \mu \int_0^\infty F(\mu, \mu) dE',
\]

and the first-order solution is

\[
F(E, \mu) = \mu \int_0^\infty F(\mu, \mu) dE' + \xi \frac{\partial}{\partial \mu} \left( 1 - \mu^2 \right) \frac{\partial F_x}{\partial \mu} \int_0^\infty \left( \frac{dE'}{E'} \right)^2 F(\mu, \mu) dE'.
\]

For \( F_0(\mu, \mu) \), this gives

\[
F_0(\mu, \mu) = \mu G(\mu) \int_0^\infty \Phi(\mu, \mu) d\tau,
\]

and including synchrotron losses (\( d \ln B/ds = C = 0 \)) is given in § II of the Appendix. For the steady state case with continuous injection, the result is given by equation (A24). This solution is valid for electrons of all energies, but it takes a simple form for relativistic energies. The behavior of nonrelativistic electrons is qualitatively similar to that of relativistic electrons.

1. Relativistic limit.——The relativistic limit of equation (A24) is given by equation (A30), which for an injected flux of the form \( F(\mu, 0) = F_0(\mu, 0) \), the equation reduces to

\[
F(\mu, s) = \frac{F_0(1 - \mu^2)}{(1 - \mu^2)^{1/2}} G(\mu),
\]

where we have defined a dimensionless depth \( \tau = s \). At a given pitch angle, \( F \rightarrow 0 \) at \( \tau = \tau_{\text{crit}} \) which reduces with increasing energy, so that higher energy particles are stripped from the beam at smaller depths. The depth \( \tau_{\text{crit}} \), corresponding to \( \tau_{\text{crit}} \) also decreases with increasing magnetic field strength through the \( B^2 \) dependence of \( S \).

At a given \( \tau_{\text{crit}} \), the flux becomes zero at a critical pitch angle \( \alpha_{\text{crit}} = \cos^{-1}(\mu_{\text{crit}}) \) given by

\[
\mu_{\text{crit}}(\tau_{\text{crit}}) = \frac{1}{2\tau_{\text{crit}} E} \left( \sqrt{1 + 4\tau_{\text{crit}}^2 E^2} - 1 \right).
\]

Note that \( \mu_{\text{crit}} \) increases as \( \tau_{\text{crit}} \) increases and approaches 1 as \( \tau_{\text{crit}} \rightarrow \infty \). Electrons with higher pitch angles are stripped from the beam as the depth increases.

In order to see the initial trend of the distribution, it is instructive to consider the small–pitch angle regime for \( \tau_{\text{crit}} \). This is carried out in Appendix E. In this case, we let \( \mu = 1 - \alpha^2/2 \) and \( G(\mu) = 2\alpha^2 e^{-2\alpha^2/\gamma^2} \), to obtain

\[
F(\mu, s) = 2 \alpha^2 F_0(E) e^{-2\alpha^2/\gamma^2},
\]

where (unless \( \delta < 1 \))

\[
\alpha^2 = \alpha_0^2 (1 + (\delta - 1)\alpha_0^2 \tau_{\text{crit}} E)^{-1},
\]

with \( \delta = d \ln F_0(d) / d \ln E \). The dispersion \( \alpha^2 \) decreases with increasing energy, depth, and magnetic field. The effects of the pitch angle term and the first-order correction to the extreme
relativistic approximation will add terms of order $1/E$ in the square brackets.

2. Spatially integrated flux.—The general expression for the flux integrated over depth is given by equation (A31). In the relativistic limit, we let $\beta \rightarrow 1$ and $\beta' \rightarrow 1$, which gives

$$F_\lambda(E, \mu) = \frac{\mu G(\mu)}{(1 - \mu^2)^2} \int E F_0(E')dE'$$

$$= \frac{K \mu G(\mu)}{(1 - \mu^2)^{\delta+1}(\delta - 1)},$$

(34)

where the second equality is for the power-law–injected flux. (Note that this expression may also be obtained by integration of eq. [29] over $\tau_c$.) For a given pitch angle, we have a spectral index of $(\delta + 1)$ for this case.

3. Total energy spectrum.—Integration of $F_\lambda$ over pitch angle (or $F_\lambda$ over depth) will give the total energy spectrum. However, if $\lim_{\mu \rightarrow 1} G(\mu) \neq 0$ (e.g., isotropic injection), the resulting expression diverges. This is because electrons with zero pitch angle never lose energy or change pitch angle; thus, with a continuous injection, there will be an infinite number of terms from $0 < \delta < \infty$. This divergence disappears if as $\mu \rightarrow 1$, $G(\mu) \rightarrow (1 - \mu^2)^2 \epsilon > 0$ and the total spectrum will be the same as that in equation (34). The divergence will also be absent in the more realistic case of finite injection time or when collisions are included.

c) Synchrotron and Collisional Losses

We need to consider both synchrotron and collisional losses when the ratio of these losses ($R$ in eq. [4]) is near unity. For nonrelativistic electrons, $R \approx 1$ only when $B$ is large. For relativistic electrons, however, synchrotron losses can be important for moderate values of $B^2$ if the density is low. There is no analytic solution for the general case including the synchrotron and collisional energy losses. Analytic solutions are possible for relativistic electrons because, as we have seen in §§ IIA and IIB, Coulomb collisions and synchrotron radiation do not alter the pitch angles of relativistic electrons.

1. Relativistic limit.—In this limit we can ignore the last two terms in equation (1). The solution of this equation for uniform field and constant plasma density (i.e., $E = 1$, $d \ln B/ds = 0$, and constant $R_0$) is given by equation (A42), which reduces to

$$F(E, \mu, \tau_c) = F_0(E_\ast(E, \mu, \tau_c)) G(\mu) \frac{1 + E_\ast(E, \mu, \tau_c)/\epsilon^2}{(1 + E_\ast/\epsilon^2)},$$

(35)

where

$$E_\ast(E, \mu, \tau_c) = E \left[ 1 + (\epsilon_c/E) \tan (\tau_c/\epsilon_c) \right] / \left[ 1 - (E/\epsilon_c) \tan (\tau_c/\epsilon_c) \right],$$

(36)

and $\epsilon_c^{-2} = R_0(1 - \mu^2)$.

Note that equation (36) is valid only for $E \tan (\tau_c/\epsilon_c) \leq \epsilon_c$. At a given pitch angle, $F \rightarrow 0$ at a depth given by

$$\tau_c = E \tau_{scr}(E, \mu) = E \tau_{scr} \tan^{-1} \frac{\epsilon_c}{E},$$

(37)

where $\tau_{scr}$ is defined in equation (30). As in the synchrotron-dominated case, for a given energy $\tau_{scr}$ increases as $\mu$ increases, becoming infinite at $\mu = 1$. Then particles with high pitch angles are stripped away, and the distribution narrows as depth increases. In the limit $\epsilon_c \ll E$ (for $R_0 \gg 1$), synchrotron losses dominate, and the flux and the critical depth reduce to equations (29) and (30), respectively. In the opposite limit ($R_0 \ll 1$, $\epsilon_c \gg E$) collisional losses dominate, and equation (35) reduces to equation (14) as it must.

2. Spatially integrated flux.—We cannot integrate the flux given in equation (36) over pitch angle due to the complex $\mu$ dependence in $E_\ast$ and $\epsilon_c$, but it is straightforward to integrate the flux over depth. We find

$$F_\lambda(E, \mu, \tau_c) = \int_0^\infty ds F(E, \mu, s) \frac{\mu G(\mu)}{(1 + E^2/\epsilon^2)}. \frac{dE}{d\tau_c}$$

(38)

where the last relation is for the power-law–injected flux. In the two limits $R_0 \gg 1$ and $R_0 \ll 1$, this equation reduces to the expressions in equations (34) and (26), respectively.

3. Total energy spectrum.—If the injected distribution is narrow (i.e., $\tau_0^2 \ll 1$) we can integrate equation (38) over pitch angle and obtain $F_{\lambda \gamma}$. We find

$$F_{\lambda \gamma}(E) = \frac{K e^{1 + \ln R^2 s^2}}{(\delta - 1) \alpha_0^2 E^{\delta - 1}} E^{\alpha - 1}$$

$$\times \left[ \ln (R_0 E^2 \alpha_0^2) - 0.577 + \sum_{k=1}^\infty \frac{-R_0 E^2 \alpha_0^2}{kk!} \right],$$

(39)

which reduces to

$$F_{\lambda \gamma}(E) \approx \begin{cases} 
K (\delta - 1) R_0 \alpha_0^2 E^{\delta - 1} \ln \left( R_0 E^2 \alpha_0^2 \right), & R_0 E^2 \alpha_0^2 \leq 1, \\
(\delta - 1) R_0 \alpha_0^2 E^{\delta + 1} \ln \left( R_0 E^2 \alpha_0^2 \right), & R_0 E^2 \alpha_0^2 \gg 1.
\end{cases}$$

(40)

Thus we have the expected spectral index for the collision-dominated case at low energies and the index for the synchrotron-dominated case (slightly modified due to the logarithmic term) for high energies, provided that $R_0 \alpha_0^2 \sim 1$. This modification is due to the fact that collisional losses dominate for electrons with very small pitch angles, $\alpha_0^2 \ll 1/R_0 E^2$.

d) Nonuniform Field ($d \ln B/ds \neq 0$)

Next we consider a nonuniform field for which $d \ln B/ds \neq 0$. We have no solution including collisional and/or synchrotron effects and a nonuniform field. (Ho 1986 has given numerical solutions of the equations of motion for the case including synchrotron losses and converging fields, but he has not solved the kinetic equation.) A solution for the case with $C = S = 0$ in equation (1) was given in LP for the flux per unit area of the loop. Our solution, which is integrated over the cross-sectional area of the loop, is different by a factor of $B_0^2/B(s)$, proportional to the change of the area with depth. In the absence of other effects, $B/(1 - \mu^2)$ is a constant which leads to the solution

$$F(E, \mu, s) = F_0(E) G \left[ \sqrt{1 - (1 - \mu^2) B_0^2/B(s)} \right],$$

(41)

which for $G(\mu) = 2\alpha_0^2 e^{-(1 - \mu^2)/\alpha_0^2}$ becomes

$$F(E, \mu, s) = \frac{2F_0(E)B_0}{\alpha_0^2 B(s)} \exp \left[ -\frac{(1 - \mu^2)}{\alpha_0^2 B(s)} \right].$$

(42)
At any point $s$, the distribution has a dispersion given by $\frac{a_{2}B(s)}{B_0}$, it is broadened by a factor of $B(s)/B_0$. The flux at $\alpha = 0$ is simultaneously decreased by a factor of $B_0/B(s)$, so that the number of electrons at a given depth integrated over pitch angle and area is constant with depth, as should be the case for zero energy losses. This result is independent of energy, and therefore the demonstration by LP of the accuracy of the numerical code remains valid.

1) Integrated fluxes.—We can integrate the solution given in equation (42) over pitch angle to obtain

$$F_{\alpha}(E, s) = F_0(E) . \tag{43}$$

It is clear that the energy dependence of the total flux $F_{TOT}$ will be the same as that for $F_0$.

III. NUMERICAL RESULTS

We now describe results from numerical solutions of equation (1). To solve the equation, we must specify the parameters of the background plasma (density and magnetic field) and the distribution of the injected electrons. We assume that electrons with the distribution given by equation (5) are injected at the top, $s = 0$, of a symmetric magnetic flux tube and solve the equation only for $s > 0$. We shall use both beam ($\alpha_1 = 0$) and pancake ($\alpha_1 = \pi/2$) distributions, but we note that the latter distributions may be inherently unstable and require acceleration perpendicular to the field lines. Electrons with $\mu < 0$ at $s = 0$ are reflected back into the flux tube to simulate the symmetric geometry. Thus the total flux at $s = 0$ is equal to $F_{\alpha}(E, 0) + F(E, -\mu, 0)$. The knowledge of the geometry of the flux tube is not necessary here but will be essential for the evaluation of the angular and spatial dependence of the emitted radiation.

For our purpose here, all we need are the values of the coefficients $d \ln B/ds$, $S$, and $C$, which can be obtained from the variation of the density and magnetic field with depth $s$, $B(s)$, and $n(s)$. If $B$ and $n$ are constant, only five constants $\delta$, $\alpha_1$, $a_0$, $n_0$, and $R_0 = S/C$, are needed for the solution. Unless otherwise specified, we assume a fully ionized hydrogen plasma with $\xi = 1$.

a) Uniform Density and Field

The parameter which determines the behavior of the electrons here is the quantity $R$ defined in equation (4). This ratio depends on $R_0$ and the energy and pitch angle of the electrons. We describe the effects of all these parameters by considering plasmas with different values of $R_0$ and discuss the spectral and angular distributions at different $\mu$ and $E$, respectively. We shall limit our discussion to electrons with energies between 10 keV and 100 MeV.

i) Collision-dominated Models

For $R_0 \ll 1$, so that $R_0 \beta^2 \gamma^2 \ll 1$ even for the highest energies of interest, collisions dominate. Numerical results for nonrelativistic electrons were given in LP and Leach (1984) and will not be reproduced here. We simply point out that the dispersion in pitch angle depends on the ratio $\gamma/\eta \approx \gamma/E^2$ for energies $\lesssim 300$ keV.

For higher energies, such a simple scaling is no longer valid, but a good approximation to the flux is given by equations (15) and (16). This is shown in Figure 2 at $\tau_0 = \eta$ for a model with $R_0 = 3 \times 10^{-3}$, $\delta = 5$, and $a_0 = 0.40$ for 16 keV, 1 MeV, and 10.6 MeV electrons. The solid lines are the numerical results for the pitch angle distribution of the electron flux $F(\alpha)/F(\alpha = 0)$, and the circles denote the analytic results (eqs. [15] and [16]), i.e., for the same model in the limit $R_0 = 0$. The pitch angle distributions broaden with increasing depth due to diffusion except for $E = 10$ MeV, where diffusion is small and almost overshadowed by the higher order pitch angle term discussed in § IIa. For all energies, the distributions are close to the analytic approximations even for angles near $\pi/2$.

In Figure 3 we show the same curves for the injected pancake distribution $\alpha_1 = \pi/2$ and $a_0 = 0.40$. Here we have no analytic approximations. As shown by this figure, the pancake character (i.e., the maximum at $\pi/2$) is lost very quickly for low
energies (at $\tau_c = \eta$). Even for the highest energy shown (10 MeV), the maximum is shifted to $x \approx \pi/4$ at $\tau_c = \eta$.

1. Spatially integrated flux, $F_x$—In Figure 4 we show $F_x$ normalized to unity at zero pitch angle for energies 300 keV and 10.6 MeV. The solid lines depict beam injection ($x_1 = 0$) with $x_2 = 0.40$, and the dashed lines depict $x_1 = \pi/2$ and $x_2 = 0.40$. Comparison of these curves with those in Figures 2 and 3 shows that for the beam injection, the flux at $\tau_c = \eta$ is a good representation of the total angular distribution $F_x$. For the pancake injection model, the maxima at both energies occur at slightly larger pitch angles for the integrated fluxes than for the flux at $\tau_c = \eta$.

2. Pitch angle integrated flux, $F_\beta$—In Figure 5 we plot $F_\beta$ versus normalized column depth $\tau_c/\eta$ for 300 keV, 1 MeV, and 10.6 MeV electrons for the beam injection model with $x_2 = 0.40$ and $\delta = 5$, and $x_2 = 0.40$, for 300 keV and 10.6 MeV.

As $R$ increases, the synchrotron losses become more and more important, and for large enough fields, they dominate even for nonrelativistic electrons. As an example, we examine models with $R_0 = 3 \times 10^3$ (or $B^2/\eta \approx 1$), which corresponds to $B = 10^4$ G and $n \approx 10^8$ cm$^{-3}$, an extreme but unlikely condition for solar flares, or to $B \approx 10^{12}$ G and $n \approx 10^{24}$ cm$^{-3}$, which may be representative of conditions in the magnetosphere of a neutron star. Except for very small pitch angles, synchrotron losses dominate for the entire range of energies considered here ($R \approx 1$ at 1 keV and $x = \pi/4$), and the distributions behave as we expect from the solution including only synchrotron losses, i.e., equation (A24), which reduces to equation (29) for $E \gg 1$. In Figure 6, we compare the numerical results (lines) with the analytic ones (circles) for 300 keV and 10.6 MeV electrons at a depth of $\tau_c = 1$ for a source distribution shown by the dashed line. The behavior of the nonrelativistic electrons is qualitatively similar to that of the relativistic electrons; as the depth increases, the distribution narrows, and for a given depth, $F \rightarrow 0$ at some critical angle $x_{cr} = \cos^{-1}(\mu_{cr})$. (See eq. (31).) Note that the numerical results do not quite agree with the analytic results on the value of the critical angle. This is due to the finite size of the angular grid used for the numerical calculations; the difference between the values of the
The angular width of the distribution decreases exponentially:
\[
\alpha^2 \approx 4\beta^2 \gamma^{-2} \Delta \approx 2\alpha_\Delta \gamma
\]
so that the flux integrated over pitch angles, \( F_\mu \propto F(\alpha = 0)\alpha^2 \), is finite.

The analytic solution breaks down not only because the effective value of the ratio \( R \) decreases as the critical angle approaches zero, but also because the gradient of the pitch angle distribution increases, and collisional diffusion can no longer be ignored. From equation (1), the ratio \( r_D \) of the diffusion term to the energy loss terms (both collisional and synchrotron) is of order
\[
r_D = \zeta E[\beta^2 \gamma^2 \alpha^2 (1 + \beta^2 \gamma^2 R_0 \alpha^2)]^{-1},
\]
so that for pitch angles less than
\[
\alpha_D = \left[ \frac{\sqrt{4\beta E \zeta R_0} + 1}{2\beta^2 \gamma^2 R_0} \right]^{1/2},
\]
r_\Delta exceeds unity, and the analytic results are not valid. The depth at which \( \alpha_D = \alpha^2 \) is thus
\[
\tau_{ad} = \frac{\beta \gamma}{2} \ln \frac{4}{\beta^2 \gamma^2 \alpha_D^2}.
\]

Critical Values for Dispersion and Depth

<table>
<thead>
<tr>
<th>Parameter</th>
<th>16 keV</th>
<th>300 keV</th>
<th>1 MeV</th>
<th>10.6 MeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_D )</td>
<td>0.09</td>
<td>0.01</td>
<td>3.5 x 10^{-3}</td>
<td>1.80 x 10^{-4}</td>
</tr>
<tr>
<td>( \tau_{ad} )</td>
<td>0.83</td>
<td>3.40</td>
<td>7.00</td>
<td>42.0</td>
</tr>
</tbody>
</table>

For depths beyond \( \tau_{ad} \), we do not expect the analytic solution to be useful. Table 2 gives values for \( \alpha_D \) and \( \tau_{ad} \) for various energies and \( R_0 = 3000 \).

Note that \( \alpha_D \) increases and \( \tau_{ad} \) decreases with decreasing energy. For 16 keV electrons, the analytic solution is incorrect even at \( \tau = 1 \), while for 10 MeV electrons the solution may be used for large values of \( \tau \). This behavior is evident in Figure 7. At \( \tau = 5 \) and 10, the numerical solutions fall away from the analytic solutions for energies less than 2 and 5 MeV, respectively.

Models with Intermediate \( R_0 \)

Next we consider a model with \( R_0 = 1.3 \) (\( B^2/n \sim 4 \times 10^{-4} \)), so that synchrotron losses are important for energies greater than the rest mass energy, and collisions dominate for \( E \lesssim 300 \) keV. For low energies, the scaling of the pitch angle distribution with \( \tau/E^2 \) described in § IIId(i) above is still valid. For relativistic electrons with energies \( \gtrsim 5 \) MeV, the synchrotron energy losses dominate, and we obtain distributions similar to those expected from equations (29) and (30). According to these equations, the pitch angle dispersion will be the same (or curves of \( F \) vs. \( \alpha \) will have the same shape) for depths \( \tau \sim 1/E \).

We have found that the numerical results agree with the analytic values of \( \alpha_D^2 \) (eq. [30]) to within 10% for \( \tau, \quad E = 1 \).

In Figure 8 we plot the integrated flux distributions \( F/F(\alpha = 0) \) for the beam (solid lines) and pancake (dashed lines) distributions with \( \alpha_D = 0.40 \) at 300 keV and 10.6 MeV. The curves are intermediate between those given in Figures 4 and 6 with the effect of synchrotron losses evident 10.6 MeV.
and the effects of collisions dominant at 300 keV. Note that the pancake injection model still has a nonzero maximum, but very few electrons with \( \alpha > \pi/2 \).

In Figure 9 we plot \( F_\theta(E, \tau) \), the flux integrated over pitch angles, versus energy for the beam and pancake distributions with \( z_0^2 = 0.40 \) at \( \tau_0 = 0.1 \) and 1.0. At larger depths, the curves fall off at low energies due to increasing collisional losses with decreasing energy, and they steepen at high energies due to increasing synchrotron losses with energy. The synchrotron losses for the pancake distribution are more noticeable because a larger proportion of the electrons have high pitch angles and therefore higher synchrotron losses.

iv) Spectral Index and Curvature

We now discuss the spectral index of the spatially integrated flux \( F_\theta(E, \mu) \). In Table 3 we give the spectral indices, slopes of power-law fits of \( F_\theta(E) \) versus \( E \), for different fields. For energies \( \lesssim 300 \) keV, we find essentially the same indices with \( m_\tau = 1.3 \). For \( \tau = 3 \times 10^{-3} \) and \( R_0 = 1.3 \), we find different behavior for the two cases. For relativistic energies (10–76 MeV), the index is \( \sim \tau - 1 \) for \( B = 0 \) and increases to slightly less than \( \tau + 1 \) for \( B \gtrsim 2000 \) G. Thus our analytic results give accurate answers for the total flux, which is important for spatially unresolved sources.

For the pancake injection, the results are the same for the nonrelativistic and semirelativistic ranges, but as was the case for the other integrated fluxes, the spectral indices are larger for relativistic energies.

b) Uniform Density, Converging Fields

We now turn to models which have nonuniform (in particular, converging) magnetic fields. This produces two changes. First, since \( \ln B(b)/ds \) is not zero, we must include the second term in equation (1). Second, the coefficient \( S \) varies along the field lines so that \( R_0 \) is no longer constant. We parameterize these models with \( R_0 \), the value of the magnetic field at \( s = 0 \), and the parameter \( S_b = \ln B(b)/ds \), which we assume to be a constant; \( b(s) = B(s)/B_0 = \exp (ss_b) \). The effects of convergence become important when \( S_b \) is of order of the coefficients \( C \) or \( S \) for collisional and synchrotron losses, or when the dimensionless column depth \( \tau_s = (S_b ds) \), which in this case is equal to \( \ln b \), is greater than \( \tau_0 \) or \( \tau_\tau \). Since we have assumed constant values for \( S_b \) and \( C, \) the relative importance of these two effects remains constant along the field lines but varies with the energy and pitch angle of the electrons, convergence becoming more important for higher energies and larger pitch angles. The coefficient \( S \), however, varies along the field lines (\( S \approx S_0 b^3 \)) so that the relative importance of the synchrotron losses increases with depth and, of course, with energy. Thus at sufficient depths. For high energies (10–76 MeV), the index is \( \sim \delta - 1 \) for \( B = 0 \) and increases to slightly less than \( \delta + 1 \) for \( B \gtrsim 2000 \) G. Thus our analytic results give accurate answers for the total flux, which is important for spatially unresolved sources.

The coefficient \( S \), however, varies along the field lines (\( S \approx S_0 b^3 \)) so that the relative importance of the synchrotron losses increases with depth and, of course, with energy. Thus at sufficient

### Table 3

<table>
<thead>
<tr>
<th>( R_0 )</th>
<th>( m_\tau )</th>
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<tbody>
<tr>
<td>3 \times 10^{-3}</td>
<td>3.20</td>
<td>3.91</td>
<td>4.05</td>
<td>4.05</td>
<td>4.05</td>
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</tr>
<tr>
<td>3 \times 10^{-3}</td>
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<td>3.94</td>
<td>4.92</td>
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<td>4.92</td>
<td>4.92</td>
</tr>
<tr>
<td>1.3</td>
<td>3.20</td>
<td>4.86</td>
<td>5.80</td>
<td>5.80</td>
<td>5.80</td>
<td>5.80</td>
</tr>
<tr>
<td>3 \times 10^{3}</td>
<td>3.50</td>
<td>6.11</td>
<td>6.02</td>
<td>6.02</td>
<td>6.02</td>
<td>6.02</td>
</tr>
</tbody>
</table>

Note.—\( m_\tau \) from 16 to 300 keV, \( m_\tau \) from 600 keV to 5.32 MeV.

### Table 4

<table>
<thead>
<tr>
<th>( R_0 )</th>
<th>( m_\tau )</th>
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<td>6.02</td>
<td>6.02</td>
<td>6.02</td>
<td>6.02</td>
</tr>
</tbody>
</table>
ciently large values of depth $s$ or energy $E$, $\tau_s = (S_0/S_s)b^2 - 1)/2$ exceeds $\tau_s$ and synchrotron losses become dominant.

i) Strong Convergence and Loss Cones

An important effect of the field convergence (when $S_b \gg C$ and $S$) is to produce pitch angle distributions with distinctive "loss cones" in the direction of the field lines. We have no analytic solution which accounts for convergence along with synchrotron radiation and collisions, so we present numerical results only.

To illustrate the interplay between these processes, we consider a model with isotropically injected flux (i.e., $\sigma_0^2 = \infty$), with $C = 2 \times 10^{-13} \text{ cm}^{-1}$, $S_0 = 6.5 \times 10^{-16} \text{ cm}^{-1}$ (e.g., $B_0 = 100 \text{ G}$, $n_0 = 10^{10} \text{ cm}^{-3}$), and $S_b = 9 \times 10^{-10} \text{ cm}^{-1}$, which is large enough to ensure that the effects of convergence dominate over synchrotron and collisional losses for $150 \text{ keV} \lesssim E \lesssim 100 \text{ MeV}$. The pitch angle distributions at the top of the loop for $16 \text{ keV}$, $1 \text{ MeV}$, and $10.6 \text{ MeV}$ electrons are shown in Figure 10 for a model with finite length $s_{\text{max}} = 2.4 \times 10^4 \text{ km}$ or $s_{\text{max}} = 2.3$. Electrons reaching $s > s_{\text{max}}$ are ignored. In the absence of collisional diffusion, electrons with initial pitch angles $\alpha_{\text{in}} < \alpha_{\text{cr}}$ with $\sin \alpha_{\text{cr}} = h_{\text{max}}^{1/2} / s_{\text{max}}$ reach $s_{\text{max}}$ and escape. The $10.6 \text{ MeV}$ curve in Figure 10 shows this rapid decrease in electron flux $\alpha \lesssim 27^\circ$ and $\alpha \gtrsim \pi - 27^\circ$.

This effect is less pronounced at lower energies due to collisional diffusion (as shown by the $16 \text{ keV}$ curve in Fig. 10) and also for large magnetic fields and high energies due to the effects of synchrotron losses. This is shown by the dashed line in Figure 10 for $10.6 \text{ MeV}$, but with $S$ increased by a factor of 40.

ii) Weak Convergence, Relativistic Electrons

In some situations, the presence of convergence can have significant consequences even when convergence is not dominant. As an example, we consider collision-dominated models with $C = 2 \times 10^{-6} \text{ cm}^{-1}$ and $S_b = 3.7 \times 10^{-9} \text{ cm}^{-1}$. In spite of the fact that $S_b \ll C$, the effect of the convergence term is approximately equal to that of the pitch angle diffusion term for relativistic electrons and can have a significant influence in broadening the pitch angle distribution when collisions are ineffective in doing so. In Figure 11 we compare the distributions of the uniform ($U$) and converging ($C$) field models with isotropic injection, $E = 10.6 \text{ MeV}$, $\tau_e = \eta$, and two values of $R_0$. Synchrotron losses dominate for $R_0 \approx 1$, so that there are few electrons with high pitch angles and the convergence has essentially no influence. For $R_0 \ll 1$, however, there are some electrons with high pitch angles at this depth, and the effect of field convergence is to increase the number of such particles even further. This effect is particularly noticeable at higher energies, but not too high an energy so that synchrotron losses dominate. Even a small amount of convergence results in a substantial number of these reflected electrons as shown by the increasing divergence of the $C$ and $U$ curves in Figure 11 for pitch angles $\alpha > \pi/2$.

To further illustrate this effect, in Table 3 we give the values of spectral indices from power-law fits to the spatially integrated flux $F_\gamma$ for the cases with small convergence discussed above. The values of the spectral index $m_\gamma$ for low energies $16-300 \text{ keV}$ are similar to those of the uniform field models. At higher energies, convergence has little effect for $\alpha < \pi/2$, and the spectral indices are again similar. But for $\alpha > \pi/2$, the indices are much smaller for the converging field cases, since there are more reflected electrons.

c) Solar Flare Models

It is believed that during the impulsive phase of a solar flare, high-energy particles ($10 \text{ keV}$ to greater than $10 \text{ MeV}$) are accelerated in a coronal magnetic loop of length $10^2-10^5 \text{ cm}$, with density $n \approx 10^{10} \text{ cm}^{-3}$ and a magnetic field of a few hundred gauss (Kundu 1983; Lu and Petrosian 1989). Below the transition region, the density increases rapidly. It is not known how the magnetic field varies with depth, but it is
suspected to increase to a few thousand G at the photosphere. Above the transition region, the plasma is fully ionized \((C = C'\) or \(\xi = 1\)), but below the transition region, the temperature decreases and the atmosphere becomes neutral.

For \(n = 10^{10}\) cm\(^{-3}\) and \(B_0 = 100\) G, the quantity \(R_0 = 3 \times 10^{-3}\) at \(s = 0\) and could be as large as \(R_0 = 0.3\) at the transition region with \(x = 0.0014\) and \(B_0 = 10^3\) G [or \(x_0(s_0) = 0.07\)]. Thus, convergence may be important, but collisional and synchrotron losses are negligible except for \(E < 20\) keV and \(E > 50\) MeV, respectively. Below the transition region, density increases rapidly, and unless the magnetic field scale height begins to decrease rapidly, synchrotron and field convergence effects become negligible. In Figure 12 we plot the pitch angle distributions for 300 keV and 10.6 MeV electrons at the top of the loop (solid lines), and at \(x = \eta\) (dashed lines) for models with isotropic injection, \(\delta = 5\) and \(R_0 = 3 \times 10^{-3}\) (\(B_0 = 100\) G and \(n = 10^{10}\) cm\(^{-3}\)) at \(s = 0\) which increases to \(R_0 = 0.3\) at the transition region. As mentioned above, the convergence will be important in the corona and at the top, we have loss cone distributions. Below the transition region, collisions dominate, and we see typical behavior at both energies.

Not surprisingly, the integrated fluxes behave as in the uniform density collision-dominated cases. Unless the magnetic field is very high above the transition region, there are no discernible effects of synchrotron losses on the integrated fluxes. Since \(x_0(s_0)/\eta < 1\) for most of the energy range, the integrated fluxes \(F_0\) and \(F_{TOT}\) are dominated by the behavior in and below the transition region, and the integrated fluxes look like those for collision-dominated models.

One quantity which is readily available from observations is the energy spectral index. To show the effects of various parameters on this quantity, in Table 5 we give values for spectral indices \(m\) (for energies between 10.6 and 76 MeV) and \(m_s\) (between 16–300 keV) of the integrated flux \(F_0\), for models with \(\delta = 5\), including: uniform field models with beam and pancake injection with \(x_0(s_0) = 0.40\) and \(R_0 = 3 \times 10^{-3}\) and 1.3, models with isotropic injection, \(B_0 = 100\) G (\(R_0 = 3 \times 10^{-3}\) at \(s = 0\)), and mirror ratios of \(b_0 = 1, 2, 5,\) and 10; strongly beamed models with \(x_0 = 0.04\), \(B_0 = 100\) G, and \(b_0 = 2\) (to show how the effects of small convergence combine with narrow beaming); and models with \(x_0 = \infty\), \(B_0 = 2000\) G [\(R_0(\text{top}) = 1.2\)], and \(b_0 = 1\) and 10. For all the weak-field cases, even for low levels of convergence, and for the sharply beamed injection the values of \(\Delta m = m - m_s\) are close to zero for \(\alpha > \pi/2\). For the high-field case, the value of \(\Delta m\) is larger for \(\alpha > \pi/2\) and nearly as large as for the uniform weak field case. This implies that we can get a high field [here \(B(s_0) = 20,000\) G] can negate the effects of convergence.

Values of the spectral indices for the total integrated flux for these models are given in Table 6. For the cases with \(B_0 = 100\) G, we find that \(F_{TOT}(E)\) behaves as in the collision-dominated uniform field cases, since \(F_{TOT}\) is dominated by the behavior of the distribution below the transition region, where the collisional scale length is much smaller than the scale of convergence. For the model with \(B_0 = 2000\) G, increased synchrotron energy losses lead to larger slopes in all regimes.

**d) Gamma-Ray Burst Models**

Gamma-ray bursts are believed to occur in the magnetospheres of neutron stars with magnetic fields ranging from \(10^{12}\) G (as deduced from absorption lines) to less than \(10^{11}\) G (Matz et al. 1985) to allow high-energy \((E > \text{few MeV})\) \(\gamma\)-rays to escape.
escape without pair production. The power-law energy spectra observed in some bursts to energies greater than 30 MeV are indications of the presence of nonthermal electrons accelerated to similarly high energies. These conditions require the examination of the transport of the relativistic electrons in variable magnetic fields and possibly inhomogeneous plasmas reaching densities of as high as $10^{30}$ electrons cm$^{-3}$ at the surface of the neutron stars. All three effects on the electron transport of the relativistic electrons considered here (synchrotron, collisions and field convergence) may play significant roles in this process.

If the electrons are accelerated in the magnetosphere, where $R_0$ is expected to be very large, they suffer synchrotron losses. They quickly lose all their perpendicular momentum and slide along the field lines with near zero pitch angles. In this phase, the synchrotron loss formulae provide an accurate description of the problem up to some column depth. As shown in § III, for large values of $\tau_s$, collisional diffusion becomes important. Since the field lines are anchored to the neutron star, the electrons hit the surface, where they undergo pitch angle diffusion and emit more synchrotron or bremsstrahlung radiation if the density is very high. Equations dealing with both synchrotron and collisional processes will be applicable. It is unlikely that field convergence will play a significant role during this transport, because the electrons will have small pitch angles and the scale height of the convergence $S_{\phi}$ (of the order of the neutron star radius) is probably much larger than the scales $C^{-1}$ or $S^{-1}$ for collisional and synchrotron process.

One can consider the problem in two steps, starting with the magnetospheric part. Consider a loop structure in the magnetospheric part. The plasma at the surface is not an ordinary plasma. The electrons may be degenerate, and there are heavy ions. Since the ratio $\zeta$ of the diffusion coefficient to the energy loss coefficient is highest by a factor equal to the atomic number of the ions, we have the situation in which synchrotron energy losses and collisional diffusion are important. The pitch angle distribution of 1 MeV electrons for a model with a density of Fe ions of $10^{27}$ cm$^{-3}$ and a magnetic field of $10^{12}$ G (or $R_0 = 1.2$) is shown in Figure 13. Each curve is marked by the value of $\tau_s \approx \tau_x$ for that depth. The flux at $\tau_x = 0$ is a delta function, but the distribution is nearly isotropic even for small depths. Since this is true, when looking at any emission the important quantity is the total electron flux $F_{\text{tot}}(E)$. The spectral index at the top of the magnetospheric loop is $\delta = 3$, and the spectral indices for the delta function flux at the surface are $m_p = 2.3$, $m_m = 3.4$, and $m_e = 4.7$. The indices for the total flux are $m_p = 2.6$, $m_m = 4.4$, and $m_e = 5.9$. As we can see, this combination of effects (collisional diffusion plus synchrotron losses) results in steepening of the spectrum at every energy, and large breaks in the spectra between nonrelativistic and relativistic energies. Such breaks have been observed in the spectra of $\gamma$-ray bursts. A detailed comparison with observations is beyond the scope of this paper and will be treated in a subsequent work.

**IV. SUMMARY**

We have extended the Fokker-Planck analysis used in LP to include ultrarelativistic electrons and effects due to both collisions and synchrotron emission. We have solved the resulting

$$F_s(E, \tau_s \to \infty) = F_0(E)[(\delta - 1)\gamma^2 + \delta \gamma + 3]^{-1}. \quad (52)$$

If we include collisions, the flux at $\alpha = 0$ does not diverge, and the width remains finite but small. Since the width is too small to include on the pitch angle grid which we use for numerical solution, for the calculation of the flux below the surface of the star we inject all the flux given by equation (52) at $\alpha = 0$.

![Figure 13](image-url)

**Fig. 13.** Electron flux vs. pitch angle for 1 MeV electrons for a model with a density of Fe ions of $10^{27}$ cm$^{-3}$ and a magnetic field of $10^{12}$ G (or $R_0 = 1.2$). The curves are labeled by the value of $\tau_s \approx \tau_x$. The width decreases and the flux for zero pitch angle increases with $\tau_s$, but the flux integrated over pitch angle is finite and is given by
kinetic equation (eq. [1]) analytically in some simple limiting cases and have shown that the numerical results for these cases agree well with the analytic results, thus verifying the accuracy of our numerical code. We have shown that the analytic results provide a useful guide for the qualitative description of the flux distribution of the nonthermal electrons and examined the effects of nonuniformities in the magnetic field, in particular converging magnetic field geometry.

Some of the features of our results are:
1. In situations dominated by collisions, equations (15) and (16) give a fairly accurate description of energy spectra of both relativistic and nonrelativistic electrons and of the flux distribution for the Gaussian pitch angle distribution of accelerated electrons. Even though these equations are small pitch angle approximations they are good for much larger angles than expected. Equations (17), (24), and (28) give the flux integrated over pitch angle, depth, and both, respectively.
2. For problems where the synchrotron process is dominant, we have obtained a complete time-dependent solution (eq. [A21]) and a steady state solution (eq. [A24]). The relativistic limit of the latter acquires a simple form given by equations (29), (32), and (34).
3. If both synchrotron and collisional losses are important, then equations (35), (38), and (39) give the fluxes in the relativistic limit. Analytic solutions are not possible for the less likely situation of nonrelativistic energies with both synchrotron and collisional losses.

The above results are for uniform magnetic fields or are applicable when the magnetic field variation scale is much larger than the scales for collisional and synchrotron losses. For cases when this is not true, one must resort to numerical solutions as described in the text.

The discussion of the behavior of nonthermal electron distributions included in this paper will be important for our future work, in which we will use the numerical solutions to calculate the expected radiation during the impulsive phase for solar flares or cosmic gamma-ray bursts. Examples describing models for these bursts will be presented, and we shall compare the radiation signature of such models with observations and determine the distribution of the accelerated electrons responsible for those radiations.

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APPENDIX

ANALYTIC SOLUTIONS

We now present some approximate analytic solutions of equation (1).

1. COLLISION-DOMINATED SOLUTIONS

Consider equation (1) with \(d \ln B/ds = 0\) and \(S = 0\). We have (with \(dT_c/C ds\))

\[
\frac{\partial \Phi}{\partial \tau_c} - \frac{\partial \Phi}{\partial E} = \frac{\xi}{E^2} \frac{\partial \Phi}{\partial \mu} \left(1 - \mu^2\right),
\]

(A1)

where \(\xi = C/C (= 1\) for a fully ionized background plasma). The small–pitch angle solution for nonrelativistic electrons is given in the text. This solution (eq. [8]) cannot be extended to high energies because as \(E\) increases, the term ignored (of order \(a^2 d\phi_c/\partial s\)) becomes more important relative to the diffusion term. For extreme relativistic electrons, we may ignore the pitch angle diffusion term on the right-hand side of equation (A1), and we have

\[
\frac{\partial \Phi}{\partial \tau_c} - \frac{\partial \Phi}{\partial E} = 0,
\]

(A2)

which for injected \(\Phi(E, \mu, 0) = \Phi_0(E)G(\mu)\) has the solution

\[
\Phi(E, \mu, \tau_c) = \Phi_0(E + \tau_c/\mu)G(\mu).
\]

(A3)

For relativistic energies, \(\beta = 1\) and \(F = c\Phi\). In the small–pitch angle regime and for a Gaussian pitch angle distribution \(G(\mu) = 2\sigma_\Phi^2 e^{-\mu^2/\sigma_\Phi^2}\), equation (A3) gives

\[
F(E, \mu, \tau_c) = F_0(E + \tau_c) \left[1 + \delta_c \tau_c \sigma_\Phi^2 \right]^{-1},
\]

(A4)

where \(\delta_c = d \ln F_0(E)/d \ln E|_{\tau_c}\) and is equal to the spectral index when \(F_0\) is a power law. Note that the term in the brackets which is of order \(\delta_c \sigma_\Phi^2\) has an effect which is opposite of the diffusion term; namely, that dispersion decreases with increasing depth \(\tau_c\).

For lower energies or smaller \(\sigma_\Phi^2\), the diffusion term becomes important, and in the intermediate energies, it increases the dispersion with depth at a rate determined by the value of \(1/2\epsilon_\Phi^2\). To account for this effect, we treat the pitch angle diffusion term as a perturbation. The fractional perturbation we denote by \(\phi_1\). In the small–pitch angle regime, \(\phi_1\) obeys the equation

\[
\frac{\partial \phi_1}{\partial \tau_c} - \frac{\partial \phi_1}{\partial E} = \frac{4\xi}{E^2 \sigma_\Phi^2} \left(\frac{\sigma_\Phi^2 - \sigma_\Phi^2}{\sigma_\Phi^2}\right),
\]

(A5)
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the solution of this is

\[ \phi_1 = \frac{4 \xi}{\alpha_0} \left( \frac{x^2 - \alpha_0^2}{\alpha_0^2} \right) \frac{\tau_c}{E(E + \tau_c)}. \]

(A7)

Combining this with equation (A4), we then obtain, in the small–pitch angle regime,

\[ F(E, \mu, \tau_c) = F_0(E + \tau_c) \frac{2 e^{-\alpha_0^2/2}}{\alpha_0^2}, \]

(A8)

where

\[ \alpha_0^2 = \alpha_0^2 + [4 \xi \tau_c / E(E + \tau_c)], \]

(A9)

and

\[ \tilde{\alpha}_c^2 = \frac{x^2}{[1 + \delta_e x_c x_e^2/(2 + \eta + \tau_c)]}. \]

(A10)

This form is similar to the nonrelativistic solution (eqs. [8]–[10] in the text). With a small modification of \( \tilde{\alpha}_c^2 \) to the form

\[ \tilde{\alpha}_c^2 = \frac{x^2}{[1 + \delta_e x_c x_e^2/(2 + \eta + \tau_c)]}, \]

(A11)

with \( \alpha_0^2 \) given by equations (9) and (10), and addition of the velocity factors as in equation (8), we can combine the solutions of the two limiting cases as

\[
F(E, \mu, \tau_c) = \left\{ \frac{\beta(E)}{\beta[E(\eta + \tau_c)]} \right\}^2 \frac{2 e^{-\alpha_0^2/2}}{\alpha_0^2} F_0[E(\eta + \tau_c)] \left[ 1 + \delta_e \tau_c x_e^2 / 2(2 + \eta + \tau_c) \right]
\]

(A12)

II. THE UNIFORM FIELD SYNCHROTRON SOLUTION

It is well known (Petrosian 1985) that the rate of change in \( \beta_\parallel = \beta \mu \), the velocity parallel to the magnetic field, due to synchrotron losses is zero; i.e., \( \beta_\parallel = 0 \). If collisional losses are negligible and the B field is uniform, it is then convenient to rewrite the kinetic equation in terms of \( \beta_\parallel \) and \( \beta_\perp = \beta(1 - \beta_\parallel^2)^{1/2} \). Furthermore, since \( \beta_\parallel \) is constant, the time dependence can easily be incorporated, and equation (A1) becomes

\[
\left( \frac{\partial}{\partial ct} + \beta_\parallel \frac{\partial}{\partial \beta_\perp} \right) \tilde{f} = \frac{\partial}{\partial \beta_\perp} [\beta_\parallel(1 - \beta_\perp^2 - \beta_\parallel^2)^{1/2} \tilde{f}].
\]

(A13)

The function \( \tilde{f} \) is now the distribution function in terms of \( (\beta_\parallel, \beta_\perp) \), and we can use the Jacobian of the transformation from \( (E, \mu) \) to \( (\beta_\parallel, \beta_\perp) \) to give us the relation between \( \tilde{f} \) and \( f \), the distribution in terms of \( (E, \mu) \);

\[
\tilde{f}(\beta_\parallel, \beta_\perp) = \tilde{g}(1 - \tilde{\mu}^2)^{1/2} f(\tilde{E}, \tilde{\mu})
\]

(A14)

The tilde denotes quantities which are to be functions of \( (\beta_\parallel, \beta_\perp) \); i.e., \( \tilde{\gamma} = (1 - \beta_\parallel^2 - \beta_\perp^2)^{-1/2}, \tilde{E} = \tilde{\gamma} - 1, \) and \( \tilde{\mu} = \beta_\parallel(\beta_\parallel^2 + \beta_\perp^2)^{1/2} \). If we make the substitution

\[
\tilde{h} = \beta_\parallel(1 - \beta_\perp^2 - \beta_\parallel^2)^{1/2} \tilde{f} = \tilde{g}(\tilde{\gamma}^2 - 1)^{1/2} (1 - \tilde{\mu}^2)^{1/2} f(\tilde{E}, \tilde{\mu}),
\]

(A15)

and define \( y = ctS \) and \( \tau_s = sS \), equation (A13) becomes

\[
\frac{\partial \tilde{h}}{\partial y} + \beta_\parallel \frac{\partial \tilde{h}}{\partial \tau_s} = \beta_\parallel(1 - \beta_\perp^2 - \beta_\parallel^2)^{1/2} \frac{\partial \tilde{h}}{\partial \beta_\perp}.
\]

(A16)

We define a new variable \( u \) by

\[
du = \frac{d\beta_\perp}{\beta_\parallel(1 - \beta_\perp^2 - \beta_\parallel^2)^{1/2}},
\]

(A17)
so that
\[ u = -\gamma_{\parallel} \text{sech}^{-1} \left( \frac{\gamma_{\parallel} \beta_{\perp}}{\gamma_{\parallel}} \right) \quad \text{and} \quad \beta_{\perp} = \frac{1}{\gamma_{\parallel} \text{sech} \left( \frac{-u}{\gamma_{\parallel}} \right)}, \] (A18)
where \( \gamma_{\parallel} \equiv (1 - \beta_{\parallel}^2)^{-1/2} \). Equation (A13) is simplified to
\[ \frac{\partial h}{\partial y} + \beta_{\parallel} \frac{\partial h}{\partial \tau_{s}} = \frac{\partial h}{\partial u}. \] (A19)
If at \( \tau_{s} = 0 \) the distribution is given by \( g(y)f_{0}(E, \mu) \) so that
\[ \hat{h}_{0}(u, 0, y) = \left[ \gamma_{\parallel}(\gamma_{\parallel}^2 - 1)^{1/2}(1 - \beta_{\parallel}^2)f_{0}(\hat{E}, \hat{\mu}) \right] \delta(u + \tau_{s} \beta_{\parallel}), \] (A20)
equation (A13) has the simple solution \( \hat{h}(u, \tau_{s}, y) = g(y - \tau_{s} \beta_{\parallel})\hat{h}_{0}(u + \tau_{s} \beta_{\parallel}) \), and the distribution function in terms of energy and pitch angle \( f = \frac{\hat{h}(u, \tau_{s}, y)}{\beta_{\parallel}^{2}(1 - \mu^{2})} \) is
\[ f(E, \mu, \tau_{s}, y) = \frac{\hat{E}(y - \tau_{s} \beta_{\parallel})}{\beta_{\parallel}^{2}(1 - \mu^{2})} \left[ \gamma_{\parallel}(\gamma_{\parallel}^2 - 1)^{1/2}(1 - \beta_{\parallel}^2)f_{0}(\hat{E}, \hat{\mu}) \right] \delta(u + \tau_{s} \beta_{\parallel}), \] (A21)
It is easy to show that
\[ \hat{E}(u + \tau_{s} \beta_{\parallel}) = \hat{E}(X) = (\gamma_{\parallel} - X)/X, \]
and
\[ \hat{\mu}(u + \tau_{s} \beta_{\parallel}) = \hat{\mu}(X) = \beta_{\parallel}^{2}(1 - \mu^{2})/X^{2} \] (A22),
where
\[ X(E, \mu, \tau_{s}) = \tanh \left( \frac{1}{\gamma_{\parallel} \beta_{\parallel}} \right) = \frac{\gamma_{\parallel} - \gamma \tanh \left( \tau_{s} \beta_{\parallel} / \gamma_{\parallel} \right)}{\gamma - \gamma \tanh \left( \tau_{s} \beta_{\parallel} / \gamma_{\parallel} \right)}. \] (A23)
Returning to the time-independent case of continuous injection, for which \( g(y) \) is a constant, we find for the flux \( F = \beta c f \)
\[ F(E, \mu, \tau_{s}) = \frac{\gamma_{\parallel}^{2} \hat{E}(X), \hat{\mu}(X), 0}{\gamma^{2}(1 - \mu^{2})} \left[ 1 - X^{2} \right] \left[ X^{2}(\gamma_{\parallel}^2 - X^2) \right], \] (A24)
where \( F(E, \mu, 0) \) is the injected flux.
At \( \tau_{s} = 0 \), \( X = \gamma_{\parallel}/\gamma \); with increasing \( \tau_{s} \), \( X \) and the flux \( F \) decrease monotonically and become zero at a depth given by
\[ \tau_{sc} = \frac{\beta_{\parallel} \gamma}{2} \ln \left( \frac{\gamma_{\parallel} + \gamma_{\parallel}}{\gamma - \gamma_{\parallel}} \right). \] (A25)
The value of \( \tau_{sc} \) decreases with increasing energy; particles with higher energy travel shorter distances, and for a given energy \( \tau_{sc} \) is smallest at \( \mu = 0 \); particles with high pitch angles lose energy more quickly than those with smaller pitch angles.
At a given depth \( \tau_{s} \), the flux approaches zero at a critical value of the pitch angle \( \alpha_{cr} = \cos^{-1} \left( \mu_{cr} \right) \), which we may calculate by solving the transcendental equation
\[ \tanh^{-1} \left( \frac{1}{\gamma_{\parallel} - \beta_{\parallel}^{2} \mu_{cr}^{2}^{1/2}} \right) = \frac{\tau_{s} \left( 1 - \beta_{\parallel}^{2} \mu_{cr}^{2}^{1/2} \right)^{1/2}}{\beta \mu_{cr}}. \] (A26)
For large depths \( \tau_{s} \to \infty \), \( \mu_{cr} \to 1 \), and the distribution becomes infinitely narrow, and at the same time, the flux at \( \mu = 1 \) gets infinitely large. For \( \mu \to 1 \), or \( \alpha^{2} \to 0 \), we have
\[ X = 1 - \frac{\beta_{\parallel}^{2} \gamma^{2} e^{2 \alpha^{2} \beta_{\parallel}}}{2}, \] (A27)
\[ \hat{E}(X) = E, \hat{\mu}(X) = \mu, \] and
\[ F(E, \mu = 1, \tau_{s}) = F(E, 1, 0)e^{2 \alpha^{2} \beta_{\parallel}}. \] (A28)
For \( \alpha_{cr} \ll 1 \), we may solve equation (A26) and find value for the critical angle given by
\[ \alpha_{cr}^{2} = \frac{4e^{2 \alpha^{2} \beta_{\parallel}}}{\beta \gamma^{2}}. \] (A29)
Even though the flux at \( \alpha = 0 \) diverges for large depths, the width becomes zero, so that the flux integrated over pitch angle, which is of order \( F(\mu = 1) \alpha_{cr}^{2} \), remains finite.
The solution has a particularly simple form for ultrarelativistic electrons. We let \( \beta \to 1 \) in equation (A21) and find \( X =
\[
\gamma \ll 1, \mu(\tau_{\text{sec}}) = \mu, \text{ and if } F(E, \mu, 0) = F_0(E)G(\mu), \text{ then}
\]
\[
F(E, \mu, s) = \frac{F_0(E(1 - \tau_{\text{sec}})/\gamma)}{(1 - \tau_{\text{sec}}^2)^2},
\]
(A30)

where \(\tau_{\text{sec}} = \mu(1 - \mu^2)/E\). This solution is discussed in § II.

In § II we discuss the flux integrated over depth for given pitch angles. This would pertain to the case for which the field lines are straight and for spatially unresolved observations. We may obtain the flux integrated over depth either by integration of the solution in equation (A30) over \(\tau_s\) or by solution of the full equation (eq. [A13]) integrated over \(\tau_s\). We find
\[
F_s(E, \mu) = \left. \frac{\beta \mu}{\gamma^2(1 - \mu^2)} \int_0^\infty dE' F_0(E')G\left(\frac{\beta \mu}{\beta'}\right) \right|_{E'}.
\]
(A31)

where we must keep \(\beta\) constant in the integrand. The integration is complex for the general case due to the complex dependence on \(\beta\) in the integrand. For an initially isotropic distribution, \(G(\beta(\beta')/\beta)\) is constant and \(F_s \propto \int_0^\infty \beta'^{-2} F_0(E')dE'\).

### III. THE RELATIVISTIC ENERGY LOSS SOLUTION

As in the collisional case, the synchrotron pitch angle change term is much less than the energy loss term for \(E \gg 1\). The relativistic electrons lose energy at constant pitch angle until they become nonrelativistic when the collisional diffusion and \(\mu_s\) terms become important. Here we ignore these terms, keeping in mind that we can add those terms as perturbations as in § I above if necessary. We take the limit \(\beta \to 1\) and let \(d \ln B/\ln s = 0\) in equation (1). If we define \(\epsilon = \epsilon \equiv C/(1 - \mu^2)\), then
\[
\mu \frac{\partial \Phi}{\partial \tau} = \left(1 + \frac{E^2}{\epsilon^2}\right) \frac{\partial \Phi}{\partial E} + 2e^{-2}E\Phi,
\]
(A32)

and if \(d\eta \equiv dE/(1 + E^2/\epsilon^2)\), then
\[
\eta = \epsilon \tan^{-1}(E/\epsilon), \quad E(\eta) = \epsilon \tan(\eta/\epsilon),
\]
(A33)

and
\[
\mu \frac{\partial \Phi}{\partial \tau} = \frac{\partial \Phi}{\partial \eta} + \frac{2}{\epsilon} \tan\left(\frac{\eta}{\epsilon}\right) \Phi.
\]
(A34)

Next we let \(\Phi(\eta, \mu, \tau) = \phi(\rho)\psi(\eta)\), where \(\rho = \eta + \tau_s/\mu_s\), so that
\[
\mu \frac{\partial \Phi}{\partial \tau} \bigg|_{\eta} = \psi \frac{\partial \Phi}{\partial \rho}, \quad \frac{\partial \Phi}{\partial \eta} \bigg|_{\tau_s} = \psi \frac{\partial \Phi}{\partial \rho} + \phi \frac{\partial \psi}{\partial \eta},
\]
(A35)

and equation (A34) reduces to
\[
\frac{d\psi}{d\eta} = -\frac{2}{\epsilon} \tan\left(\frac{\eta}{\epsilon}\right) \psi
\]
(A36)

which has the solution
\[
\psi(\eta) = \psi_0 \sec^{-2}\left(\eta/\epsilon\right).
\]
(A37)

This gives
\[
\Phi(\eta, \mu, \tau) = \phi(\rho)\psi_0 \sec^{-2}\left(\eta/\epsilon\right),
\]
(A38)

where \(\psi_0\) is a constant of integration.

For the initial condition
\[
\Phi(E, \mu, 0) = \phi(\rho)\psi_0 = c^{-1} F_0[E(\eta)]G(\mu),
\]
(A39)

we then find
\[
\phi(\rho)\psi_0 = c^{-1} G(\mu)F_0[\epsilon(\eta) \tan(\eta/\epsilon)] \sec^2(\eta/\epsilon).
\]
(A40)

Thus
\[
\phi(\rho) = \frac{cF_0[\epsilon(\eta) \tan(\rho/\epsilon)]G(\mu) \sec^2(\rho/\epsilon)}{c\psi_0},
\]
(A41)

and finally
\[
F(E, \mu, \tau_s) = c\Phi(E, \mu, \tau_s) = \frac{c\phi(\rho)\psi_0}{\sec^2(\eta/\epsilon)} = \frac{cF_0[\epsilon(\eta) \tan(\rho/\epsilon)]G(\mu) \sec^2(\rho/\epsilon)}{c\psi_0}.
\]
(A42)
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We substitute for \(\rho\) and \(\eta\) to obtain the flux in terms of \(E\) and \(\mu\):

\[
F(E, \mu, \tau_c) = \frac{F_0[E_*(E, \mu, \tau_c)]G(\mu)}{(1 + E^2/\epsilon_c^2)},
\]

where

\[
E_*(E, \mu, \tau_c) = \epsilon_c \left[ \frac{E + \epsilon_c \tan (\epsilon_c/\mu \epsilon_c)}{\epsilon_0 - E \tan (\epsilon_c/\mu \epsilon_c)} \right].
\]

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