AN EQUATION FOR THE EVOLUTION OF SOLAR AND STELLAR FLARE LOOPS

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ABSTRACT

An ordinary differential equation describing the evolution of a coronal loop subjected to a spatially uniform but time-varying heating rate is discussed. We assume that the duration of heating is long compared to the sound transit time through the loop, which is assumed to have uniform cross section area. The form of the equation changes as the loop evolves through three stages: "strong evaporation," "scaling law behavior," and "strong condensation." Solutions to the equation may be used to compute the time dependence of the average coronal temperature and emission measure for an assumed temporal variation of the flare heating rate. The results computed from our model agree reasonably well with recent published numerical simulations and may be obtained with far less computational effort. We then use our model to study the 1980 May 21 solar flare observed by SMM and the giant 1985 April 12 flare observed on the star AD Leo.

Subject headings: hydromagnetics — stars: flare — Sun: corona — Sun: flares

I. INTRODUCTION

Flares on the Sun and other stars result from the release of large amounts of energy into the corona and chromosphere. Much of the energy is released along closed magnetic field lines and results, through the phenomenon of "chromospheric evaporation," in the formation of hot, dense X-ray emitting coronal loops. The time scales over which energy is released can range from the very short (some individual hard X-ray bursts during the impulsive phase of solar flares are just a few milliseconds long) to many hours for some of the largest flares. Most flares have a long, slow decay period which is known as the "gradual phase." It is generally believed that flare energy release continues through both the impulsive and gradual phases.

Theoretical studies of the atmospheric response to flare energy release have focused on solving the equations of gas-dynamics (the continuity, momentum, and energy equations) for one-dimensional plasma flow along strong magnetic field lines subject to some assumption about how flare energy is released (e.g., Livshits et al. 1981; Somov, Syrovatskii, and Spektor 1981; Pallavicini et al. 1983; Nagai and Emslie 1984; MacNeice et al. 1984; Fisher, Canfield, and McClymont 1985a, b, c [hereafter FMCa, b, c]; Karpen and DeVore 1987; Mariska, Emslie, and Li 1989). Although such studies constitute the most thorough and accurate means of studying flare loop evolution, the equations are difficult and time consuming to solve. Consequently, most flare loop calculations have focused on the response of a flare loop to a single, short burst of energy and have followed the loop evolution for only 1 or 2 minutes. Although much has been learned from these studies, only limited work has been done on computing flare loop evolution over the course of an entire flare. In general, this is beyond the scope of present computational resources.

In this paper, our goal is to develop a simple model for the evolution of a coronal loop which describes how the average coronal temperature, density, and emission measure evolve during a flare for a specified coronal energy input rate. We are primarily interested in describing flare loop evolution on time scales long compared to the hydrodynamic time scale (the sound transit time through a coronal loop, which is approximately 1 minute for typical solar flare loops), and we therefore assume that the momentum equation can be replaced by an assumption of quasi-hydrostatic equilibrium. Our model is therefore not valid for flares with brief, intense bursts of impulsive phase heating, but it is intended as a tool to study flares with extended periods of impulsive phase heating and to study loop evolution during the gradual phase of flares.

The remainder of this paper is organized as follows: in § II, we start from the energy equation for a loop in which the pressure is uniform but time varying, and we derive the loop-integrated energy equation relating the time derivative of the loop pressure to the spatially averaged heating and cooling rates. In § III, we describe numerical solutions to the energy equation, which we use to explore the behavior of the spatially averaged cooling rate with respect to global loop variables. We find that the average cooling rate depends almost exclusively on the total column depth of plasma in the loop, and that this cooling rate is given by that from an "equivalent static loop" containing the same mass of plasma as the evolving loop. Section IV consists of a brief review of the properties of static loop models. In § V, we use the cooling rate result of § III to convert the loop-averaged energy equation of § II to a first-order differential equation for the total column depth of the loop as a function of time. We introduce three different regimes of loop evolution, "strong evaporation," "scaling law behavior," and "strong condensation." These three regimes are characterized by differing ratios of the logarithmic pressure time derivative to the logarithmic column depth time derivative. In Appendices A and B, we discuss the "strong evaporation" and "strong condensation" limits in more detail. Our model, based on solutions to the differential equation described in § V, is then compared in § VI with a specific set of numerical simulations of loop evolution carried out by Palavicini et al. (1983). In § VII, we use our model to compute the time dependence of the...
average coronal temperature and emission measure for two observed flares, the solar flare of 1980 May 21 observed by the Solar Maximum Mission, and the huge flare of 1985 April 12 observed on the star AD Leo. We summarize the model in § VIII and point out its potential uses and its limitations.

II. THE INTEGRATED ENERGY EQUATION

We are primarily interested in flare evolution on fairly long time scales. Observations of plasma flows in coronal loops during flares have shown that strong, upward velocities on the order of the sound speed are present during the first minute or two after flare onset (Antonucci, Gabriel, and Dennis 1984); however, the velocities decrease rapidly to subsonic values. The reason for this is clear when one examines gasdynamic simulations of solar flares (e.g., Nagai and Emslie 1984; Pallavicini et al. 1983). After roughly one loop hydrodynamic time scale, an approximate hydrostatic equilibrium is established in the corona, and the subsequent flows are substantially subsonic. Since the pressure scale height in the flare corona is generally large compared to flare loop lengths, we will therefore assume that the pressure \( P \) in the corona is uniform, although changing with time as evaporation or condensation occurs. The validity of this assumption must be checked post facto when we compute flow velocities with our model.

By assuming uniform pressure, the momentum equation is eliminated from the set of gasdynamics equations, and only the energy and continuity equations remain. When we also invoke the assumption of uniform loop cross section, the energy equation can then be written

\[
\frac{3}{2} \dot{P} = Q - R - \frac{5}{2} P \frac{d\nu}{dz} - \frac{dF_c}{dz},
\]

where \( \dot{P} \) is the time derivative of the coronal pressure, \( Q \) is the volumetric heating rate, \( R \) is the radiative loss rate, \( \nu \) is the plasma velocity,

\[
F_c = -\kappa_0 T^{5/2} \frac{dT}{dz},
\]

is the conductive flux, and \( z \) is the distance measured from one of the chromospheric footpoints toward the loop apex. The plasma is assumed to be fully ionized and obeys the equation of state \( P = \frac{2}{5} \kappa n T \). The coefficient \( \kappa_0 \approx 10^{-6} \) in cgs units. Note that our expression (2) for the conductive flux (the Spitzer-Harm result) neglects a host of nonlocal heat conduction and saturation effects, which have been demonstrated to be important early in the impulsive phase (FCMa; Fisher 1986; Karpen and DeVore 1987; Ljepojevic and MacNeice 1988). However, we expect that a few hydrodynamic time scales after flare onset these effects will become unimportant because the coronal density will increase substantially due to chromospheric evaporation. This increases the number of coronal electrons available to carry the heat flux, reducing the relative importance of nonclassical effects. Nonclassical effects are therefore not included in this paper. The radiative loss rate is assumed to have the form \( R = n^2 \Lambda(T) \), where we take \( \Lambda(T) = AT^\alpha \), with \( T \) being the plasma temperature. The power-law index is taken to be \( \alpha = -\frac{1}{3} \), consistent with the radiative loss rate at the range of temperatures found within a flaring coronal loop (Raymond, Cox, and Smith 1976). Appropriate values for \( A \) are discussed in § VI and VII.

Since the pressure \( P \) is assumed uniform, its time derivative is uniform as well. Assuming plasma flows symmetrically in both legs of the flare loop, the heat flux and velocity must both vanish at the loop apex. Integrating equation (1) from footpoint to apex, the heat flux vanishes if the integration begins at the topmost layer of the chromosphere. The enthalpy term does not vanish when integrated from the chromosphere, but its integral can be shown to be given by

\[
\frac{3}{2} P_{ch} \approx \frac{3}{2} Q \frac{T_{ch}}{T_{co}} L,
\]

where \( T_{ch} \) and \( T_{co} \) are the temperatures in the chromosphere and corona, respectively, \( \Omega \) (discussed in § V) is a constant of order unity, and \( L \) is the apex to footpoint loop length. Since the temperature ratio is much less than unity, the enthalpy flux will be much less than \((3/2)P \) integrated over the loop length. We therefore neglect the enthalpy flux in the integrated equation.

Dividing the remaining integrated terms by \( L \), we find

\[
\frac{3}{2} \dot{P} = \langle Q \rangle - \langle R \rangle,
\]

where \( \langle Q \rangle \) and \( \langle R \rangle \) are the spatially averaged heating and loss rates. In principle, equation (3) provides a way of solving for the loop evolution with time, if we can express \( \langle Q \rangle \) and \( \langle R \rangle \) in terms of global loop variables (e.g., the loop length \( L \), the total column depth, or the average temperature). There are various theories for how flare energy is released in a loop, but for simplicity and generality, we assume throughout that \( Q \) is uniform in the coronal loop, but that its time dependence is specified \( a \) priori by the researcher. For example, we will assume in § VIIa that \( Q(t) \) is determined by the observed hard X-ray count rate for a specific flare. In the following section of the paper, we integrate the energy and continuity equations numerically in order to investigate how \( \langle R \rangle \) depends on global loop variables.

III. PROPERTIES OF NUMERICALLY INTEGRATED LOOP MODELS

Several numerical studies of the thermodynamic structure of evolving coronal loops have been carried out (e.g., Craig and McClymont 1986; McClymont and Craig 1987) with the approximation that flows driven by chromospheric evaporation can be regarded as steady. In the work described in this section of the paper, we relax this assumption and allow for time dependence in an approximate fashion. We compute numerically integrated solutions of the energy equation (1) by making a simplifying assumption regarding the continuity equation

\[
\frac{\partial}{\partial z} (n \nu) = 0.
\]

Because of the partial time derivative in equation (4), it is not possible in general to integrate this system of equations as ordinary differential equations. However, we remove this difficulty by making the approximation that the ratio \( n/\nu \) is uniform throughout the loop wherever the continuity equation is used. This should be a good approximation, since the loop will respond globally to the addition of mass by evaporation or the removal of mass by condensation. Then we can replace \( n/\nu \) by \( \dot{N}_t/N_t \), where \( \dot{N}_t \) is the total mass flux into or out of the base of the loop (i.e., the evaporation or condensation rate), and \( N_t \) is the total amount of mass in the loop at some instant in time. With this approximation and the fact that the pressure in the loop is uniform, equation (4) can be written

\[
\frac{\partial \nu}{\partial z} = -\frac{v}{n} \frac{\partial}{\partial z} \left( \frac{\dot{N}_t}{N_t} \right).
\]
Equation (5) has the formal solution

$$v(z) = \frac{\dot{N}_t}{\dot{n}(z)} \left[ 1 - \frac{N(z)}{N_f} \right],$$

(6)

where $N(z) = \int_z^L n(z')dz'$.

As a price for eliminating the partial time derivative from equation (4), we have introduced two unknown parameters, $\dot{N}_t$ and $N_f$. We next introduce a parameter $\Omega$ which relates $\dot{N}_t$ to $\dot{P}$ by

$$\frac{\dot{N}_t}{N_f} = \frac{\dot{P}}{P}.$$  

(7)

We defer a more detailed discussion of $\Omega$ to § V and Appendices A and B, but note that our motivation is to characterize the dynamics of the loop by the single parameter $\dot{N}_t$. For the moment, we regard $\Omega$ as a fixed constant. This allows us to express $\dot{P}$ in terms of $N_f$ in equation (1).

Our approach to calculating detailed temperature and density structures for evaporating and condensing loops is to integrate equations (1), (2), and (5) from the base of the loop to the loop apex, which is defined to be the point where the conductive flux vanishes. We use a fourth-order Runge-Kutta method to integrate the equations. At the beginning of the integration procedure, we specify at the bottom boundary a fixed temperature $T_	ext{bot}$ and flux $F_	ext{bot}$. $T_	ext{bot}$ must be chosen low enough that the neglected portion of the loop makes a negligible contribution to the total cooling power, yet high enough that the plasma remains fully ionized and that our approximation to the optically thin cooling law is reasonable. These considerations have led us to adopt $T_	ext{bot} = 10^5$ K. One must also choose an initial value of $F_	ext{bot}$. We choose a negligible but nonzero value of $F_	ext{bot}$ to avoid a singularity at the initial point in the integration. This rather arbitrary assumption is only reasonable if the maximum conductive flux in the loop is much greater than the flux one would realistically encounter at $10^5$ K. The flaring loops we consider in this paper (with coronal temperatures in excess of $10^6$ K) do satisfy this criterion. One must also note that by fixing the origin of our coordinate system to a specific temperature, the coordinate system is not truly fixed in space, meaning that strictly speaking the velocity at the loop apex would not be zero in our scheme. However, owing to the fact that the actual distance between the loop apex and the $T = 10^5$ K point will not move substantially during a flare, the error we make will be small. We henceforth ignore this effect and assume that our coordinate system is also fixed in space. The plasma velocity at the bottom boundary can be related to our loop parameters by $v_	ext{bot} = N_f/\dot{N}_t = 2kT_	ext{bot}N_f/P$.

The free parameters which can be specified initially, and which are not varied during calculation of the loop structure are $\Omega$, the heating rate $Q$, and the total column depth $N_f$. The other two parameters (the pressure $P$ and the evaporation rate $\dot{N}_t$) are unknown ahead of time, but they are adjusted until the correct boundary conditions are achieved, namely that the conductive flux and velocity vanish at $z = L$.

We carry out the adjustments by a nested sequence of iterations. During the inner iteration sequence, $\dot{N}_t$ is held fixed. The equations are integrated upward until the conductive flux vanishes. This point is the loop apex, which occurs at a distance $z_0$ from the footpoint. In general the velocity $v(z_0)$ will not be zero. The pressure $P$ and hence the initial velocity $v_	ext{bot}$ are then reset to new values so that $v(z_0)$ will be closer to zero when the equations are reintegrated. The inner iteration has converged when $v(z_0) = 0$. From equation (6), we observe that the integrated column depth will then be exactly equal to the parameter $N_f$. The outer iteration sequence adjusts $\dot{N}_t$ until the correct loop length is obtained, i.e., the numerically integrated loop length $z_0$ has converged to $L$. We use an approximate Newton-Raphson scheme (i.e. the secant method) to carry out each of the two iteration sequences for determining $P$ and $N_f$. We demand an accuracy of one part in $10^4$ in both iterations. Typically four to six inner iterations are necessary to get the correct $P$ for a given $N_f$, and three to five outer iterations are required to adjust $\dot{N}_t$ to get the correct loop length. Thus, typically 20 loop integrations are required for a self-consistent loop model.

Before going on to discuss the results of our numerical calculations, we must first comment on how the parameter $\Omega$ affects our solutions. Throughout the preceding discussion, we have treated $\Omega$ as a fixed but arbitrary constant appearing in the energy and continuity equations. It will become clear in § V that the correct choice of $\Omega$ depends on the state of the loop, i.e., whether it is evaporating, condensing, or nearly static. However, this introduces a problem of self-consistency. For example, in order to investigate the properties of a condensing loop, one must choose a value of $\Omega$ which is appropriate for condensation. However, in order to estimate what $\Omega$ should be for condensation, we need to know the thermodynamic structure in the loop. To resolve this circular dilemma, we anticipate results obtained in § V and the Appendices.

We find that strongly evaporating loops generally have values of $\Omega$ close to unity, nearly static loops have an $\Omega$ near $\frac{1}{2}$, and strongly condensing loops have a value of $\Omega$ about 0.2–0.3. It is interesting to note that no solution exists for a condensing loop if a value of $\Omega$ appropriate for strong evaporation or nearly static loops is imposed. This suggests that these values of $\Omega$ are not physically consistent with condensation.

Within the context of the approximations and assumptions we have made, our numerical calculations show that the average radiative loss rate from an evolving loop is a function primarily of the total column depth $N_f$ of the loop over a very wide range of conditions, including both strong evaporation and condensation. Furthermore, the loss rate can be approximated to an accuracy of a few percent by the average loss rate from an "equivalent static loop" (described in § IV) of the same column depth and length. To demonstrate the degree to which this approximation holds, we compute the relative difference between the average loss rate computed from the numerical solutions described in this section of the paper to that from equivalent static loops of the same $N_f$. The results are shown in Figure 1. Note that over a range of loop masses spanning two orders of magnitude and representing both strong evaporation and strong condensation, the average loss rate from the equivalent static loops differs from the numerically computed loss rate by less than 4%. The approximation still holds even when the parameter $\Omega$ is varied (at least in those cases where our algorithm is still able to find solutions) and therefore seems to be quite robust. Since the equivalent static loops approximate the loss rate of the numerical solutions so well over a dynamically interesting range, we proceed assuming that the average loss rate of any loop is equal to that from its equivalent static loop. This allows us to easily estimate $\langle R \rangle$ in terms of the single global variable $N_f$, as will be described in § IV.

**IV. EQUIVALENT STATIC LOOPS**

Because of the importance of the "equivalent static loops," we review briefly the determination of their structure. As men-

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tioned earlier, we define the equivalent static loop to be the static loop with the same length and column depth as the evolving loop. The energy equation (1) in the absence of flows becomes

$$Q_{\text{eff}} - R - \frac{dF_z}{dz} = 0,$$  \hspace{1cm} (8)

where $Q_{\text{eff}}$ is the heating rate it would take to maintain the equivalent static loop in equilibrium. The solution of equation (8) is straightforward. In its integrated form, it leads to the well-known loop scaling law (Craig, McClymont, and Underwood 1978; Rosner, Tucker, and Vaiana 1978). For our choice of parameters of the loss function, its precise form is

$$L P_* T_A^{11/4} = \beta_1 \left[ 2 k \kappa_0 (a + \frac{1}{2}) A^1/2 (2 - a) \right] \equiv C,$$  \hspace{1cm} (9)

where $L$ is the apex to footpoint loop length, $\kappa_0$ is the Spitzer coefficient, $T_A$ is the static loop apex temperature, $P_*$ is the static loop pressure, $\beta_1 \equiv B(11/4 - a/2)/(2 - a), 1/2$, and $B(a, b)$ is the beta function. The position within the loop is related to the temperature by

$$\frac{z}{L} = I_e \left( \frac{11/4 - a/2}{2 - a}, \frac{1}{2} \right),$$  \hspace{1cm} (10)

where $t \equiv (T/T_A)^{2 - x}$, $I_e(a, b)$ is the incomplete beta function, and $z$ is measured from the loop footpoint toward the apex. It is important to note that the apex temperature $T_A$, and the pressure $P_*$ correspond to the equivalent static loop and do not correspond to those variables in the evaporating or condensing loop.

In the course of deriving the scaling law, one also finds the relationship between the apex temperature and the average loss rate (also equal to $Q_{\text{eff}}$)

$$\langle R \rangle = \frac{7}{4} \left( \frac{\beta_1}{2 - a} \right)^2 \frac{\kappa_0 T_A^{14/7}}{L^2}. $$  \hspace{1cm} (11)

From equations (9), (10), and (11), and the equation of state, the column depth $N_i$, is related to the apex density $n_{a*}$ and loop length $L$ by

$$N_i = \frac{\beta_2}{\beta_1} \frac{\kappa_0}{L} n_{a*} L^2,$$  \hspace{1cm} (12)

where $\beta_2 = B((7/4 - a/2)/(2 - a), 1/2)$. We can then eliminate all the other static loop variables in terms of the column depth $N_i$. We find

$$\langle R \rangle = \frac{7}{4} \left( \frac{\beta_1}{2 - a} \right)^2 \frac{\kappa_0}{L^2} \frac{(2k\beta_1)^{14/7 - 2a}}{C\beta_2} N_i^{14/7 - 2a} \equiv \beta \Delta N_i^{14/7 - 2a}.$$  \hspace{1cm} (13)

It is this expression for $\langle R \rangle$ that we find adequately approximates the losses from the dynamically evolving loops, e.g., $R_{\text{est}}$ in Figure 1.

V. THE LOOP EVOLUTION EQUATION

Having found an approximation for $\langle R \rangle$, we now return to the loop evolution equation (3). We seek first to express equation (3) in terms of the evaporation rate $N_t$ rather than $P$, since we have found that the average loss rate from the loop is primarily a function of its column depth. Converting $P$ to $N_t$ is therefore necessary to obtain a differential equation which involves only one global loop quantity.

The key to relating $P$ to $N_t$ is the ability to estimate the parameter $\Omega \equiv (N_t/N_i)/(P/P)$ for the various phases of a flare. During the rise phase, we expect that the loop will undergo strong evaporation, while during the decay phase, the loop is expected to undergo condensation. To illustrate the relative importance of the various terms in the energy equation for loops exhibiting strong evaporation ($Q \gg \langle R \rangle$), strong condensation ($Q \ll \langle R \rangle$), and behavior near equilibrium ($Q \approx \langle R \rangle$), we show in Figures 2-4 three different loops computed numerically with the procedure described in § III. The top panel of each figure shows the temperature and velocity as a function of length along the loop. The bottom panel shows the relative
Fig. 2.—The upper panel shows the dependence of the temperature and velocity with length along the loop for a strongly evaporating loop. The lower panel shows the variation of the various terms in the energy equation for the same loop.

Fig. 3.—The upper panel shows the dependence of the temperature and velocity with length along the loop for a strongly condensing loop. The lower panel shows the variation of the various terms in the energy equation for the same loop.
strengths of the heating rate, the radiative loss rate, the enthalpy flux divergence, and the conductive flux divergence as functions of length along the loop. These figures clearly illustrate that the dominant terms in the energy equation are different depending upon which energy regime the loop is in. Focusing on the dominant energy mechanisms, we then performed detailed calculations of $Q$ for strongly evaporating and strongly condensing loops (discussed below). We find that during strong evaporation, $Q$ is generally close to unity, while for strong condensation $Q$ is generally in the range 0.2–0.3. For the case of strong evaporation (see Appendix A), the loop should condense, i.e., the loop pressure will then increase in proportion to the time integral of $Q$. What is not clear a priori is whether the pressure increase is due mainly to a temperature rise, or to evaporation and subsequent density increase. In Appendix A, we solve the energy and continuity equations for the loop with no radiative losses present. Our treatment is similar to that of Antiochos and Sturrock (1978), expect that they kept the pressure constant, while we allow the pressure to vary. We find that the apex temperature is determined by a balance between conductive flux (which is proportional to $QL$) and enthalpy flux, with the expression for the apex temperature in terms of $Q$ given by equation (A14). We then derive an expression for $\Omega$ (eq. [A16]) which in general depends on the time derivative of the heating rate. We find that $\Omega$ is usually close to unity, meaning that most of the pressure increase is due to evaporation. The only exception to this is early in the flare, if the heating rate increases sufficiently rapidly. This leads to a rapid temperature increase, which briefly depresses the value of $\Omega$ below unity. We must caution that our model breaks down if the heating rate changes on a time scale faster than the hydrodynamic time scale $\tau_{\text{hy}} \equiv L/c_s$, because we neglect the momentum equation in our model.

After a substantial amount of heat has been deposited in a flare loop, a correspondingly large amount of plasma will have been evaporated. When the heating rate begins to decrease, the average radiative loss rate may become quite large compared to the heating rate. In this case, the loop should condense, i.e., the column depth and pressure should decrease. Numerical gasdynamical simulations relevant to our model (those with durations much longer than the hydrodynamic time scale) have been carried out by Pallavicini et al. (1983) (hereafter PPSVALR), and Craig, Robb, and Rollo (1982). These simulations show that when flare heating ceases, the coronal temperature first cools on a conductive time scale while evaporation continues. After roughly one conductive cooling time the evaporation ceases, and the pressure, temperature, and density all begin to decrease. Examination of Figure 9 from PPSVALR shows that during this condensation phase, $(h/n)/(\dot{P}/P) = \Omega \approx 0.3$. From our own numerically integrated model of a condensing loop shown in Figure 3, it is evident that throughout most of the corona, both direct heating and thermal conduction have ceased to play the most important roles in the energy equation. The largest terms are radiative losses, the $(3/2)\dot{P}$ term, and enthalpy flux divergence. In Appendix B, we develop an analytical model of condensing loops where direct heating and thermal conduction are omitted from the energy equation. We find that for these loops, $\Omega$ must obey the constraint $\Omega < 0.43$, which is consistent with our analysis of the numerical simulations of PPSVALR.

During an intermediate period between strongly evaporating and strongly condensing loop conditions, the loop is in a state where the average loss rate is roughly equal to the heating rate. We therefore expect that the loop will evolve near the static scaling law curve relating the pressure and column depth. Equations (9) and (12) and the equation of state give the static scaling law result $P \propto N^1/4 - \alpha/2)/(1/2 + \alpha/2)$, Thus $\Omega = (7/4 - \alpha/2)/(11/4 - \alpha/2) = \frac{3}{2}$ for $\alpha = \frac{1}{2}$. We have identified three different regimes of loop evolution: strong evaporation, scaling law behavior, and strong condensation. Each regime is characterized by different expressions for $\Omega$ and the loop pressure. We now use these results to study flare loop evolution.

For the case of strong evaporation (see Appendix A), the loop pressure is given by

$$P = 2^{1/3} \eta k T_A(t) N_{\gamma} / L,$$

where $\eta = 0.4656$ and $T_A(t)$ is given by equation (A14). The quantity $\Omega$ is given by equation (A16). For the scaling law and strong condensation regimes, $\Omega$ is assumed to be constant.
within each regime, so that from equation (7) we can write
\[ P = P_0 \left( \frac{N_t}{N_0} \right)^{1/3}, \tag{15} \]
where \( N_0 \) and \( P_0 \) are constants chosen to make the pressure and average coronal temperature evolve continuously across the transition from one regime into either one of these two regimes. This is done by simply choosing \( N_0 \) and \( P_0 \) in the new regime to be the last values of \( N_t \) and \( P \) from the previous regime. We arbitrarily choose to make the division between the three loop regimes in the following way: if \( \langle R \rangle < (1/2)Q(t) \), then the loop is in the strong evaporation regime, if \( \langle R \rangle > 2Q(t) \), then the loop is in the strong condensation phase, and otherwise the loop is in the scaling law phase.

The loop evolution equation is formally identical for the three regimes. Making use of equations (7) and (13), we can reduce equation (3) to the first-order ordinary differential equation for the column depth as a function of time, i.e.,
\[ \dot{N}_t = \frac{2\Omega_0 N_t}{3P} \left[ Q(t) - KN_t \right], \tag{16} \]
where \( K \) and \( \lambda \) were defined in equation (13), and \( P \) is determined by equations (14) or (15). For a specified heating rate \( Q(t) \), the solution is well defined and easily obtained by simply integrating equation (16) numerically, taking care to switch the value of \( \Omega \) when necessary, and resetting \( N_0 \) and \( P_0 \) between regimes so that \( N_t \) and \( P \) are continuous (a necessary exception is the transition from the initial loop model to the strong evaporation regime, where the coronal temperature and pressure are allowed to change discontinuously. This would correspond to a jump in the coronal temperature on the heating time scale, if our model was valid on these short time scales. The most common example of this is the transition from the initial loop model to the strong evaporation regime when the onset of flare heating is sudden). The column depth and the pressure are otherwise continuous, and the pressure has a continuous first derivative. We define an average coronal temperature, \( T_c \), by
\[ T_c = \frac{PL}{2kN_t}, \tag{17} \]
which is also continuous, although its derivative can jump discontinuously as the loop moves from one regime to another.

It is also useful to compute the “equivalent apex temperature,” the apex temperature of the equivalent static loop with the same column depth as the evolving loop. As an example, suppose we wish to compute the effects of X-ray heating from the corona on the flare chromosphere, requiring that we solve the energy equation for the chromospheric plasma. We assume that the chromosphere and corona are both in hydrostatic equilibrium, with the corona parameterized by a static loop model with some apex temperature. The radiative losses from the dynamic corona are characterized by the equivalent static loop, which would have the temperature \( T_{c*} \) rather than \( T_c \). Since the coronal losses supply the X-rays to heat the chromosphere, \( T_{c*} \) better characterizes the heating from the evolving loop than does \( T_c \). From equations (9) and (12), we find
\[ T_{c*} = \left( \frac{\beta_2}{\beta_1} \frac{2kN_t}{C} \right)^{1/(7/4 - n/2)}. \tag{18} \]

Finally, X-ray emission from flares is often used to derive the coronal emission measure \( EM \), defined to be the volume integral of \( n^2 \). Our model allows us to estimate \( EM \) to facilitate comparisons with flare observations (see § VIIa). We find
\[ EM = \frac{\pi d^2}{L} N_t^2 \tag{19} \]
where \( d \) is the diameter of the loop cross section. It is well known that for a given column depth \( N_t \), the emission measure is minimized by distributing the plasma uniformly, as equation (19) assumes, meaning that equation (19) actually underestimates the emission measure. One can make use of the analytical models in Appendices A and B and the static loop solution to compute the emission measure in these limits. We find that for strong evaporation, the emission measure is roughly 50% larger than that predicted by equation (19); for static loops it is roughly 25% larger; and for strongly condensing loops, it is roughly 10% larger. Although these differences might seem important, remember that over the course of a flare calculation, the emission measure is likely to change by two orders of magnitude, and on such a scale these differences will not appear significant. Furthermore, if one attempted to apply these corrections, the emission measure would make discontinuous jumps as the loop switched between regimes. We therefore use equation (19) to estimate EM without further corrections.

VI. COMPARISONS WITH NUMERICAL SIMULATIONS

As a test of our loop evolution equation, we have compared our predicted evolution of the loop density, pressure, and apex temperature with that from the gasdynamic simulations of PPSVALR, in particular the simulations shown in their Figures 9 and 11. We begin by reviewing their calculations. The preflare loop in PPSVALR has an apex to footpoint length of \( 2 \times 10^7 \) cm and an apex temperature of 3.2 \( \times 10^6 \) K, corresponding to a preflare heating rate of approximately 0.01 ergs cm\(^{-3}\) s\(^{-1}\). The spatial dependence of the flare heating rate is assumed to be a Gaussian centered at the loop apex, with the full width at half-maximum occurring a distance \( 5 \times 10^6 \) cm from the apex. The amplitude of the heating rate is 10 ergs cm\(^{-3}\) s\(^{-1}\) at the loop apex. The time dependence of the heating is chosen to be a square pulse starting at \( t = 0 \) and ending after a time \( t_f \). For \( t > t_f \), the heating is returned to its preflare value. The simulations shown in PPSVALR’s Figure 9 (also reproduced in Fig. 5 of this paper) show the evolution of the average coronal pressure (squares) and apex temperature (triangles) with time for \( t_f = 100 \) s. PPSVALR’s Figure 11 (also reproduced in Fig. 6 in this paper) shows the same variables for flare heating of the same magnitude, but with \( t_f = 300 \) s. Both simulations follow the loop evolution for an elapsed time of some 1000 s.

There are several important features of these two numerical simulations. In both cases, the apex temperature jumps to near \( 2 \times 10^7 \) K within 10 s and remains at that level for the duration of the flare heating period. After the first 10 s, the coronal density (given by PPSVALR only for the \( t_f = 100 \) s simulation) and the coronal pressure increase more or less linearly until \( t = t_f \). During this phase, the loop in both simulations undergoes rapid chromospheric evaporation. After the heating ceases, the loop enters a transient period of evaporative cooling. The large conductive flux produced by the heating continues to drive evaporation, but since it is no longer sustained by flare heating, the coronal plasma now cools until
there is insufficient conductive flux to drive evaporation. The evaporative cooling phase lasts roughly one conductive cooling time, which is less than 100 s for the $\tau_{fl} = 100$ s simulation. The evaporative cooling phase is followed by a period of condensation, where the temperature, pressure, and density all decrease slowly in time. At this point, there is a qualitative difference between the $\tau_{fl} = 100$ s and the $\tau_{fl} = 300$ s simulations. In the former case, the temperature undershoots its preflare value after the condensation phase but remains at coronal values and appears to be slowly recovering to the preflare temperature. In the latter case, the longer heating time resulted in a greater coronal density that causes a catastrophic cooling of the entire corona to below $10^5$ K. PPSVALR note that this is the only one of their simulations which exhibits this behavior (probably

Fig. 5.—Comparison of coronal pressure and apex temperature obtained from our model (solutions of eq. [16] plus eqs. [A14] and [B7]) and the simulations shown in Fig. 9 of PPSVALR ($\tau_{fl} = 100$ s). Solid curve is our computed value of the apex temperature, while triangles are the apex temperature from PPSVALR. Dashed curve is our computed values of the pressure, while squares represent those from PPSVALR.

Fig. 6.—Comparison of coronal pressure and apex temperature obtained from our model (solutions of eq. [16] plus eqs. [A14] and [B7]) and the simulations shown in Fig. 11 of PPSVALR ($\tau_{fl} = 300$ s). Solid curve is our computed value of the apex temperature, while triangles are the apex temperature from PPSVALR. Dashed curve is our computed values of the pressure, while squares represent those from PPSVALR.
because it is the only one with sufficiently high coronal density. We feel that this behavior is to be expected if the coronal density has risen above some critical value, and that the observational consequence would be the formation of cool postflare loops (but see comments at the end of Appendix B).

The results of our calculations based in equation (16) are overlaid on Figures 5 and 6. The solid curves show our computed apex temperatures (cf. triangles), and the dashed curves show our computed coronal pressures (cf. squares). Because of the sudden onset and shut off of the flare heating rate, the loop in our model is in either the strong evaporation or strong condensation phase throughout the calculation, skipping over the intermediate scaling law phase (the loop apex temperatures from our model are therefore computed from eqs. [A14] or [B7], depending on the regime).

In our calculations, we have tried to the greatest extent possible to emulate the conditions in the gasdynamic simulations. However, there are two assumptions we were forced to make. First, PPSVALR use a nonuniform heating rate, which we average over the loop in order to obtain a uniform heating rate of 3.13 ergs cm$^{-3}$ s$^{-1}$ for use in our model. Second, PPSVALR use a tabulated cooling curve $A(T)$ (Raymond, Cox, and Smith 1976) and not a simple power law of the form $A = AT^n$ which is used in our model (see § II). Our approach is to set the coefficient $A$ such that our model does an optimal job of duplicating the pressure decay behavior of the condensation phase in the simulations. This value of $A (= 2.2 \times 10^{-19}$ in cgs units) also fits the calculation of Raymond, Cox, and Smith (1976) fairly well at transition region temperatures and temperatures near $3 \times 10^6$ K, but it is otherwise generally higher than their curve. We use the cooling law index $n = -\frac{1}{2}$. Finally, we find that although we can adjust the parameter $\Omega$ for the strong evaporation phase, our computed apex temperatures agree almost exactly with the values of PPSVALR; and our computed pressure variations, although lower by a factor of 1.5-2.0 than those obtained by PPSVALR, exhibit almost identical time variation (since we only have access to the PPSVALR's published results and not the details of their gasdynamic code, it is impossible to ascertain why our pressures are less than theirs by this relatively constant factor).

During the condensation phase, we find that the general trends are correctly predicted by our model, but there are discrepancies in the details. For example, in Figure 6 our temperature decay during the condensation phase matches the simulation results very well. On the other hand, our calculation of the pressure decrease seems at first too rapid, and then later too slow, although the general trend is matched. Our computed pressures are generally factors of 3 lower than the simulations over most of the condensation phase. In contrast, Figure 5 shows that our computed temperature drops off somewhat too slowly compared to the simulation. At the worst point, the apex temperature is a factor of 2 too high. The computed pressures, on the other hand, have essentially the same time variation as those from the simulation, though our values are still somewhat smaller (see comment above). We note that our model correctly predicts the cooling catastrophe seen in Figure 6. However, it also predicts a similar cooling catastrophe for the case shown in Figure 5 (occurring at a time beyond that shown on the figure), which is not seen in the gasdynamic simulations. We believe this is due to our model not having sufficiently realistic physics to properly return to the preflare state after the undershooting phenomenon (this would correspond to yet another "regime" which does not exist in our model). In general, our model should not be used once the temperature during condensation has dropped below the preflare value. Since the strong condensation phase as idealized in Appendix B neglects conduction entirely, our model cannot correctly return the loop to its preflare state if strong condensation is still occurring at low coronal temperatures. Our model does return the loop to its approximate preflare state if the loop enters the scaling law regime, but this seems to occur only if flare heating decays reasonably slowly. Finally, our model is incapable of reproducing the transient evaporative cooling stage seen in the simulations, since we do not include a phase corresponding to decreasing temperature and pressure with continuing evaporation. Although this transient stage only lasts about 10% of the total simulation time, we feel that this may explain why our temperature and pressure variations computed during the condensation phase did not fit the simulations as well as those found during the evaporation phase.

VII. EXAMPLES OF FLARE LOOP EVOLUTION

One of our principal goals in developing the loop evolution equation is to calculate the behavior of the flare corona for a proposed flare heating rate, and then to test the computed results by comparing with actual flare observations. By iterative comparison, we then hope to arrive at a self-consistent picture of how and when the energy is released during a flare. To illustrate this process, we consider two specific examples. First, we discuss the 1980 May 21 solar flare observed by SMM; we then consider the huge stellar flare of 1985 April 12 observed on the M dwarf AD Leo.

a) The Flare of 1980 May 21

Antonucci, Gabriel, and Dennis (1984, hereafter AGD) have analyzed the data from a number of solar flares observed in soft and hard X-rays with the bent crystal spectrometer (BCS) and the hard X-ray burst spectrometer (HXIBS), respectively, on the Solar Maximum Mission (SMM) spacecraft. BCS spectra taken in a Ca XIX line can be used to determine the temperature and emission measure of the flaring coronal plasma as a function of time. AGD show the detailed time histories of the coronal temperature, emission measure, and hard X-ray count rate for four flares. From images of the flaring regions taken with the hard X-ray imaging spectrometer (HXIS), they have estimated the footpoint areas and coronal volumes involved in the flares. Finally, by combining the geometrical data with observed temperatures and emission measures, AGD have estimated the total energy in the coronal plasma over the course of these flares. These data are summarized in their Table 2. AGD claim that the soft X-ray time history can be understood by equating the rate of coronal energy increase to the flux of nonthermal electrons implied by the hard X-rays. In this section of the paper, we will test this hypothesis by using their flare geometry, the hard X-ray count rate, and the loop evolution equation to compute time-dependent behavior of the coronal temperature and emission measure. This can then be checked for consistency with the
We have chosen the flare 1980 May 21 discussed in AGD (their Fig. 4) for our comparison.

According to AGD, the May 21 flare had hard X-ray footpoints which were separated by 40°. If we assume the magnetic loop connecting the footpoints is semicircular, this implies an apex to footpoint loop length of $L = 2.36 \times 10^9$ cm. We assume that the coronal heating rate $Q$ is proportional to the HXRBS count rate (with modifications described below). We normalize the heating rate by specifying the total energy per unit volume $e_{\text{tot}}$ released over the course of the flare, i.e., $e_{\text{tot}} = \int_0^t Q(t) \, dt$. We let $e_{\text{tot}}$ and the area of the loop footpoints be free parameters which we vary in order to match the observed behavior of the temperature and emission measure. We set the radiative loss coefficient $A$ to $1.2 \times 10^{-19}$ in cgs units and $\alpha = -\frac{1}{3}$, since these values give good overall agreement with Raymond, Cox, and Smith (1976) at flaring coronal temperatures. We set $\Omega$ (somewhat arbitrarily) to 0.26 for strong condensation.

Our model is unable to cope with all parts of the $Q(t)$ behavior given by the HXRBS count rate in Figure 4 of AGD. The reason for this is that the first hard X-ray peak rises and falls too quickly for our model to be valid. Both the rise and fall occur on time scales comparable to the sound transit time through the loop, meaning that our neglect of the momentum equation is not justified. A manifestation of the lack of validity is that during the decay of the first X-ray peak, while the loop is still in the “strong evaporation” regime, the value of $\Omega$ as computed from equation (A16) becomes singular because $Q/Q$ becomes sufficiently negative to cause the denominator to vanish. However, our model should apply over the rest of the flare. Our solution to this difficulty is to smear out the first peak and the following dip such that the time-integrated heating rate over the peak and dip remains the same, and to realize that the solution we obtain during the first peak corresponds to a longer, but less intense burst. We have otherwise kept the heating rate proportional to the HXRBS count rate shown in Figure 4 of AGD. The time history of the heating rate used in our calculations is shown as the lowest (dashed) curve in the upper panel of Figure 7.

We have found that a reasonably good match to the observations over most of the period of hard X-ray emission is obtained by setting $e_{\text{tot}}$ to $1.822 \times 10^9$ erg cm$^{-3}$ and the diameter $d$ of the coronal loop to $1.2 \times 10^9$ cm. In the top panel of Figure 7, we have plotted both the observed coronal temperature and the average coronal temperature computed from our model; in the bottom panel, we have plotted both the observed emission measure and the approximate emission measure computed from our model. The observed quantities were taken directly from Figure 4 of AGD. The time origin in Figure 7 corresponds to the time at the left edge of the plots in Figure 4 of AGD, i.e., 20:52 UT. The emission measure from our model is computed from equation (19). We found that to match the initial emission measure, we had to assume the preflare loop had a quiescent heating rate of 0.05 ergs cm$^{-3}$ s$^{-1}$, corresponding to an average preflare coronal temperature of $4 \times 10^6$ K.

There are a number of interesting features in Figure 7. During the initial part of the flare corresponding to the first hard X-ray burst at $t \approx 250$ s, the computed coronal temperature is much higher than that observed. Since we have smeared out the first hard X-ray peak, we expect that the temperature computed from the true burst would be even higher than that which we compute in Figure 7. This implies that during the first part of the flare, the actual coronal heating rate is significantly less than that we infer from the hard X-ray burst. However, from $t = 400$ s to $t = 900$ s, the observed and computed temperatures agree very well. We believe that the key to understanding this discrepancy is to note that by $t = 400$ s, both the observed and computed emission measure have increased by over an order of magnitude from flare onset. If the hard X-ray count rate is interpreted in terms of the thick-target model, then we expect that flare heating is driven by nonthermal electrons accelerated in the corona. If the initial column depth of the corona is small, then most of the energy will be deposited in the chromosphere, where it is radiated away. However, as more evaporation takes place, an increasingly larger fraction of the thick target energy is deposited in the coronal plasma. By $t = 400$ s, most of the energy is apparently being deposited in the corona. After $t = 900$ s, the hard X-ray count rate and hence our assumed heating rate decrease to a minimal level. Although the observed coronal temperature does decrease during this time period, our computed temperature decreases much more quickly. Similarly, we find that...
after \( t = 900 \) s, the observed emission measure becomes significantly greater than our computed values. We conclude that there must be additional coronal heating after the impulsive phase which keeps the loop from cooling and condensing as quickly as our calculations indicate.

Finally, we find that our choices of \( \epsilon_{\text{tot}} \) and \( d \) give good agreement with the impulsive phase flare energetics estimates made by AGD. The total energy deposited in the coronal plasma over the course of our calculations is given by \( \Delta E = \epsilon_{\text{tot}} (\pi/2) d^2 L \), which for our choice of \( \epsilon_{\text{tot}}, d, \) and \( L \) gives \( \Delta E = 9.85 \times 10^{30} \) ergs. This is consistent with the AGD estimate of the total energy increase in the coronal plasma at the end of the impulsive phase, including radiation and conduction, which is \( 8-16 \times 10^{30} \) ergs. Our total energy also agrees very well with the AGD estimate of the total energy in non-thermal electrons derived from hard X-rays, which is \( 10 \times 10^{30} \) ergs.

This example provides a clear illustration of how our model can be used to help interpret flare observations. In addition to the calculations already discussed here, it would be a straightforward matter to determine the degree of post-impulsive heating necessary to match the observed late behavior of the temperature and emission measure. Such calculations are beyond the scope of this paper, since our main objective is simply to describe the model itself and how it can be used.

b) The Flare of 1985 April 12 on AD Leonis

One of the largest flares ever observed on the M dwarf star AD Leo is the event of 1985 April 12 (Pettersen, Hawley, and Anderson 1986; Hawley 1989). Optical and ultraviolet spectroscopy and optical photometry were obtained during the more than 4 hr long event. A comprehensive overview of these observations is given by Hawley (1989). Hawley (1989) concludes that the total radiated energy from the flare exceeds \( 10^{34} \) ergs and takes place in a loop or loops with an apex to footpoint length of roughly \( 10^{10} \) cm. From an inferred flare area of some 1%-10% of the stellar surface, Hawley (1989) estimates that the average energy density \( \epsilon_{\text{tot}} \) deposited in the loop is roughly \( 10^2-10^3 \) ergs cm\(^{-3}\) over the course of the flare. We will assume a value of \( \epsilon_{\text{tot}} = 3 \times 10^3 \) ergs cm\(^{-3}\) here. To apply our model, we first construct a proposed heating history based on the temporal behavior of the observed \( U \)-band light curve and the Balmer line and \( \text{Ca} \)\(\text{II}\) resonance line fluxes. We then compute the behavior of the coronal temperature, pressure, and emission measure as functions of time. Had X-ray observations been available, we could then have checked our predictions directly with X-ray temperatures and emission measures. Since they were not available, we must construct time-dependent models of the flare chromosphere (where the lines are formed) in order to check the self-consistency of the proposed heating. The chromospheric models are beyond the scope of this paper, but they are discussed in Hawley (1989). We will use her results to facilitate comparisons between our model and the observations.

In the AD Leo event, the white-light emission peaks and decays very quickly compared to the line emission. The \( U \)-band flux reaches its maximum roughly \( \tau_s \approx 500 \) s after flare onset and then drops off quickly. The Balmer lines, on the other hand, peak about \( 1000 \) s after flare onset and subsequently decay on a time scale of roughly \( 2000 \) s. The \( \text{Ca} \)\(\text{II}\) flux reaches its peak about \( 1500 \) s after onset and also decays on a time scale of about \( 2000 \) s. In the case of white-light flares on the Sun, there is a demonstrated correlation between hard X-rays and white-light emission (Kane et al. 1985). Assuming that the same correlation applies to the AD Leo flare, the \( U \)-band emission can be used as a proxy for impulsive phase heating. We therefore assume that the coronal heating rate rises linearly with time to a peak at \( t \approx 500 \) s. Because there is continued flare emission even 4 hr after flare onset, it is clear that there must be some continued heating after the impulsive phase. Since the emission-line fluxes decay on the time scale of roughly \( \tau_s \approx 2000 \) s, we assume that the coronal heating rate decreases exponentially on this time scale after its peak at \( 500 \) s. This leads us to propose a heating rate of the form

\[
Q(t) = \frac{\epsilon_{\text{tot}}}{\tau_s + \tau_s/2} \quad t < \tau_s, \\
Q(t) = \frac{\epsilon_{\text{tot}}}{\tau_s} \exp \left( \frac{t - \tau_s}{\tau_s} \right) \quad t > \tau_s.
\]

Note that \( Q(t) \) is continuous (but its derivative is not) and that \( \int_0^\tau \! Q(t) \, dt = \epsilon_{\text{tot}} \).

From the values of \( \epsilon_{\text{tot}} \) and \( \tau_s \) given in the text, we compute the evolution of a flare loop using equation (16). For this calculation, we set \( L = 10^{16} \) cm, and again set \( \Omega = 0.26 \) for the strong condensation phase. Assuming that the optical flare occurs in roughly 5% of the projected star area, we estimate the diameter of the coronal loop to be \( 9 \times 10^8 \) cm, which results in a pulsed looking coronal loop. A preflare heating rate of 0.001 ergs cm\(^{-3}\) s\(^{-1}\) gives an average preflare coronal temperature of \( 3 \times 10^9 \) K, consistent with that determined by Hawley (1989). The coefficient \( A \) of the radiative loss rate (which affects the constant \( K \) in eq. [16]) was set to 1.2 \( \times 10^{-12} \) ergs units to be consistent with the value used for the SMM flare analyzed in § VIIa and that used in the chromospheric modeling described by Hawley (1989). The predicted loop evolution is shown by the solid and dashed curves in Figure 8. The solid and dashed curves in the upper panel show the average coronal temperature \( T_c \) and the equivalent static loop apex temperature \( T_{\text{apex}} \), respectively. The dashed and solid curves in the lower panel show the temporal behavior of the loop pressure and emission measure, respectively. We have marked the upper panel to show when the loop is in the three evolutionary regimes. During the time period marked "SE" (from 0-1200 s) the loop undergoes strong evaporation, during the period marked "SL" (from 1200-4400 s) the loop is in the scaling law regime, and during the period marked "SC" (from 4400 s on) the loop undergoes strong condensation. One of the shortcomings of our model is apparent in this figure. Because we use sharp definitions of the three different loop phases, the average coronal temperature, although continuous, exhibits kinks with time as the loop crosses the threshold between regimes at 1200 and 4400 s. The kinks are also present, but not as noticeable, in the behavior of the emission measure. The pressure does not exhibit kinks because equation (3) guarantees that \( P \) will be continuous. Since the sharp boundaries between the loop regimes are somewhat unphysical, we believe it is acceptable for the user to connect the curves between the regimes smoothly. However, we have not done so in this example. Note that there is also a kink in the average coronal temperature at 500 s during the strong evaporation phase. This is due to the cusp in the heating rate in equation (20) at 500 s. The kink could be smoothed out by choosing a heating rate with a continuous derivative.

The most salient result of our curves shown in Figure 8 is that both the coronal temperature and emission measure are
Fig. 8.—Predicted coronal evolution for the 1985 April 12 flare on AD Leo. Solid and dashed curves in the top panel show the time evolution of the average coronal temperature and the equivalent static loop apex temperature, respectively. The three loop regimes (strong evaporation, scaling law behavior, and strong condensation) are marked with the symbols SE, SL, and SC. Dashed and solid curves in the bottom panel show the evolution of the pressure and coronal emission measure, respectively. The diameter \( d \) of the loop is assumed to be \( 9 \times 10^9 \) cm, and the apex to footpoint loop length \( L \) is assumed to be \( 10^{10} \) cm, resulting in a pudgy looking loop. The squares and triangles in the upper panel represent the coronal temperature as determined from inverting the \( H_\gamma \) and \( Ca^{II} K \) line fluxes, respectively, through the chromospheric models of Hawley (1989).

Elevated for a relatively long time. Although the coronal temperature barely exceeds \( 2.5 \times 10^7 \) K at the heating peak, it stays above \( 2 \times 10^7 \) K for nearly 5000 s. The emission measure peaks at over \( 10^{52} \) cm\(^{-3} \) about 2200 s into the flare. It exceeds \( 10^{51} \) cm\(^{-3} \) for nearly the entire calculation, except the first 500 s. About 7000 s into the flare, the average coronal temperature drops below \( 10^7 \) K and thereafter begins to plummet rapidly, due to the continually decreasing heating rate and the increasing cooling rate as the temperature drops. As mentioned earlier, we stop our calculations when the temperature drops below its preflare value, since our model then becomes unphysical. This occurs at just over 8000 s.

The squares and triangles in the top panel of Figure 8 show the predicted coronal temperature as a function of time for the AD Leo flare, based on the observed line fluxes in \( H_\gamma \) and \( Ca^{II} K \) and the chromospheric models computed by Hawley (1989). These models are self-consistent calculations of chromospheric conditions at the base of the static coronal loop of fixed apex temperature. The observed line fluxes were inverted through the models to give the predicted coronal temperature. It is immediately evident that the qualitative behavior of the equivalent apex temperature (\( T_A^* \)) predicted by our model and that obtained through the observations and chromospheric models are in good agreement. For the \( H_\gamma \) line, there is even good quantitative agreement, considering the uncertainty in the total flare energy, etc. The difference in the temperatures predicted from the \( H_\gamma \) line and the \( Ca^{II} K \) line may be explained by an area difference (Hawley 1989).

The observations of chromospheric emission show continued evidence for a hot dense corona after 8000 s, in contrast to our calculations. Our interpretation of this disagreement is that there is continued heating above the rate we specified in equation (20) at late times in the flare. The assumption that the decay in the heating rate could be determined by modeling the decay rate of the line fluxes as an exponential was not borne out at the latest times of the flare in our calculations. A differ-
ent time dependence for the heating late in the flare may be required. Determining this, however, is beyond the scope of this paper.

Finally, we point out that the equivalent static loop temperature (used in the chromospheric modeling) is roughly equal to the average coronal temperature from 1000–6000 s. During this period, the use of equivalent static loops (for example, to compute X-ray heating) for chromospheric modeling should be a good approximation. During the first part of the flare, $T_{\text{e}}$ is considerably less than $T_\text{c}$, so that the equivalent static loop X-ray spectra might be unduly soft. On the other hand, during the last 1000 s of the flare, the average coronal temperature plummeted, while $T_{\text{e}}$ drops only slowly. During this time, the X-ray spectrum computed from the equivalent static loop is liable to be unrealistically hard. The large difference between $T_\text{c}$ and $T_{\text{e}}$ during the early and late parts of the flare calculation also shows that one should expect significant departures from the loop scaling law at correspondingly early or late times in flare observations.

VIII. SUMMARY

Our goal has been to develop a reasonably simple algorithm for calculating the response of a coronal loop to the release of flare energy over rather long time scales. We began in § II by deriving the loop integrated energy equation (3) upon which our model is based. From this equation, it was clear that we needed to express the mean cooling rate from the evolving loop in terms of global loop variables. In § III we developed a numerical model for calculating the structure of evaporating and condensing loops. This model makes use of an approximate solution of the continuity equation which assumes that $h/n$ is uniform in the loop and is equal to $N_2/N_\text{e}$. We found that the average cooling rate is primarily a function of the total column depth of plasma within the loop, for both evaporating and condensing loops. Specifically, we found that the cooling rate is approximated rather well by the “equivalent static loop” of the same length and column depth as the evolving loop. The structure of static loops was reviewed in § IV.

We sought next to relate the rate of pressure change to the evaporation rate, since the integrated energy equation involves $\dot{P}$ but the mean cooling rate depends primarily on the column depth. We express the relationship as $\dot{N}_2/N_\text{e} = \Omega \dot{P}/P$, where $\Omega$ is evaluated in Appendices A and B. In Appendix A, we found from an analytical model that in a loop undergoing strong evaporation due to a large flare heating rate $Q(t)$, $\Omega$ is given by equation (A16). In general, this value is close to unity unless $Q(t)$ is changing rapidly. In Appendix B we study analytical models of strongly condensing loops, from which we conclude that $\Omega \approx 0.4$. When we analyze the numerical simulations of PPSVALR, we find that $\Omega \approx 0.2-0.3$ for strongly condensing loops. We note in § V that loops with a rough equilibrium between heating and losses will be reasonably close to those predicted by the static scaling law. These loops will have a value of $\Omega \approx \frac{1}{3}$.

In § V we synthesize the above results into a single ordinary differential equation (eq. [16]) which describes how a loop evolves during a flare. We divide loop evolution into three phases based on the ratio of flare heating to the mean loss rate: strong evaporation, scaling law behavior, and strong condensation. In § VI, we compare our model with two of the gasdynamic simulations of PPSVALR. Finally, in § VII we use our model to study the 1980 May 21 solar flare observed by SMM and to investigate what the coronal evolution may have been for the huge flare of 1985 April 12 on the star AD Leo.

To what extent have we succeeded in meeting the goal of our work? Attempts to characterize systems having many degrees of freedom by just a few parameters can be dangerous and our model is therefore compromised to some degree by the limitations of assumptions which were necessary to derive the model in the first place. Glaring examples of these include (1) the assumption that we are only interested in changes on time scales greater than the loop sound transit time (necessary to neglect the momentum equation); (2) the assumption that the flare heating rate is uniform in the loop; and (3) the assumption that a flare loop will at any given time be in one of the three well-defined regimes. Although we believe that there are distinct phases of loop evolution, the interface between them will not be as sharp as was illustrated for example in Figure 8. These kinks in the temporal behavior of the loop remain a defect of the model. Another shortcoming of our model is its failure to realistically duplicate the last stages of flare loop evolution, the recovery to the preflare state. We therefore feel that our model is no longer useful once the average coronal temperature has dropped below its preflare value.

However, our model is capable of reproducing most of the features seen in the gasdynamics simulations of PPSVALR with relatively little effort in terms of programming and CPU time. It does a particularly good job of computing the state of the loop when strong chromospheric evaporation is occurring. It reproduces the qualitative features of the condensation phase faithfully, and it correctly predicts the cooling catastrophe seen in Figure 6. Because of the relative simplicity of our model, we expect it to be useful in studying flare evolution over long time scales where the use of gasdynamic simulations would not be feasible. Future gasdynamic simulations over long time scales will also benefit by using our model as a first approximation to the expected flare behavior.

There are a number of directions which could be taken to improve the work described in this paper. Two obvious candidates include (1) a more careful integration of the energy equation with the pressure held uniform, but the continuity equation solved in full, rather than using the approximation we make in § III. A particular point of inquiry should be whether the robustness of our average cooling rate result (eq. [13]) holds when the continuity equation is solved exactly; (2) the role of magnetic field convergence in altering the results discussed in this paper.

We will make available, upon request, the computer program we use to solve the loop evolution equation (eq. [16]) with the understanding that any publications making use of the program will be sent to us prior to publication.

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STRONGLY EVAPORATING LOOPS

In Figure 2, we show the various terms in the energy equation for a loop undergoing strong evaporation. It is clear from the figure that over most of the corona, the dominant terms in the energy equation are flare heating, enthalpy flux, and conductive flux. Radiative losses play an important role only at the very base of the loop. These results motivate us to solve the energy equation (1) in a loop where radiative losses are absent. From equation (3), we first obtain \( Q = 3/2P \), which reduces the energy equation to

\[
\frac{dF}{dz} + \frac{5}{2} P \frac{dv}{dz} = 0 .
\]

Thus \( F_e + 5/2Pv = C \), where \( C \) is a constant of integration. If the conductive flux and the velocity must both vanish at the loop apex \( z = L \), then \( C = 0 \), and we have

\[
F_e = -\frac{5}{2}Pv .
\]

The equation is identical to that derived by Antiochos and Sturrock (1978) in their treatment of evaporative cooling. While they solved the equation by separating variables, we proceed by substituting our approximate solution (6) for the continuity equation (4) to find the velocity \( v \). When we substitute equation (2) for the conductive flux, we then find

\[
\kappa_0 T^{5/2} \frac{d T}{d N} = 10 \frac{k^2 T^2 N}{P} \left( 1 - \frac{N}{N} \right) .
\]

Assuming that the temperature at the base of the loop is small compared to the apex temperature, this equation has the solution

\[
T^{3/2}(x) = \frac{15k^2 N}{\kappa_0 P} \left( x - \frac{1}{2} x^2 \right) ,
\]

where \( x = N/N \). The apex temperature \( T_A \) is given by

\[
T_A^{3/2} = \frac{15k^2 N_t}{2\kappa_0 P} .
\]

We use the boundary condition \( T = T_A \) at \( z = L \) to eliminate \( N_t \) as follows. We first define \( L = \int_0^N dN/n(N) \), then we use the equation of state and the fact that the pressure is uniform to express \( n(N) \) in terms of \( T(N) \). The resulting expression for \( L \) is

\[
L = \left( \frac{15k^2 N_t N_c}{\kappa_0 P} \right)^{2/3} \frac{\eta k N_t}{P} ,
\]

where \( \eta = \int_0^1 (x - x^2/2)^{2/3} dx = 2^{2/3} B(5/3, 5/3) = 0.4656 \). Using equation (A7) to eliminate \( N_t \) in terms of \( N_t, P, \) and \( L \), equation (A6) becomes

\[
P = 2^{2/3}\eta k T_A N_t/L .
\]

From equations (A6), (A7), and (A8), the ratio \( N_t/N \) is given by

\[
\frac{N_t}{N} = \frac{\zeta \kappa_0 T_A^{3/2}}{P L^2} ,
\]

where

\[
\zeta = \frac{15}{2} 2^{1/3} \eta^2 = 0.2913 .
\]

We are now able to determine the time-dependent behavior of the loop. We assume that heating rate does not change much on a heating time scale \( \tau_H = (3/2)P/Q \), which means that over some period of time, the solution to equation (3) can be written

\[
P(t) = (2/3)Q t ,
\]

and therefore \( P/P = 1/t \). We also have

\[
\frac{\dot{P}}{P} = \frac{N_t}{N_t} + \frac{\dot{T}_A}{T_A} .
\]

Using equations (A9), (A11), and (A12), we derive the following differential equation for the time behavior of the apex temperature:

\[
\frac{d \ln T_A}{d \ln t} = \left( 1 - \frac{3}{2} \frac{\kappa_0 T_A^{3/2}}{Q L^2} \right) .
\]
The equation clearly shows there are two distinct types of behavior depending on the ratio of the conductive cooling rate to the flare heating rate. When the second term on the right-hand side of equation (A13) is small compared to unity, the temperature increases linearly in time in proportion to the flare heating rate. When this term approaches unity, the temperature saturates at an upper limit, given by

$$T_a = \left( \frac{2/3Q^2}{\zeta \kappa_0} \right)^{2/3}.$$  \hspace{1cm} (A14)

It is the second, saturated regime, with the apex temperature given by equation (A14), that is relevant here, since our analysis is only valid on time scales long compared to the hydrodynamic time scale. This is generally much longer than the heating time scale, which is the time the gas in the loop would spend in the first regime before the temperature saturates. Changes on that time scale are outside the scope of our analysis here and must be done by explicitly solving the gasdynamic equations. We therefore assume the apex temperature is given by equation (A14) when the loop is undergoing strong evaporation. The time dependence of the temperature is therefore directly related to the time dependence of the flare heating rate. From equation (A14), we have

$$\frac{\dot{T}_a}{T_a} = \frac{2}{70/0} - (A15).$$

We then find $\Omega$ from equations (7), (A9), and (A12) to be

$$\Omega = \left( 1 + \frac{2}{Q} \frac{P}{\zeta \kappa_0} T_a^{3/2} \right)^{-1}.$$ \hspace{1cm} (A16)

For $Q$ fixed in time or changing only slowly, this implies that $\Omega = 1$, whereas if $Q$ is increasing or decreasing sufficiently rapidly, $\Omega$ becomes respectively smaller or larger than one. Our experience has been that during the early part of the flare, $\Omega$ can become noticeably less than one if the heating rate ramps up quickly, but thereafter stays near unity.

**APPENDIX B**

**STRONGLY CONDENSING LOOPS**

Figure 3 shows a numerically integrated loop that is undergoing strong condensation. The dominant terms in the energy equation throughout most of the corona are enthalpy flux divergence, the $(3/2)P$ term, and radiative losses. The heating rate is negligible, and the conductive flux divergence, while not negligibly small, is dominant only at the very base of the loop. In the same spirit as Appendix A, we now solve the energy equation for a condensing loop, omitting the heating and conduction terms. The energy equation (1) then becomes

$$\frac{3}{2} \dot{P} + n^2 A T^* + \frac{5}{2} P \frac{dv}{dz} = 0.$$ \hspace{1cm} (B1)

As before, we assume that $h/n$ is uniform in the loop, which yields equation (6) for the velocity. Using equations (5) and (6) to evaluate the velocity derivative, we find that the enthalpy flux divergence becomes

$$\frac{5}{2} P \frac{dv}{dz} = 5kN_t \left( 1 - \frac{N(z)}{N_t} \right) \frac{dT}{dz} - \frac{5}{2} P \frac{\dot{N}_t}{N_t}.$$ \hspace{1cm} (B2)

Again, $\dot{N}_t$ is an unknown quantity, which we will later relate to $\dot{P}$ through the parameter $\Omega$. When equation (B2) is substituted into equation (B1), we can divide by $n$ and once again switch our independent variable from $z$ to $N$. We then find that the temperature derivative is

$$\frac{dT}{dN} = \frac{(2kT/P)[(5P\dot{N}_t/2N_t) - (3/2)\dot{P}] - (P/2kT^2)A T^*]}{5kN_t(1 - N/N_t)}. \hspace{1cm} (B3)$$

One important point can be made immediately from equation (B3). The boundary condition that the temperature gradient vanish at the loop apex requires that

$$\frac{5}{2} P \frac{\dot{N}_t}{N_t} - \frac{3}{2} \dot{P} = \frac{P}{(2k)} A T_a^{3/2} = R_A,$$ \hspace{1cm} (B4)

where $R_A$ is the loss rate at the loop apex. If we eliminate $\dot{N}_t/N_t$ in terms of $\dot{P}/P$ using equation (7), and note that $\langle R \rangle = -3/2\dot{P}$ because $\langle R \rangle \gg Q$, equation (B4) becomes

$$\Omega = 3 \left( 1 - \frac{R_A}{\langle R \rangle} \right). \hspace{1cm} (B5)$$

Equation (B5) shows that $\Omega$ is bounded above by $\frac{3}{5}$. Our more stringent upper limit on $\Omega$ (discussed below) is significantly less than this.
FISHER AND HAWLEY

Equation (B3) is separable, and after some manipulation making use of equations (B4) and (7), we find

\[ T(x) = T_0 \left[ 1 - (1 - \alpha) \left( \frac{3}{5} - Q(2 - a) \right) \right]^{1/(2 - a)}, \tag{B6} \]

where \( x = N/N_c \). If we require that the integrated length of the loop be \( L \), then we find in a calculation analogous to that carried out in Appendix A that

\[ P = 2akT_0 N_c / L, \tag{B7} \]

where

\[ \sigma = \frac{\Omega}{(3/5 - \Omega)(2 - a)} \left( \frac{3 - \alpha}{2 - \alpha} \right) \left( \frac{\Omega}{(3/5 - \Omega)(2 - a)} \right). \tag{B8} \]

One can use the temperature solution \( T(x) \) to calculate the spatially averaged loss rate \( \langle R \rangle \). Unfortunately, this integration leads to a result that is much too close to exactly the same result as equation (B5) and hence yields no new information concerning \( \Omega \). However, we can derive some limits on \( \Omega \) from estimates of the ratio \( R_d / \langle R \rangle \). For example, we know that the conductive flux, and hence the temperature gradient in a condensing loop of a given mass will be less than that in a static loop of the same mass. This means that \( R_d / \langle R \rangle \) must be smaller in a static loop than it would be in a condensing loop. This ratio is easily calculable for static loops; we find by simplifying equation (6) of Fisher and Hawley (1989) that \( R_d / \langle R \rangle = 3 + 2\alpha / 7 \). Using this as a lower limit for the ratio in equation (B5), we finally obtain

\[ \Omega < \frac{12 - 6\alpha}{35} = \frac{3}{7} \approx 0.43 \quad \text{for} \quad \alpha = -\frac{1}{2}. \tag{B9} \]

This strict upper limit is consistent with the value of \( \Omega \) we find by analyzing the numerical simulations of PPSVALR (i.e., 0.2–0.3), and it gives us good reason to believe this is the appropriate range of \( \Omega \) to use for the strong condensation phase.

Finally, we comment on the thermal stability of our condensing loop solutions. It has been established that static coronal loops are thermally stable (McClymont and Craig 1985a, b) for a wide range of coronal heating dependencies, including the simple volumetrically uniform case considered in this paper. The physical mechanism which prevents thermal runaway in static loops is heat conduction between the thermally unstable plasma in the corona and the thermally stable plasma below \( 10^7 \) K. For condensing loops, however, heat conduction is only a moderately important term in the energy equation (see Fig. 3). It is therefore unclear whether our solutions are thermally unstable or not; or if unstable, whether they are unstable for all stages of condensation or only during the latter stages. For example, it seems likely that perturbations on a short length scale might prove unstable (Antiochos 1976; Antiochos and Sturrock 1976). On the other hand, perhaps the enthalpy flux term would provide some stabilizing influence. We suggest that further study of the thermal stability of condensing loops would provide for interesting future work.

REFERENCES